NOTES

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A Note on the Effros Theorem

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1. INTRODUCTION. *All topological spaces under discussion here are assumed to be separable and metrizable.* An *action* of a topological group G on a space X is a continuous map

$$(g, x) \mapsto gx \colon G \times X \to X$$

such that $ex = x$ for every x in X and $g(hx) = (gh)x$ for g and h in G and x in X (here e denotes the neutral element of G). It is easily seen that for each g in G the map $x \mapsto gx$ is a homeomorphism of X whose inverse is the map $x \mapsto g^{-1}x$.

If x belongs to X and U is a subset of G, then $Ux = \{gx : g \in U\}$. The action of G on X is *transitive* if $Gx = X$ for every x in X. It is *micro-transitive* if for every x in X and every neighborhood U of e in G the set Ux is a neighborhood of x in X.

A metric on a space X is *admissible* if it generates the topology on X. A space is *Polish* if it has an admissible *complete* metric.

Theorem 1.1 (Open Mapping Principle, Version A). *Suppose that a Polish group G acts transitively on a space X. Then the following statements are equivalent:*

(A) *G acts micro-transitively on X.*

(B) *X is Polish.*

(C) *X is of the second category.*

The implication (B) $\Rightarrow$ (C) is simply the Baire Category Theorem for Polish spaces, and (A) $\Rightarrow$ (B) is a consequence of Hausdorff's theorem [**8**] that an open continuous image of a topologically complete space is topologically complete.

This extremely useful result was first proved by Effros [**7**] using a Borel selection argument. Simpler proofs were found independently by Ancel [**1**], Hohti [**9**], and Toruńczyk (unpublished). The proof of Ancel and Toruńczyk is based on an ingenious technique of Homma [**10**], while Hohti uses an open mapping theorem due to Dektjarev [**6**].

The Open Mapping Principle implies Effros's Theorem 2.1 of [**7**] as well as the classical Open Mapping Theorem of functional analysis (for separable Banach spaces). For let B and E be separable Banach spaces, and let $\alpha \colon B \to E$ be a continuous linear surjection. We think of B as a topological group, and define an action of B on E by $(x, y) \mapsto \alpha(x) + y$. This action is transitive, since if y and y' in E and x in B are such that $\alpha(x) = y' - y$, then $(x, y) \mapsto y'$. So by Theorem 1.1, the map $B \to E$ defined by $x \mapsto \alpha(x) + 0$ is open.

The Open Mapping Principle also implies that for every homogeneous compactum (X, ϱ) and every $\varepsilon > 0$ there exists $\delta > 0$ such that, if x and y in X satisfy $\varrho(x, y) < \delta$, then there is a homeomorphism $f \colon X \to X$ such that $f(x) = y$ and such that f

moves no point more than ε. (This goes part way towards explaining the word "micro-transitive.") This interesting and surprising fact, first discovered by Ungar [**15**], was used with great success by continuum theorists in their study of homogeneous continua. See Ancel [**1**] and Charatonik and Maćkowiak [**5**] for details and further references.

A space is *analytic* if it is a continuous image of a Polish space. It is well known that an absolute Borel set is analytic and that a Borel subspace of an analytic space is analytic. For information on analytic spaces see, for example, Kechris [**11**].

The aim of this note is to present a direct and completely elementary proof of the following generalization of Theorem 1.1.

Theorem 1.2 (Open Mapping Principle, Version B). *Suppose that an analytic group G acts transitively on a space X. If X is of the second category, then G acts micro-transitively on X.*

This result is similar to the result of Charatonik and Maćkowiak [**5**] asserting that a Borel subgroup of the group of all homeomorphisms of a compact space acts micro-transitively provided that it acts transitively. Their proof is based on the classical and nontrivial result of Lusin and Sierpiński that, if A is an analytic subspace of a space X, then there is an open subset U of X such that $(U \setminus A) \cup (A \setminus U)$ is meager. Our proof of Theorem 1.2 is direct and can be explained rather easily to students familiar with the Baire Category Theorem. Theorem 1.2 may very well have been observed earlier, although besides the Charatonik-Maćkowiak paper we could not find a reference.

2. MEAGER SETS.

A subset A of a space X is called *meager* or *first category* in X if it is contained in a countable union of nowhere dense subsets of X. If X is not of the first category, then X is of the *second category*. A countable union of meager sets is meager, as is any subset of a meager set. The empty set is clearly meager. A subset A of a space X is *nowhere meager* in X provided that no nonempty relatively open subset of A is meager in X. Finally, we call a subset of X *fat* in X if it is both nowhere meager and dense in X. Observe that if A is fat then so is every larger set.

Lemma 2.1. *Suppose that S and A are subsets of X with S fat and A meager. Then $S \setminus A$ is fat.*

Proof. Let V be a nonempty open subset of X. If $V \cap (S \setminus A)$ is meager, then so is the nonempty relatively open subset $V \cap S$ of S. Indeed, we simply observe that $V \cap S$ is contained in the meager set $(V \cap (S \setminus A)) \cup A$. Thus $V \cap (S \setminus A)$ is not meager, which proves simultaneously that $S \setminus A$ is dense and nowhere meager. ■

Proposition 2.2 is our main tool in the proof of Theorem 1.2. It is a consequence of the Baire Category Theorem (in the relevant setting) and a result of Levi [**13**]. Since we promised a completely elementary proof of Theorem 1.2, we present a simple and direct proof of Proposition 2.2.

Proposition 2.2. *If both S and T are analytic fat subspaces of X, then $S \cap T \neq \emptyset$.*

Proof. Let $\alpha\colon P \to S$ be a continuous surjection, where P is Polish. We denote by $\mathcal{U}$ the collection of all open subsets U of P such that $\alpha(U)$ is meager in X. We claim that $\alpha(\bigcup \mathcal{U})$ is meager in X. To see this, let $\mathcal{V}$ be a countable subcollection of $\mathcal{U}$ such that

$\bigcup \mathcal{V} = \bigcup \mathcal{U}$ (see [**14**, A.2.3]). Since

$$\alpha\left(\bigcup \mathcal{U}\right) = \alpha\left(\bigcup \mathcal{V}\right) = \bigcup\{\alpha(V) : V \in \mathcal{V}\},$$

this establishes our claim. If $P' = P \setminus \bigcup \mathcal{U}$, then P' is Polish (as a closed subset of P), whence $\alpha(P')$ is analytic. In addition, $\alpha(P')$ contains $S \setminus \alpha(\bigcup \mathcal{U})$, hence is fat by Lemma 2.1. Finally, if W' is a nonempty and (relatively) open subset of P', then $\alpha(W')$ is not meager in X. For let W be an open subset of P such that $W \cap P' = W'$. Then $W \notin \mathcal{U}$, hence $\alpha(W)$ is not meager but $\alpha(W) \subseteq \alpha(W') \cup \alpha(\bigcup \mathcal{U})$ and $\alpha(\bigcup \mathcal{U})$ is meager, so $\alpha(W')$ is not meager.

These considerations prove that we may assume without loss of generality that $\alpha\colon P \to S$ has the additional property that, for every nonempty open subset V of P, $\alpha(V)$ is not meager in X. Accordingly, $\overline{\alpha(V)}$ is not nowhere dense (i.e., it has nonempty interior). We may assume that there are a Polish Q and $\beta\colon Q \to T$ with similar properties.

All our metrics are bounded by 2^{-1}, and on P (respectively, Q) we use complete metrics. By induction on n we construct a nonempty open subset U_n of P, a nonempty open subset V_n of Q, and a nonempty open subset W_n of X, having the following properties:

(1) $\operatorname{diam} U_n < 2^{-n}$, $\operatorname{diam} V_n < 2^{-n}$, and $\operatorname{diam} W_n < 2^{-n}$;
(2) $\overline{U}_{n+1} \subseteq U_n$, $\overline{V}_{n+1} \subseteq V_n$, $\overline{W}_{n+1} \subseteq W_n$;
(3) $\overline{W}_{n+1} \subseteq \overline{\alpha(U_n)} \subseteq \overline{\beta(V_n)} \subseteq W_n$.

Put $U_1 = P$, $V_1 = Q$, and $W_1 = X$. Suppose that U_n, V_n, and W_n have been found. Pick a nonempty open subset W of X such that $\operatorname{diam} W < 2^{-(n+1)}$ and $\overline{W} \subseteq \overline{\alpha(U_n)}$. Since $\beta^{-1}(W) \cap V_n$ is nonempty, we may pick a nonempty open subset F of Q such that $\overline{F} \subseteq V_n$, $\operatorname{diam} F < 2^{-(n+1)}$, and $\overline{\beta(F)} \subseteq W$. Let W' be a nonempty open subset of X that is contained in $\overline{\beta(F)}$. Since $\alpha^{-1}(W') \cap U_n$ is nonempty, we may pick a nonempty open subset E of P such that $\overline{E} \subseteq U_n$, $\operatorname{diam} E < 2^{-(n+1)}$, and $\overline{\alpha(E)} \subseteq W'$. We conclude that $U_{n+1} = E$, $V_{n+1} = F$, and $W_{n+1} = W$ satisfy our inductive requirements.

Since the metrics on P and Q are complete, there exist p in $\bigcap_{n=1}^{\infty} U_n$ and q in $\bigcap_{n=1}^{\infty} V_n$. Since $\operatorname{diam} W_n < 2^{-n}$ for every n, $\alpha(p) = \beta(q)$, so $S \cap T \neq \emptyset$. ∎

A slightly more complicated argument shows that the intersection of countably many analytic fat subspaces is fat.

3. PROOF OF THEOREM 1.2.

If G is a topological group, then for subsets H and K of G we define

$$H^{-1} = \{h^{-1} : h \in H\},\ HK = \{hk : H \in H, k \in K\}.$$

The neutral element e of G has a neighborhood base consisting of open sets U_n ($n = 1, 2, 3, \dots$) with the following properties:

(i) U_n is symmetric (i.e., $U_n^{-1} = U_n$) and $U_1 = G$;
(ii) $U_{n+1} \subseteq U_{n+1}^2 \subseteq U_n$.

In the remainder of this note, G is a fixed topological group whose neutral element e has a neighborhood base (U_n) satisfying (i) and (ii). We also assume that G acts *transitively* on the space X and that X is of the second category.

For each x in X define the map $\gamma_x : G \to X$ by $\gamma_x(g) = gx$. It is clear that each γ_x is continuous. It is a simple exercise to show that G acts micro-transitively on X if and only if $\gamma_x : G \to X$ is open for every x in X (this explains the terminology "Open Mapping Principle"; see Ancel [**1**, sec. 3, Lemma 1]).

Observe that since U_n is symmetric, x belongs to $U_n y$ if and only if y belongs to $U_n x$. Also notice that $U_1 x = X$ for every x in X.

Our proof of Theorem 1.2 is very similar to the standard proof of the Open Mapping Theorem in functional analysis (see Brown and Page [**2**, pp. 316–317]). First we prove that for every x in X and n in $\mathbb{N}$ the set $\overline{U_n x}$ is a neighborhood of x (the elementary argument for this is well known (see Ancel [**1**] and Hohti [**9**])). Then we proceed to prove that the interior of $\overline{U_{n+1} x}$ is contained in $U_n x$, which finishes the proof.

Lemma 3.1. *For every x in X, n in $\mathbb{N}$, open subset V of X, and z in $V \cap U_n x$ there exists m in $\mathbb{N}$ such that $U_m z \subseteq V \cap U_n x$.*

Proof. There is an element h of U_n such that $\gamma_x(h) = hx = z$. The set $E = \gamma_x^{-1}(V)$ is an open neighborhood of h, so $Eh^{-1} \cap U_n h^{-1}$ is a neighborhood of e. Pick m so large that $U_m \subseteq Eh^{-1} \cap U_n h^{-1}$. We claim that m is as required. To see this, pick an arbitrary element p of $U_m z$. There is an element g in U_m such that $gz = p$. Thus $\gamma_x(gh) = (gh)x = p$ and gh belongs to $E \cap U_n$, which proves that p is in $V \cap U_n x$. ■

Lemma 3.2. *If x is in X and n is in $\mathbb{N}$, then $U_n x$ is not meager in X.*

Proof. Since G is Lindelöf, there is a countable subset F of G such that $FU_n = G$. If φ lies in F, then $(\varphi U_n)x$ is the image of $U_n x$ under the homeomorphism $p \mapsto \varphi p$ of X. If $U_n x$ is meager, then X is meager in itself. Indeed,

$$X = \bigcup \{(\varphi U_n)x : \varphi \in F\},$$

because the action of G on X is transitive. This contradicts the fact that X is of the second category. ■

Corollary 3.3. *For every x in X and n in $\mathbb{N}$ the set $U_n x$ is nowhere meager in X.*

Proof. This follows immediately from Lemmas 3.1 and 3.2. ■

Lemma 3.4. *For every x in X and n in $\mathbb{N}$ the interior of the set $\overline{U_n x}$ is dense in $\overline{U_n x}$. In addition, x belongs to the interior of $\overline{U_n x}$.*

Proof. Let V be an open subset of X that intersects $\overline{U_n x}$. Then by Corollary 3.3 $V \cap U_n x$ is not meager. Hence $\overline{V \cap U_n x}$ is not nowhere dense (i.e., has nonempty interior). This proves that the interior of $\overline{U_n x}$ is dense in $\overline{U_n x}$.

Let V be a nonempty open subset of X that is contained in $\overline{U_{n+1} x}$. Then V intersects $U_{n+1} x$, say hx belongs to V for some h in U_{n+1}. So

$$x \in h^{-1} V \subseteq h^{-1} \overline{U_{n+1} x} = \overline{h^{-1} U_{n+1} x} = \overline{\{(h^{-1}\varphi)x : \varphi \in U_{n+1}\}} \subseteq \overline{U_n x}$$

by (i) and (ii) and the fact that the map $p \mapsto h^{-1} p$ is a homeomorphism. ■

Observe that thus far we have used only the hypothesis that X is of the second category. To finish the proof of Theorem 1.2, we now assume additionally that G is analytic. The continuous map γ_x maps U_n onto $U_n x$. Since U_n is an open subspace of the analytic space G, it is analytic. As a consequence, $U_n x$ is analytic.

Proposition 3.5. *If x belongs to X and n to $\mathbb{N}$, then the interior of $\overline{U_{n+1}x}$ is contained in $U_n x$.*

Proof. Let z be an arbitrary element of the interior V of $\overline{U_{n+1}x}$. Consider the set $U_{n+1}z$, and let W be the interior of $\overline{U_{n+1}z}$. By Lemma 3.4, $E = V \cap W$ is an open neighborhood of z. Since $E \subseteq \overline{U_{n+1}x}$, $U_{n+1}x \cap E$ is dense in E. Similarly, $U_{n+1}z \cap E$ is dense in E. Since $U_{n+1}x$ and $U_{n+1}z$ are analytic, so are $U_{n+1}x \cap E$ and $U_{n+1}z \cap E$ (recall that open subspaces of analytic spaces are analytic). From Corollary 3.3 and Proposition 2.2 it follows that $U_{n+1}x \cap E$ and $U_{n+1}z \cap E$ intersect, say that both contain the element y. Pick elements g and h in U_{n+1} such that $gx = y$ and $hz = y$, and put $\varphi = h^{-1}g$. Then clearly $\varphi x = z$ and $\varphi \in U_{n+1}U_{n+1} \subseteq U_n$ by (i) and (ii). We conclude that z lies in $U_n x$. ■

Lemma 3.4 and Proposition 3.5 yield:

Corollary 3.6. *If x is in X and n in $\mathbb{N}$, then $U_n x$ is a neighborhood of x.*

Observe that Corollary 3.6 and Lemma 3.1 imply that $U_n x$ is open in X for every n.

Remarks. We finish this section by making some remarks.

(1) Roman Pol asked whether the assumptions in Theorem 1.2 actually imply that the space X is Polish. The answer to this question is no. A proof will be published elsewhere.

(2) Let H be an analytic topological group of the first category that admits a continuous homomorphism α onto a Polish group G. Define an action of H on G by $(x, y) \mapsto \alpha(x)y$. It is easy to see that this action is transitive. Theorem 1.2 applies in this situation, whereas Theorem 1.1 doesn't. Examples are easily found. For instance, let

$$H = \{x \in \mathbb{R}^\infty : (\exists N \in \mathbb{N})(\forall n > N)(x_n = 0)\}$$

and $G = \mathbb{R}$, and let $\alpha\colon H \to G$ denote the projection $\alpha(x) = x_1$. Observe that H is analytic, for it is an F_σ-subset of $\mathbb{R}^\infty$.

(3) There are several related results in the literature. See, for example, Problems R and T in Kelley [**12**, chap. 6], Byczkowski and Pol [**3**], and Schwartz's version of the Open Mapping Theorem [**4**, Theorem 1.2.40]. We are indebted to Roman Pol for this information. We repeat for emphasis that Charatonik and Maćkowiak [**5**] proved that a Borel subgroup of the group of all homeomorphisms of a compact space acts micro-transitively provided that it acts transitively.

(4) Theorem 1.2 fails if X is first category. This is well known. A simple example is the following. Let $\mathbb{Q}$ denote the additive group of rational numbers, endowed with the Euclidean topology. In addition, let $\mathbb{Q}_d$ denote the same group with the discrete topology. Consider the action $(p, q) \mapsto p + q$ of $\mathbb{Q}_d \times \mathbb{Q} \to \mathbb{Q}$. This is a transitive action, but it is not micro-transitive, since $\mathbb{Q}_d$ is discrete whereas $\mathbb{Q}$ is not.

(5) Nowhere in the proof of Theorem 1.2 did we use the full strength of the continuity of the action. In fact, we used only the fact that the action is *separately*

continuous (i.e., the maps $g \mapsto gx : G \to X$ for x in X and $x \mapsto gx : X \to X$ for g in G are all continuous). As a consequence, we we actually proved a stronger result than stated.

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An Elementary Proof of Jacobi's Six Squares Theorem

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Let $r_k(n)$ denote the number of representations of the positive integer n as a sum of k integral squares. Then Jacobi's six squares theorem (see [**10**, pp. 159–170]) asserts that

$$r_6(n) = 4\left(\sum_{\substack{d|k \\ d\equiv 3 \pmod 4}} d^2 - \sum_{\substack{d|k \\ d\equiv 1 \pmod 4}} d^2\right) + 16\left(\sum_{\substack{d|k \\ \frac{k}{d}\equiv 1 \pmod 4}} d^2 - \sum_{\substack{d|k \\ \frac{k}{d}\equiv 3 \pmod 4}} d^2\right). \quad (1)$$

Many proofs have been discovered for Jacobi's two, four, six, and eight squares theorems. (See, for example, [**1**], [**4**], [**5**], [**7**], [**8**], and [**11**].)

In [**1**], S. Bhargava and C. Adiga showed how Jacobi's two and four square theorems can be obtained from Ramanujan's famous ${}_1\psi_1$ summation formula. In [**8**], with the aid of an identity of K. Venkatachaliengar, S. Cooper and H. Y. Lam obtained Jacobi's six