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A note on the exact boundary controllability for an imperfect transmission problem

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Abstract

In this note, we consider a hyperbolic system of equations in a domain made up of two components. We prescribe a homogeneous Dirichlet condition on the exterior boundary and a jump of the displacement proportional to the conormal derivatives on the interface. This last condition is the mathematical interpretation of an imperfect interface. We apply a control on the external boundary and, by means of the Hilbert Uniqueness Method, introduced by J. L. Lions, we study the related boundary exact controllability problem. The key point is to derive an observability inequality by using the so called Lagrange multipliers method, and then to construct the exact control through the solution of an adjoint problem. Eventually, we prove a lower bound for the control time which depends on the geometry of the domain, on the coefficients matrix and on the proportionality between the jump of the solution and the conormal derivatives on the interface.

Keywords Exact controllability · Second order hyperbolic equations · Imperfect interface condition · HUM

Mathematics Subject Classification M35B27 · 35Q93 · 93B05 · 35B37 · 35L20

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1 Introduction

Let us consider an open bounded subset Ω of $\mathbb{R}^n$, $n \geq 2$, and let $\Omega_1 = \Omega \setminus \bar{\Omega}_2$, where Ω_2 is an open and bounded set such that $\Omega_2 \subset\subset \Omega$. Denote by $\partial\Omega$ and $\Gamma = \partial\Omega_2$ the external and the interface boundaries respectively (see Fig. 1).

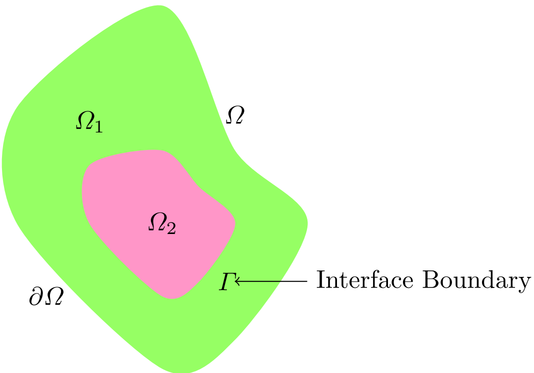
In the above mentioned domain we want to study the boundary exact controllability problem for a hyperbolic system of equations with appropriate boundary and interface conditions on $\partial\Omega$ and on Γ. This evolution system describes the wave propagation in a composite made up of two components having very different coefficients of propagation. More precisely, we prescribe a homogeneous Dirichlet condition on the exterior boundary and a jump of the displacement proportional to the conormal derivatives on the interface. The discontinuity of the solution is the mathematical interpretation of imperfect interface (see [1,3,5–13,15,16,21,22,25–31] and references therein).

The boundary exact controllability problem consists in finding a suitable control, acting on the external boundary or even on just a part of it, driving the trajectories of an evolution system to a desired state at a certain time $T > 0$, for all initial data. Eventually, it reduces to prove an estimate for the energy of an uncontrolled system, at time $t = 0$, through partial measurements of its solution done on the boundary control set. This estimate, known as observability inequality, yields an upper bound for the norm of the initial data of the uncontrolled problem.

In general, the observability inequality does not hold for arbitrary T or control regions. Indeed, the part of the boundary where the control is acting has to satisfy certain geometric conditions. Moreover, as usual in the hyperbolic case, due to the finite speed of propagation of waves, one needs to require that T is sufficiently large. For instance, in the class of regular domains, when using the microlocal approach, one can achieve the observability inequality if and only if every ray of geometric optics, propagating into the domain and reflecting on its boundary, enters the control region in time less than the control time T (see [2]).

In this note, we prove the above mentioned observability inequality by means of Lagrange multipliers method. This is done making use of some results of Monsurrò et al. [28], where the exact internal controllability of the same system of equations is studied.

Fig. 1 $\Omega = \Omega_1 \cup \overline{\Omega}_2$ with $\Omega_2 \subset\subset \Omega$



Usually, when dealing with controllability problems, one fixes a point $x^0 \in \mathbb{R}^n$ which can be viewed as an observer, and which determines the control action region. In the framework of boundary controllability, this control region could be either the entire external boundary of Ω or just a part of it, according to the shape of the domain.

In our context, we need to choose x^0 inside Ω_2 and to assume that Ω_2 is star-shaped with respect to it. This particular choice of x^0 will influence the control time too (see Lemma 5). Moreover, due to the imperfect transmission condition, the control time will depend also on the coefficients matrix and on the proportionality between the jump of the solution and the conormal derivatives on the interface. See Sect. 3 for more details.

The paper is organized as follows. Section 2 is devoted to the introduction of the imperfect interface problem and of the appropriate functional spaces required for its solution (see also [12,25]). Moreover, as usual when studying controllability problems, since the initial data are in a weak space, we need to apply the so called transposition method (see [20], Chapter 3, Section 9). Finally, we give the definition of exact controllability. In Sect. 3, by using a crucial identity proved in Lemma 3.2 of [28] and by adapting to our case some arguments as in [18,19], we obtain the observability inequality and find the lower bound for the control time T (see Lemma 5). To this aim, we apply the above mentioned identity in order to establish two fundamental inequalities, given in Lemmas 2 and 3. Lemma 4 justifies the particular choice of the point x^0. In Sect. 4, we find the exact control by using the Hilbert Uniqueness Method (HUM for short) which is a constructive method introduced by Lions [17,18]. The key point is to define a suitable functional which, as a consequence of the observability estimate, turns out to be an isomorphism. Let us recall that the control obtained by HUM is also the energy minimizing control.

In [18], Chapter 6, J. L. Lions studies for the first time the exact controllability, via HUM, for the wave equation with transmission conditions. More precisely, he considers a Dirichlet problem with a matrix constant on each component of the domain and a control set on part of the external boundary. For the case of a Neumann boundary exact controllability problem in the same framework we quote here [24]. For what concerns the internal exact controllability of hyperbolic problems in composites with imperfect interface we refer to Faella et al. [14], Monsurrò and Perugia [29] and Monsurrò et al. [28].

2 Setting of the problem

Let us consider an open bounded subset Ω of $\mathbb{R}^n$, $n \geq 2$, and let $\Omega_1 = \Omega \setminus \bar{\Omega}_2$, where Ω_2 is an open and bounded set such that $\Omega_2 \subset\subset \Omega$. Denote by $\partial\Omega$ and $\Gamma = \partial\Omega_2$ the external and the interface boundaries respectively. Suppose that the interface is Lipschitz continuous. By construction one has

$$\partial\Omega \cap \Gamma = \emptyset. \tag{2.1}$$

Let $T > 0$ and define $Q_1 = \Omega_1 \times (0, T)$, $Q_2 = \Omega_2 \times (0, T)$, $\Sigma = \partial\Omega \times (0, T)$ and $\Sigma_\Gamma = \Gamma \times (0, T)$.

Given a control ζ, we consider the following boundary exact controllability problem defined in the domain Ω

$$\begin{cases} u_1'' - \operatorname{div}(A(x)\nabla u_1) = 0 & \text{in } Q_1, \\ u_2'' - \operatorname{div}(A(x)\nabla u_2) = 0 & \text{in } Q_2, \\ A(x)\nabla u_1\, n_1 = -A(x)\nabla u_2\, n_2 & \text{on } \Sigma_\Gamma, \\ A(x)\nabla u_1\, n_1 = -h(x)(u_1 - u_2) & \text{on } \Sigma_\Gamma, \\ u_1 = \zeta & \text{on } \Sigma, \\ u_1(0) = U_1^0, \quad u_1'(0) = U_1^1 & \text{in } \Omega_1, \\ u_2(0) = U_2^0, \quad u_2'(0) = U_2^1 & \text{in } \Omega_2, \end{cases} \tag{2.2}$$

where n_i's are the unitary outward normals to Ω_i, for $i = 1, 2$.

Let us define a class of function spaces which takes into account the geometry of the domain as well as the boundary and interfacial conditions, suitable for the solutions of this kind of interface problems. They were introduced for the first time in [25] in the analogous stationary framework (see also [12] for more details). First of all, let

$$V = \{v_1 \in H^1(\Omega_1) \mid v_1 = 0 \text{ on } \partial\Omega\},$$

which, as proved in [4], is a Banach space endowed with the norm

$$\|v_1\|_V = \|\nabla v_1\|_{L^2(\Omega_1)}.$$

By (2.1), V can be defined as the closure, with respect to the $H^1(\Omega_1)$-norm, of the set of the functions in $C^\infty(\Omega_1)$ with a compact support contained in Ω. Indeed the condition on $\partial\Omega$ in the definition of V must be interpreted in a density sense, since we do not assume any regularity on $\partial\Omega$. We set

$$H_\Gamma = \left\{ v = (v_1, v_2) \mid v_1 \in V \text{ and } v_2 \in H^1(\Omega_2) \right\}. \tag{2.3}$$

The space H_Γ is a separable and reflexive Hilbert space when equipped with the norm

$$\|v\|_{H_\Gamma}^2 = \|\nabla v_1\|_{L^2(\Omega_1)}^2 + \|\nabla v_2\|_{L^2(\Omega_2)}^2 + \|v_1 - v_2\|_{L^2(\Gamma)}^2$$

and, as evidenced in [7], it can be identified with $V \times H^1(\Omega_2)$ endowed with its usual norm. The dual of H_Γ is denoted by $(H_\Gamma)'$ and observe that the norms of $(H_\Gamma)'$ and $V' \times (H^1(\Omega_2))'$ are equivalent (see [9]). Moreover, if $v = (v_1, v_2) \in (H_\Gamma)'$ and $u = (u_1, u_2) \in H_\Gamma$, then

$$\langle v, u \rangle_{(H_\Gamma)', H_\Gamma} = \langle v_1, u_1 \rangle_{V', V} + \langle v_2, u_2 \rangle_{(H^1(\Omega_2))', H^1(\Omega_2)}.$$

As proved in [7,8], $(H_\Gamma, L^2(\Omega_1) \times L^2(\Omega_2), (H_\Gamma)')$ is an evolution triple. Let us set

$$W = \left\{ v = (v_1, v_2) \in L^2\left(0, T; V \times H^1(\Omega_2)\right) \text{ s.t.} \right.$$

$$v' = \left(v_1', v_2'\right) \in L^2\left(0, T; L^2(\Omega_1) \times L^2(\Omega_2)\right)\Big\}, \tag{2.4}$$

which is a Hilbert space if equipped with the graph norm given by

$$\|v\|_W = \|v_1\|_{L^2(0,T;V)} + \|v_2\|_{L^2(0,T;H^1(\Omega_2))} + \left\|v_1'\right\|_{L^2(0,T;L^2(\Omega_1))} + \left\|v_2'\right\|_{L^2(0,T;L^2(\Omega_2))}.$$

For what concerns the initial data and the control problem (2.2), we suppose that

$$\begin{cases} \text{(i) } U^0 = \left(U_1^0, U_2^0\right) \in L^2(\Omega_1) \times L^2(\Omega_2), \\ \text{(ii) } U^1 = \left(U_1^1, U_2^1\right) \in (H_\Gamma)', \end{cases} \tag{2.5}$$

and

$$\zeta \in L^2(\Sigma). \tag{2.6}$$

We also assume that A is a symmetric matrix field such that

$$\begin{cases} (i)\ a_{ij} \in \left(W^{1,\infty}(\Omega)\right)^{n^2}, \quad 1 \le i, j, k \le n, \\ (ii)\ (A(x)\lambda, \lambda) \ge \alpha|\lambda|^2, \ |A(x)\lambda| \le \beta|\lambda|, \end{cases} \tag{2.7}$$

for every $\lambda \in \mathbb{R}^n$ and a.e. in Ω, where $\alpha, \beta \in \mathbb{R}$, with $0 < \alpha < \beta$, and put

$$M = \max_{1 \le i,j,k \le n} \max_{x \in \Omega} \left|\frac{\partial a_{ij}}{\partial x_k}\right|. \tag{2.8}$$

Furthermore, the function h satisfies

$$h \in L^\infty(\Gamma) \text{ and } \exists\, h_0 \in \mathbb{R} \text{ such that } 0 < h_0 < h(x) \text{ a.e. in } \Gamma. \tag{2.9}$$

Since the initial data are in a weak space, we need to apply the transposition method to define in an appropriate way the solution of problem (2.2) (see [20], Chapter 3, Section 9). Thus, for every $g = (g_1, g_2) \in L^2(0, T; L^2(\Omega_1) \times L^2(\Omega_2))$, we consider the following backward problem

$$\begin{cases} \psi_1'' - \operatorname{div}(A(x)\nabla\psi_1) = g_1 & \text{in } Q_1, \\ \psi_2'' - \operatorname{div}(A(x)\nabla\psi_2) = g_2 & \text{in } Q_2, \\ A(x)\nabla\psi_1\, n_1 = -A(x)\nabla\psi_2\, n_2 & \text{on } \Sigma_\Gamma, \\ A(x)\nabla\psi_1\, n_1 = -h(x)(\psi_1 - \psi_2) & \text{on } \Sigma_\Gamma, \\ \psi_1 = 0 & \text{on } \Sigma, \\ \psi_1(T) = \psi_1'(T) = 0 & \text{in } \Omega_1, \\ \psi_2(T) = \psi_2'(T) = 0 & \text{in } \Omega_2. \end{cases} \tag{2.10}$$

The existence and uniqueness of the weak solution in W of problem (2.10) are proved in [7]. For sake of simplicity, in the sequel we omit the explicit dependence on the space variable x in the matrix A and in the function h when considering integrals. Now, we give the following notion of transposition solution.

Definition 1 For any fixed $\left(U^0, U^1\right) \in (L^2(\Omega_1) \times L^2(\Omega_2)) \times (H_\Gamma)'$, we say that a function $u = (u_1, u_2) \in L^2(0, T; L^2(\Omega_1) \times L^2(\Omega_2))$ is a solution of problem (2.2), in the sense of transposition, if it satisfies

$$\begin{aligned}\int_{Q_1} u_1 g_1 dxdt + \int_{Q_2} u_2 g_2 dxdt = &-\int_{\Omega_1} U_1^0 \psi_1'(0) dx + \left\langle U_1^1, \psi_1(0)\right\rangle_{V',V} \\ &- \int_{\Omega_2} U_2^0 \psi_2'(0) dx + \left\langle U_2^1, \psi_2(0)\right\rangle_{(H^1(\Omega_2))', H^1(\Omega_2)} \\ &- \int_{\Sigma} A \nabla \psi_1 \, n_1 \, \zeta \;\; d\sigma_x dt \end{aligned} \tag{2.11}$$

for all $g = (g_1, g_2) \in L^2(0, T; L^2(\Omega_1) \times L^2(\Omega_2))$ and ψ is the unique solution of problem (2.10) corresponding to g.

Remark 1 Observe that, since $\psi_1 = 0$ on Σ, one has that $\nabla \psi_1 = n_1 \dfrac{\partial \psi_1}{\partial n_1}$ on Σ. Hence (2.11) reads as follows

$$\begin{aligned}\int_{Q_1} u_1 g_1 dxdt + \int_{Q_2} u_2 g_2 dxdt = &-\int_{\Omega_1} U_1^0 \psi_1'(0) dx + \left\langle U_1^1, \psi_1(0)\right\rangle_{V',V} \\ &- \int_{\Omega_2} U_2^0 \psi_2'(0) dx + \left\langle U_2^1, \psi_2(0)\right\rangle_{(H^1(\Omega_2))', H^1(\Omega_2)} \\ &- \int_{\Sigma} A \, n_1 \, n_1 \frac{\partial \psi_1}{\partial n_1} \zeta \;\; d\sigma_x \; dt. \end{aligned} \tag{2.12}$$

Problem (2.2) admits a unique solution $u \in C\left([0, T]; L^2(\Omega_1) \times L^2(\Omega_2)\right) \cap C^1\left([0, T]; (H_\Gamma)'\right)$ satisfying the estimate

$$\begin{aligned}\|u\|_{L^\infty(0,T;L^2(\Omega_1)\times L^2(\Omega_2))} + \|u'\|_{L^\infty(0,T;(H_\Gamma)')} \leq C(&\|U^0\|_{L^2(\Omega_1)\times L^2(\Omega_2)} \\ &+ \|U^1\|_{(H_\Gamma)'} + \|\zeta\|_{L^2(\Sigma)}),\end{aligned} \tag{2.13}$$

with C as a positive constant (see [20], Chapter 3, Section 9, Theorems 9.3 and 9.4).

It is clear that the solution of problem (2.2) depends on the control ζ, i.e. $u(\zeta) = (u_1(\zeta), u_2(\zeta))$. Nevertheless, for sake of simplicity, in the sequel we omit this explicit dependence.

Definition 2 System (2.2) is exactly controllable at time $T > 0$, if for every $\left(U^0, U^1\right), \left(Z^0, Z^1\right)$ in $(L^2(\Omega_1) \times L^2(\Omega_2)) \times (H_\Gamma)'$, there exists a control ζ belonging to $L^2(\Sigma)$ such that the corresponding solution u of problem (2.2) satisfies

$$u(T) = Z^0, \;\; u'(T) = Z^1.$$

Remark 2 It is well known that for a linear system reversible in time, exact controllability is equivalent to null controllability (see, for instance, [32]). Hence, it is enough to prove the existence of a control $\zeta \in L^2(\Sigma)$ of (2.2) such that $u(T) = u'(T) = 0$.

From literature it is renowned that it is not possible to achieve controllability without additional requirements. Hence, in this paper, we prove a boundary controllability result under the further geometrical assumption that Ω_2 is starshaped with respect to a point $x^0 \in \Omega_2$. More precisely, we prove that system (2.2) is exactly controllable, for a suitable time $T > 0$ sufficiently large, taking a control acting on a part of the external boundary $\partial\Omega$, or even on the entire external boundary, according to the shape of the domain (see Theorem 2). We use a constructive method known as the Hilbert Uniqueness Method, HUM for short, introduced by Lions in [17,18]. Eventually HUM reduces to the derivation of a delicate estimate from below, known as observability estimate, for an uncontrolled problem, see (3.1). This estimate can be obtained by proving some preliminary fundamental results based on the so called Lagrange multipliers method. These results are proved in the following section. We point out that the control obtained by HUM is also the energy minimizing control, hence it is unique.

3 The observability inequality

In this section, we consider an imperfect transmission problem similar to (2.2) presenting a homogeneous Dirichlet condition on the external boundary and more regular initial data. Namely, for $T > 0$, let us introduce the following problem

$$\begin{cases} z_1'' - \operatorname{div}(A(x)\nabla z_1) = 0 & \text{in } Q_1, \\ z_2'' - \operatorname{div}(A(x)\nabla z_2) = 0 & \text{in } Q_2, \\ A(x)\nabla z_1\, n_1 = -A(x)\nabla z_2\, n_2 & \text{on } \Sigma_\Gamma, \\ A(x)\nabla z_1\, n_1 = -h(x)(z_1 - z_2) & \text{on } \Sigma_\Gamma, \\ z_1 = 0 & \text{on } \Sigma, \\ z_1(0) = z_1^0, \quad z_1'(0) = z_1^1 & \text{in } \Omega_1, \\ z_2(0) = z_2^0, \quad z_2'(0) = z_2^1 & \text{in } \Omega_2, \end{cases} \tag{3.1}$$

with

$$\begin{cases} z^0 = \left(z_1^0, z_2^0\right) \in H_\Gamma, \\ z^1 = \left(z_1^1, z_2^1\right) \in L^2(\Omega_1) \times L^2(\Omega_2), \end{cases} \tag{3.2}$$

where n_i's are the unitary outward normals to Ω_i, $i = 1, 2$. In view of (2.7) and (2.9), as already observed in [7], problem (3.1) admits a unique weak solution in W and its variational formulation is given by

$$\begin{cases} \text{Find } z = (z_1, z_2) \text{ in } W \text{ such that} \\ \langle z_1'', v_1 \rangle_{V',V} + \langle z_2'', v_2 \rangle_{(H^1(\Omega_2))',H^1(\Omega_2)} + \displaystyle\int_{\Omega_1} A\nabla z_1 \nabla v_1 \, dx + \int_{\Omega_2} A\nabla z_2 \nabla v_2 \, dx \\ + \displaystyle\int_{\Gamma} h(z_1 - z_2)(v_1 - v_2) \, d\sigma_x = 0, \quad \forall \, (v_1, v_2) \in V \times H^1(\Omega_2) \text{ in } \mathcal{D}'(0, T), \\ z_1(0) = z_1^0, \quad z_1'(0) = z_1^1 \quad \text{in } \Omega_1, \\ z_2(0) = z_2^0, \quad z_2'(0) = z_2^1 \quad \text{in } \Omega_2. \end{cases} \tag{3.3}$$

More precisely it holds the following result.

Theorem 1 [7] *Let $T > 0$ and H_Γ and W be defined as in (2.3) and (2.4). Under hypotheses (2.7), (2.9) and (3.2), problem (3.1) admits a unique weak solution $z \in W$. Moreover, there exists a positive constant C, such that*

$$\|z\|_{L^\infty(0,T;H_\Gamma)} + \left\|z'\right\|_{L^\infty(0,T;L^2(\Omega_1)\times L^2(\Omega_2))} \le C \left(\left\|z^0\right\|_{H_\Gamma} + \left\|z^1\right\|_{L^2(\Omega_1)\times L^2(\Omega_2)} \right). \tag{3.4}$$

In fact, under the same hypotheses of Theorem 1, the solution of problem (3.1) has further regularity

$$z \in C\left([0,T]; H_\Gamma\right), \; z' \in C\left([0,T]; L^2(\Omega_1) \times L^2(\Omega_2)\right),$$

see [20], Chapter 3, Theorem 8.2 for more details. Hence the initial values $z(0)$ and $z'(0)$ are meaningful in the appropriate spaces. Let us recall a fundamental identity, proved in [28], crucial when establishing the inverse inequalities involved in the exact controllability problem. For clearness sake, from now on we use the repeated index summation convention.

Lemma 1 [28] *Let $q = (q_1, \dots q_n)$ be a vector field in $(W^{1,\infty}(\Omega))^n$ and let $z = (z_1, z_2)$ be the solution of problem (3.1)–(3.2). Then, the following identity holds*

$$\begin{aligned} &\frac{1}{2}\int_{\Sigma} A n_1 n_1 \left(\frac{\partial z_1}{\partial n_1}\right)^2 q_k n_{1k} \, d\sigma_x \, dt + \frac{1}{2}\sum_{i=1}^{2}\int_{\Sigma_\Gamma} A n_i n_i \left(\frac{\partial z_i}{\partial n_i}\right)^2 q_k n_{ik} \, d\sigma_x \, dt \\ &\quad - \int_{\Sigma_\Gamma} h\left(z_1 - z_2\right) q_k \left(\nabla_\sigma (z_1 - z_2)\right)_k \, d\sigma_x \, dt \\ &\quad + \frac{1}{2}\sum_{i=1}^{2}\int_{\Sigma_\Gamma} \left(|z_i'|^2 - A\nabla_\sigma z_i \nabla_\sigma z_i\right) q_k n_{ik} \, d\sigma_x \, dt \end{aligned}$$

$$= \sum_{i=1}^{2} \left(z_i', q_k \frac{\partial z_i}{\partial x_k} \right)_{\Omega_i} \Bigg|_0^T + \frac{1}{2} \sum_{i=1}^{2} \int_{Q_i} \left(|z_i'|^2 - A\nabla z_i \nabla z_i \right) \frac{\partial q_k}{\partial x_k} \, dx \, dt$$
$$+ \sum_{i=1}^{2} \int_{Q_i} A\nabla z_i \nabla q_k \frac{\partial z_i}{\partial x_k} \, dx \, dt - \frac{1}{2} \sum_{i=1}^{2} \int_{Q_i} q_k \sum_{l,j=1}^{n} \frac{\partial a_{lj}}{\partial x_k} \frac{\partial z_i}{\partial x_l} \frac{\partial z_i}{\partial x_j} \, dx \, dt, \tag{3.5}$$

where

$$\left(z_i', q_k \frac{\partial z_i}{\partial x_k} \right)_{\Omega_i} = \int_{\Omega_i} z_i'(t) q_k \frac{\partial z_i(t)}{\partial x_k} \, dx$$

and $\nabla_\sigma z_i = (\sigma_j z_i)_{j=1}^n$ *denotes the tangential gradient of* z_i *on* Γ *for* $i = 1, 2$ *(see, for instance,* [18]*, p. 137).*

In order to prove the direct inequality, stated in the next Lemma 2, we apply the above identity for a particular choice of the vector field q. At first, let us denote by $E(t)$, the energy of the problem (3.1)–(3.2) which is defined as

$$E(t) = \frac{1}{2} \Bigg[\int_{\Omega_1} |z_1'(t)|^2 dx + \int_{\Omega_2} |z_2'(t)|^2 dx + \int_{\Omega_1} A\nabla z_1(t) \nabla z_1(t) dx$$
$$+ \int_{\Omega_2} A\nabla z_2(t) \nabla z_2(t) dx + \int_{\Gamma} h \, |z_1(t) - z_2(t)|^2 \, d\sigma_x \Bigg]. \tag{3.6}$$

It is easy to see by a direct derivation that the energy defined in (3.6) is conserved (see [8], Lemma 4.1), i.e.

$$E(t) = E(0), \quad \forall t \in [0, T]. \tag{3.7}$$

Lemma 2 *Let* $z = (z_1, z_2)$ *be the solution of problem* (3.1)–(3.2)*. Then, for any* $T > 0$*, it holds*

$$\int_{\Sigma} \left(\frac{\partial z_1}{\partial n_1} \right)^2 d\sigma_x \, dt \le C E(0), \tag{3.8}$$

with C *as a positive constant.*

Proof Let us choose $q_k = \tau_k$ in (3.5) for $k = 1, .., n$, where $\tau = (\tau_k)_{k=1,..,n} \in (C^1(\overline{\Omega}))^n$ is such that

$$\begin{cases} (i) & \tau = n_1 \quad \text{on } \partial\Omega, \\ (ii) & \tau = 0 \quad \text{in } \Omega_2, \\ (iii) & \|\tau\|_{(L^\infty(\Omega_1))^n} \le 1. \end{cases} \tag{3.9}$$

The existence of such a vectorial field is proved in [18]. By (3.9)(i) and (3.9)(ii) we get

$$
\begin{aligned}
&\frac{1}{2}\int_{\Sigma} An_1n_1\left(\frac{\partial z_1}{\partial n_1}\right)^2 d\sigma_x\, dt \\
&= \left(z_1', \tau_k\frac{\partial z_1}{\partial x_k}\right)_{\Omega_1}\Bigg|_0^T + \frac{1}{2}\int_{Q_1}\left(|z_1'|^2 - A\nabla z_1\nabla z_1\right)\frac{\partial \tau_k}{\partial x_k}\, dx\, dt \\
&\quad + \int_{Q_1} A\nabla z_1\nabla\tau_k\frac{\partial z_1}{\partial x_k}\, dx\, dt - \frac{1}{2}\int_{Q_1}\tau_k\sum_{l,j=1}^{n}\frac{\partial a_{lj}}{\partial x_k}\frac{\partial z_1}{\partial x_l}\frac{\partial z_1}{\partial x_j}\, dx\, dt. \qquad (3.10)
\end{aligned}
$$

Passing to the absolute value, by (2.7), (3.9)(iii), Young inequality, (3.6), the conservation law and since $\tau \in (C^1(\overline{\Omega}))^n$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\left|\int_{\Sigma} An_1n_1\left(\frac{\partial z_1}{\partial n_1}\right)^2 d\sigma_x\, dt\right| \\
&\le \frac{1}{2}\int_{\Omega_1}\left|z_1'(0)\right|^2 dx + \frac{1}{2}\int_{\Omega_1}\left|z_1'(T)\right|^2 dx + \frac{1}{2}\int_{\Omega_1}|\nabla z_1(0)|^2\, dx \\
&\quad + \frac{1}{2}\int_{\Omega_1}|\nabla z_1(T)|^2\, dx + C_1\int_{Q_1}\left(|z_1'|^2 + A\nabla z_1\nabla z_1\right) dx\, dt \\
&\quad + C_2\int_{Q_1} A\nabla z_1\nabla z_1\, dx\, dt + C_3\int_{Q_1}|\nabla z_1|^2\, dx\, dt \\
&\le 2\max\left(1, \frac{1}{\alpha}\right)E(0) + C_4\int_{Q_1}\left(|z_1'|^2 + |\nabla z_1|^2\right) dx\, dt \le CE(0),
\end{aligned}
$$

where the last inequality is a consequence of estimate (3.4). Then, hypothesis (2.7)(ii) gives the result. □

At this point, in order to derive the observability estimate, we adapt to our context some arguments introduced in [18,19] (see also [28]). Here, for $x^0 \in \mathbb{R}^n$, we set

$$
m(x) = x - x^0 = (x_k - x_k^0)_{k=1}^n. \qquad (3.11)
$$

Lemma 3 *Let $z = (z_1, z_2)$ be the solution of problem (3.1)–(3.2). Then, for any $T > 0$, it holds*

$$
\begin{aligned}
&\frac{1}{2}\left|\sum_{i=1}^{2}\int_{\Sigma_\Gamma} An_in_i\left(\frac{\partial z_i}{\partial n_i}\right)^2 m_kn_{ik}\, d\sigma_x\, dt - \int_{\Sigma_\Gamma} h\,(z_1 - z_2)\, m_k\,(\nabla_\sigma(z_1 - z_2))_k\, d\sigma_x\, dt\right. \\
&\quad \left. + \frac{1}{2}\sum_{i=1}^{2}\int_{\Sigma_\Gamma}\left(|z_i'|^2 - A\nabla_\sigma z_i\nabla_\sigma z_i\right) m_kn_{ik}\, d\sigma_x\, dt\right| \le CE(0), \qquad (3.12)
\end{aligned}
$$

with C as a positive constant.

Proof Let us choose $q_k = m_k\, w$ in (3.5) for $k = 1, \dots, n$, where $w \in C^1(\mathbb{R}^n)$ is such that

$$\begin{cases} (i) & \operatorname{supp} w \subset \overline{\Omega}_2, \\ (ii) & 0 \le w \le 1 \quad \text{in } \Omega_2, \\ (iii) & w = 1 \quad \text{on } \Gamma, \\ (iv) & \|\nabla w\|_{L^\infty(\mathbb{R}^n)} \le 1. \end{cases} \tag{3.13}$$

The existence of such a vectorial field is proved in [18]. By (3.13)(i)–(iii) we get

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{2}\int_{\Sigma_\Gamma} An_i n_i \left(\frac{\partial z_i}{\partial n_i}\right)^2 m_k n_{ik}\, d\sigma_x\, dt - \int_{\Sigma_\Gamma} h\,(z_1 - z_2)\, m_k\,(\nabla_\sigma(z_1 - z_2))_k\, d\sigma_x\, dt \\ &\quad + \frac{1}{2}\sum_{i=1}^{2}\int_{\Sigma_\Gamma} \left(|z_i'|^2 - A\nabla_\sigma z_i \nabla_\sigma z_i\right) m_k n_{ik}\, d\sigma_x\, dt \\ &= \left(z_2', m_k w \frac{\partial z_2}{\partial x_k}\right)_{\Omega_2}\Bigg|_0^T + \frac{n}{2}\int_{Q_2}\left(|z_2'|^2 - A\nabla z_2 \nabla z_2\right) w\, dx\, dt \\ &\quad + \frac{1}{2}\int_{Q_2}\left(|z_2'|^2 - A\nabla z_2\nabla z_2\right) m_k \frac{\partial w}{\partial x_k}\, dx\, dt + \int_{Q_2} A\nabla z_2 \nabla w \frac{\partial z_2}{\partial x_k} m_k\, dx\, dt \\ &\quad + \int_{Q_2} A\nabla z_2 \nabla z_2 w\, dx\, dt - \frac{1}{2}\int_{Q_2} m_k w \sum_{l,j=1}^{n} \frac{\partial a_{lj}}{\partial x_k}\frac{\partial z_2}{\partial x_l}\frac{\partial z_2}{\partial x_j}\, dx\, dt. \end{aligned} \tag{3.14}$$

Passing to the absolute value, by (3.13)(iv), Young inequality, (3.6), the conservation law and (3.4), we obtain

$$\begin{aligned} &\Bigg|\frac{1}{2}\sum_{i=1}^{2}\int_{\Sigma_1} An_i n_i \left(\frac{\partial z_i}{\partial n_i}\right)^2 m_k n_{ik}\, d\sigma_x\, dt - \int_{\Sigma_\Gamma} h\,(z_1 - z_2)\, m_k\,(\nabla_\sigma(z_1 - z_2))_k\, d\sigma_x\, dt \\ &\quad + \frac{1}{2}\sum_{i=1}^{2}\int_{\Sigma_\Gamma}\left(|z_i'|^2 - A\nabla_\sigma z_i\nabla_\sigma z_i\right) m_k n_{ik}\, d\sigma_x\, dt\Bigg| \\ &\le \frac{1}{2}\int_{\Omega_2}\left|z_2'(0)\right|^2 dx + \frac{1}{2}\int_{\Omega_2}\left|z_2'(T)\right|^2 dx + C_1\int_{\Omega_2}|\nabla z_2(0)|^2\, dx \\ &\quad + C_1\int_{\Omega_2}|\nabla z_2(T)|^2\, dx + \frac{n}{2}\int_{Q_2}\left(|z_2'|^2 + A\nabla z_2\nabla z_2\right) dx\, dt \\ &\quad + C_2\int_{Q_2}\left(|z_2'|^2 + A\nabla z_2\nabla z_2\right) dx\, dt \\ &\quad + C_3\int_{Q_2} A\nabla z_2\nabla z_2\, dx\, dt + C_4\int_{Q_2}|\nabla z_2|^2\, dx\, dt \le CE(0). \end{aligned}$$

where C_i's are positive constants independent of T.

Usually, in the context of controllability problems, the point x^0 can be interpreted as an observer and its choice is strictly related to the region where the control is acting. Having this in mind, we define

$$\partial\Omega(x^0) = \{x \in \partial\Omega : m(x)n_1(x) = m_k(x)n_{1k}(x) > 0\} \tag{3.15}$$

and

$$\Sigma(x^0) = \partial\Omega(x^0) \times (0, T). \tag{3.16}$$

Further, we put

$$R(x^0) = \max_{x\in\overline{\Omega}} |m(x)| = \max_{x\in\overline{\Omega}} \left|\sum_{k=1}^{n}(x_k - x_k^0)^2\right|^{\frac{1}{2}}. \tag{3.17}$$

In our case, as observed in [28], due to the geometry of the domain, x^0 must be chosen such that the set Ω_2 is star-shaped with respect to x^0. As we will see later on, this choice will also influence the control time (see Lemma 5). By Lemma 3.3 in [28], we can easily obtain the following result.

Lemma 4 *Let us suppose that Ω_2 is starshaped with respect to a point $x^0 \in \Omega_2$. Let $z = (z_1, z_2)$ be the solution of problem (3.1)–(3.2). Then, for any $T > 0$, it holds*

$$\begin{aligned}
&\frac{R(x^0)\beta}{2}\int_{\Sigma(x^0)}\left(\frac{\partial z_1}{\partial n_1}\right)^2 d\sigma_x\, dt + \frac{1}{2}\sum_{i=1}^{2}\int_{\Sigma_\Gamma} An_i n_i\left(\frac{\partial z_i}{\partial n_i}\right)^2 m_k n_{ik}\, d\sigma_x\, dt\\
&\quad - \int_{\Sigma_\Gamma} h(x)\,(z_1 - z_2)\, m_k\,(\nabla_\sigma(z_1 - z_2))_k\, d\sigma_x\, dt\\
&\quad + \frac{1}{2}\sum_{i=1}^{2}\int_{\Sigma_\Gamma}\left(|z_i'|^2 - A\nabla_\sigma z_i \nabla_\sigma z_i\right) m_k n_{ik}\, d\sigma_x\, dt\\
&\geq \left[T\left(1 - \frac{nR(x^0)M}{\alpha}\right) - 2\max\left(\frac{R(x^0)}{\sqrt{\alpha}}, \frac{(n-1)\sqrt{\alpha}}{2h_0}\right)\right]E(0),
\end{aligned} \tag{3.18}$$

with M given in (2.8).

Combining the results of Lemmas 4 and 3, we obtain the claimed observability inequality.

Lemma 5 *Let us suppose that Ω_2 is starshaped with respect to a point $x^0 \in \Omega_2$ such that*

$$R(x^0) < \frac{\alpha}{nM}, \tag{3.19}$$

with α and M respectively given by (2.7) *and* (2.8)*. Let* $z=(z_1,z_2)$ *be the solution of the problem* (3.1)–(3.2)*. Then, there exists* $T_0>0$ *such that*

$$E(0)\le C(T)\int_{\Sigma(x^0)}\left(\frac{\partial z_1}{\partial n_1}\right)^2\,d\sigma_x\,dt, \tag{3.20}$$

for T *large enough so that*

$$\frac{T-T_0}{T}>\frac{nR(x^0)M}{\alpha}. \tag{3.21}$$

Proof By (3.18) and (3.12), we get

$$\begin{aligned}
&\left[T\left(1-\frac{nR(x^0)M}{\alpha}\right)-2\max\left(\frac{R(x^0)}{\sqrt{\alpha}},\frac{(n-1)\sqrt{\alpha}}{2h_0}\right)\right]E(0)\\
&\le\frac{R(x^0)\beta}{2}\int_{\Sigma(x_0)}\left(\frac{\partial z_1}{\partial n_1}\right)^2\,d\sigma_x\,dt+\frac{1}{2}\sum_{i=1}^{2}\int_{\Sigma_1}An_in_i\left(\frac{\partial z_i}{\partial n_i}\right)^2 m_kn_{ik}\,d\sigma_x\,dt\\
&\quad-\int_{\Sigma_\Gamma}h(x)\,(z_1-z_2)\,m_k\,(\nabla_\sigma(z_1-z_2))_k\,d\sigma_x\,dt\\
&\quad+\frac{1}{2}\sum_{i=1}^{2}\int_{\Sigma_\Gamma}\left(|z_i'|^2-A\nabla_\sigma z_i\nabla_\sigma z_i\right)m_kn_{ik}\,d\sigma_x\,dt\\
&\le\frac{R(x^0)\beta}{2}\int_{\Sigma(x^0)}\left(\frac{\partial z_1}{\partial n_1}\right)^2\,d\sigma_x\,dt+CE(0),
\end{aligned}$$

where C is the positive constant given in Lemma 3. Denoting

$$T_0=2\max\left(\frac{R(x^0)}{\sqrt{\alpha}},\frac{(n-1)\sqrt{\alpha}}{2h_0}\right)+C, \tag{3.22}$$

we obtain

$$\left[T\left(1-\frac{nR(x^0)M}{\alpha}\right)-T_0\right]E(0)\le\frac{R(x^0)\beta}{2}\int_{\Sigma(x^0)}\left(\frac{\partial z_1}{\partial n_1}\right)^2\,d\sigma_x\,dt.$$

Following an argument similar as in [23], if (3.19) is satisfied and if T is large enough so that (3.21) holds, then $T\left(1-\frac{nR(x^0)M}{\alpha}\right)-T_0$ is positive and we get the required result. □

As a consequence of the direct and inverse inequalities, we immediately get the following equivalence.

Corollary 1 *Under the hypotheses of Lemma 5, there exists $T_0 > 0$ such that*

$$E(0) \leq C_1(T) \int_{\Sigma(x_0)} \left(\frac{\partial z_1}{\partial n_1} \right)^2 d\sigma_x \, dt \leq C_2(T) \, E(0), \tag{3.23}$$

for T large enough so that

$$\frac{T - T_0}{T} > \frac{nR(x^0)M}{\alpha}.$$

4 The boundary exact controllability result

In this section, let us prove the boundary exact controllability of system (2.2) by means of the Hilbert Uniqueness Method introduced by Lions (see [17,18]).

Theorem 2 *Let us suppose that Ω_2 is starshaped with respect to a point $x^0 \in \Omega_2$ such that*

$$R(x^0) < \frac{\alpha}{nM},$$

with α, M and $R(x^0)$ respectively given by (2.7), (2.8) and (3.17). Under assumptions (2.7) and (2.9), for any given $\left(U^0, U^1\right)$ in $\left(L^2(\Omega_1) \times L^2(\Omega_2)\right) \times (H_\Gamma)'$, there exist a control $\zeta \in L^2(\Sigma(x^0))$ and a time $T_0 > 0$ such that the corresponding solution of problem (2.2) satisfies

$$u(T) = u'(T) = 0, \tag{4.1}$$

for T large enough so that

$$\frac{T - T_0}{T} > \frac{nR(x^0)M}{\alpha}. \tag{4.2}$$

Proof Let T_0 be as in Lemma 5 and T as in (4.2). Let z be the solution of (3.1) and consider the following backward problem

$$\begin{cases} \theta_1'' - \operatorname{div}(A(x)\nabla\theta_1) = 0 & \text{in } Q_1, \\ \theta_2'' - \operatorname{div}(A(x)\nabla\theta_2) = 0 & \text{in } Q_2, \\ A(x)\nabla\theta_1 n_1 = -A(x)\nabla\theta_2 n_2 & \text{on } \Sigma_\Gamma, \\ A(x)\nabla\theta_1 n_1 = -h(x)(\theta_1 - \theta_2) & \text{on } \Sigma_\Gamma, \\ \theta_1 = \begin{cases} \dfrac{\partial z_1}{\partial n_1} & \text{on } \Sigma(x^0), \\ 0 & \text{on } \Sigma \setminus \Sigma(x^0) \end{cases} & \\ \theta_1(T) = \theta_1'(T) = 0 & \text{in } \Omega_1, \\ \theta_2(T) = \theta_2'(T) = 0 & \text{in } \Omega_2. \end{cases} \tag{4.3}$$

Similar to the solution of problem (2.2), the solution $\theta = (\theta_1, \theta_2)$ of problem (4.3) is also defined by the transposition method. Hence, it satisfies

$$\int_{\Sigma(x^0)} A\nabla z_1 n_1 \frac{\partial z_1}{\partial n_1}\ d\sigma_x\ dt = \left\langle \theta_1'(0), z_1^0 \right\rangle_{V',V} - \int_{\Omega_1} z_1^1 \theta_1(0) dx + \left\langle \theta_2'(0), z_2^0 \right\rangle_{(H^1(\Omega_2))', H^1(\Omega_2)} - \int_{\Omega_2} z_2^1 \theta_2(0) dx, \tag{4.4}$$

where z is the solution of (3.1) for z^0, z^1 as in (3.2). Now, since $z_1 = 0$ on Σ, we see that $\nabla z_1 = n_1 \dfrac{\partial z_1}{\partial n_1}$ on Σ. Hence (4.4) can be rewritten as

$$\int_{\Sigma(x^0)} A n_1 \frac{\partial z_1}{\partial n_1} n_1 \frac{\partial z_1}{\partial n_1}\ d\sigma_x\ dt = \left\langle \theta_1'(0), z_1^0 \right\rangle_{V',V} - \int_{\Omega_1} z_1^1 \theta_1(0) dx + \left\langle \theta_2'(0), z_2^0 \right\rangle_{(H^1(\Omega_2))', H^1(\Omega_2)} - \int_{\Omega_2} z_2^1 \theta_2(0) dx. \tag{4.5}$$

Inspired by HUM method, we introduce the linear operator

$$\Lambda : H_\Gamma \times \left(L^2(\Omega_1) \times L^2(\Omega_2)\right) \to (H_\Gamma)' \times \left(L^2(\Omega_1) \times L^2(\Omega_2)\right) \tag{4.6}$$

by setting for all $\left(z^0, z^1\right) \in H_\Gamma \times \left(L^2(\Omega_1) \times L^2(\Omega_2)\right)$,

$$\Lambda\left(z^0, z^1\right) = \left(\theta'(0), -\theta(0)\right), \tag{4.7}$$

where θ is the unique solution of problem (4.3). It is easy to verify that

$$\left\langle \Lambda\left(z^0, z^1\right), \left(z^0, z^1\right) \right\rangle = \left\langle \left(\theta'(0), -\theta(0)\right), \left(z^0, z^1\right) \right\rangle = \left\langle \theta_1'(0), z_1^0 \right\rangle_{V',V} - \int_{\Omega_1} z_1^1 \theta_1(0) dx + \left\langle \theta_2'(0), z_2^0 \right\rangle_{(H^1(\Omega_2))', H^1(\Omega_2)} - \int_{\Omega_2} z_2^1 \theta_2(0) dx, \tag{4.8}$$

for every $\left(z^0, z^1\right) \in H_\Gamma \times \left(L^2(\Omega_1) \times L^2(\Omega_2)\right)$.

Putting together (4.5) and (4.8), we can obtain the following explicit formula for the operator Λ

$$\left\langle \Lambda\left(z^0, z^1\right), \left(z^0, z^1\right) \right\rangle = \int_{\Sigma(x^0)} A n_1 n_1 \left(\frac{\partial z_1}{\partial n_1}\right)^2\ d\sigma_x\ dt. \tag{4.9}$$

Observe that, by (2.7),

$$\alpha \int_{\Sigma(x^0)} \left(\frac{\partial z_1}{\partial n_1}\right)^2 d\sigma_x\, dt \le \int_{\Sigma(x^0)} A n_1 n_1 \left(\frac{\partial z_1}{\partial n_1}\right)^2 d\sigma_x\, dt \le \beta \int_{\Sigma(x^0)} \left(\frac{\partial z_1}{\partial n_1}\right)^2 d\sigma_x\, dt.$$

In view of Corollary 1, the right hand side of (4.9) defines a norm on $H_\Gamma \times \left(L^2(\Omega_1) \times L^2(\Omega_2)\right)$ equivalent to the inner one. Therefore, Λ is an isomorphism between $H_\Gamma \times \left(L^2(\Omega_1) \times L^2(\Omega_2)\right)$ and $(H_\Gamma)' \times \left(L^2(\Omega_1) \times L^2(\Omega_2)\right)$. Thus, if (U^0, U^1) are the initial conditions of problem (2.2), the equation

$$\Lambda\left(z^0, z^1\right) = (U^1, -U^0)$$

has a unique solution. This motivates us to take the control ζ in (2.2) as

$$\zeta = \begin{cases} \dfrac{\partial z_1}{\partial n_1} & \text{on } \Sigma(x^0), \\ 0 & \text{on } \Sigma\backslash\Sigma(x^0). \end{cases} \tag{4.10}$$

By uniqueness, we see that $u = \theta$ and therefore, we have the null controllability of problem (2.2) and hence its exact controllability. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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