A NOTE ON THE EXISTENCE OF A SOLUTION TO A PROBLEM OF STEFAN*

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When certain metals are heated slowly, the temperature rises until it reaches a critical temperature at which the structure of the metal changes from one crystalline form to another. As for example, iron changes from α to β crystals at 1643°F. Accompanying this change of crystalline form is a latent heat of recrystallization. In order to study the process we investigate the associated mathematical problem, which requires the solution of a partial differential equation in a region with an undetermined boundary. Our analysis establishes the existence and uniqueness of the solution. In a previous paper[2] this problem is treated from the point of view of computing the solution.

Suppose a metal slab having two infinite parallel faces is brought uniformly to the critical temperature and then heated by a uniform source covering the front face while an insulator covers the back face. Under these conditions, the new crystals are first formed at the front face, and the interface between the new and old crystals travels from the front face to the back face. Mathematically the problem can be stated as follows, where $u = 0$ is taken as the critical temperature: Find the temperature, $u = u(x, t)$, and the curve, $x = x(t)$, which satisfy the following conditions

$$u_t = \alpha^2 u_{xx} \qquad \text{for } 0 < x < x(t) \tag{1}$$

$$u = 0 \qquad \text{for } x = x(t) \tag{2}$$

$$-Ax^{\cdot}(t) = u_x[x(t), t] \qquad \text{where } A > 0 \tag{3}$$

$$x(0) = 0 \tag{4}$$

$$u_x(0, t) = -g \tag{5}$$

where g is a constant >0.

In this notation

$$u_t = \partial u/\partial t, \qquad u_{xx} = \partial^2 u/\partial x^2, \qquad x^{\cdot}(t) = dx(t)/dt,$$

α^2 is the coefficient of thermal diffusivity, $A = \rho H/k$ where ρ is the density of the metal, H is the latent heat of recrystallization, and k is the coefficient of thermal conductivity.

We simplify the notation by introducing new variables as follows:

$$\begin{aligned} v(y, \tau) &= u(x, t)/A\alpha^2 \\ y &= gx/A\alpha^2 \\ \tau &= g^2 t/A^2\alpha^2; \end{aligned} \tag{6}$$

*Received Sept. 7, 1950.

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[2]G. W. Evans II, E. Isaacson, and J. K. L. Mac Donald, *Stefan-like problems*, to be published in the Q. Appl. Math. 8, 312-319 (1950).

and, then, by renaming v by u, τ by t, and y by x, the mathematical statement of the problem is:

Find $u = u(x, t)$ and $x = x(t)$ where the temperature $u(x, t)$ satisfies the following equation

$$u_{xx} = u_t \qquad \text{for } 0 < x < x(t) \tag{7}$$

with the boundary conditions

$$u = 0 \qquad \text{for } x = x(t) \tag{8}$$

$$x^{\cdot}(t) = -u_x[x(t), t] \tag{9}$$

$$x(0) = 0 \tag{10}$$

$$u_x(0, t) = -1. \tag{11}$$

In this discussion we will use a theorem of Dr. Louis Nirenberg[3] on the parabolic equation. For the requirements of this paper, a restricted statement of this theorem is given below as our principal lemma.

Lemma 1: Let R be a simply connected region in the x,t-plane where $0 \leq t \leq T$ with a part of the boundary of R being $t = T$ and the remaining part of the boundary being given locally by a curve $x = \beta(t)$. Furthermore, let $u(x, t)$ be a continuous and bounded solution of the heat conduction equation having continuous derivatives satisfying $u_{xx} = u_t$ in the interior of R, and let the solution be continuable into the region for which $t > T$. If $u(x, t)$ assumes its maximum or minimum in R, say at a point (ξ, τ), other than at a point of the boundary $x = \beta(t)$, then u is a constant in the subregion described as follows: the subregion consists of all points of R which may be reached by a continuous curve $t = f(s)$, $x = g(s)$, where $t = f(s)$ is a monotonic nonincreasing function of s, starting from any point in R that lies on the line $t = \tau$.

Furthermore, we assume the following two lemmas[4]:

Lemma 2: There exists a bounded solution, u, of $u_{xx} = u_t$ with bounded continuous first derivatives in the interior of the region $0 \leq t \leq T$, $0 \leq x \leq X(t)$ where $X(t) \geq 0$ is a curve with $X(0) = 0$, and u assumes the following boundary values: $u_x(0, t) = -1$ and $u[X(t), t] = 0$.

Lemma 3: There exists a bounded solution, u, of $u_{xx} = u_t$ with bounded continuous first derivatives in the interior of the region $0 \leq t \leq T$, $0 \leq x \leq X(t)$ where $X(t) \geq 0$ is a curve with $X(0) = 0$, and u assumes the following boundary values: $u_x(0, t) = 0$ and $u[X(t), t] = u(t) \geq 0$.

We now establish the existence of the solution $x = x(t)$, $u = u(x, t)$ of the problem given in (7)-(11). The proof consists in applying an iteration scheme to the equation

$$x(t) = t - \int_0^{x(t)} u(x, t)\, dx. \tag{12}$$

[3]This theorem is still to be published; but a similar theorem which is not as general, but which would be satisfactory for this paper, was proved by Mauro Picone (*Sul problema della propagazione del calore in un mezzo privo di frontiera, conducttore, isotropo e omogeneo*, Math. Ann. **101** (1929)).

[4]The problems of Lemmas 2 and 3 are of the type which were considered in a more general way by W. Sternberg (*Uber die Gleichung der Warmeleitung*, Math. Ann. **101** (1929)).

This equation is derived by evaluating

$$\int_0^t \int_0^{x(t)} (u_{xx} - u_t)\, dx\, dt = 0$$

with the boundary conditions (8), (9), (10), and (11). Our procedure determines $x(t)$ and $u(x, t)$ in the following way: let

$$x_n(t) = t - \int_0^{x_{n-1}(t)} u_{n-1}(x, t)\, dx \tag{13}$$

with $u_{n-1}(x, t)$ the solution of Lemma 2 for $X(t) = x_{n-1}(t)$. This procedure has been chosen because, after it has been shown that $u(x, t) = \mathrm{Lim}_{n\to\infty} [u_{n-1}(x, t)]$ is the solution of Lemma 2 with $X(t) = x(t) = \mathrm{Lim}_{n\to\infty} [x_n(t)]$, we may differentiate Eq. (12) with respect to t and find that

$$u_x[x(t), t] = -x^{\cdot}(t)$$

which is the boundary condition (9) of our problem not contained in Lemma 2.

The $x_n(t)$ for $n = 0, 1, 2, 3, \cdots$ are monotonic non-decreasing functions of t, i.e.

$$x_n^{\cdot}(t) \geq 0.$$

This may be seen by differentiating Eq. (13) with respect to t giving

$$x_n^{\cdot}(t) = -u_{n-1_x}[x_{n-1}(t), t].$$

And, it remains to show that $u_{n-1_x}[x_{n-1}(t), t] \leq 0$. By Lemma 1, $u_{n-1}(x, t)$ must have its maximum and minimum value along $x = 0$ since $u_{n-1}(x, t) \equiv C$ cannot satisfy the condition $u_{n-1_x}(0, t) = -1$. Furthermore, one can show $0 \leq u_{n-1}(0, t) < M$ where M is the upperbound of $u_{n-1}(x, t)$ of Lemma 2; and since $u_{n-1}[x_{n-1}(t), t] = 0$, then $u_{n-1_x}[x_{n-1}(t), t] \leq 0$.

To show that $0 \leq u_n(x, t) \leq M$ for $n = 0, 1, 2, 3, \cdots$, form $v_n(x, t) = u_n(x, t) + x$ where u_n satisfies the conditions of Lemma 2. $v_n(x, t)$, then, satisfies the equation

$$v_{n_{xx}}(x, t) = v_{n_t}(x, t) \tag{14}$$

with the boundary conditions

$$v_{n_x}(0, t) = 0 \qquad \text{and} \qquad v_n[x_n(t), t] = x_n(t)$$

and is a solution of Lemma 3. Since $v_{n_x}(0, t) = 0$, we reflect the solution about $x = 0$ giving the boundary conditions:

$$v_n[x_n(t), t] = x_n(t) \qquad \text{and} \qquad v_n[-x_n(t), t] = x_n(t).$$

Applying Lemma 1 to $v_n(x, t)$ in the region between the two curves $x = -x_n(t)$ and $x = x_n(t)$, we see that $v_n(x, t)$ must assume its maximum and minimum along $x = x_n(t)$, i.e., $v_n(x, t) \not\equiv C$ since $v_n[x_n(t), t] = x_n(t) \not\equiv C$. For any given $t = \tau$

$$0 \leq v_n(x, t) \leq \text{maximum } [x_n(t)] \qquad \text{for} \qquad 0 \leq t \leq \tau$$

or

$$-x \leq u_n(x, t) \leq \text{maximum } [x_n(t)] - x \qquad \text{for} \qquad 0 \leq t \leq \tau.$$

But, $u_n(x, t)$ is defined only for $x \geq 0$ and assumes its maximum and minimum along $x = 0$, therefore

$$0 \leq u_n(x, t) \leq \text{maximum } [x_n(t)] - x \qquad \text{for} \qquad 0 \leq t \leq \tau.$$

Using, now, the monotonity of $x_n(t)$, we have

$$0 \leq u_n(x, t) \leq x_n(t) - x. \tag{15}$$

To show that the $\text{Lim}_{n\to\infty} [x_n(t)]$ exists, it is sufficient to show that

$$| x_{n+m}(t) - x_n(t) | \to 0 \qquad \text{as } n \to \infty$$

uniformly in t for sufficiently small t, say all $t < T_0$. For this, we choose $x_0(t) = t$, then

$$x_1(t) = t - \int_0^t u_0(x, t)\, dx$$

and

$$x_1(t) - x_0(t) = - \int_0^t u_0(x, t)\, dx. \tag{16}$$

From Eq. (16) and the inequality (15), $x_1(t)$ is seen to lie to the left of $x_0(t) = t$ in the x,t-plane where the x-axis is taken in the horizontal direction. Similarly,

$$x_2(t) - x_1(t) = - \int_0^{x_1(t)} [u_1(x, t) - u_0(x, t)]\, dx + \int_{x_1(t)}^t u_0(x, t)\, dx \geq 0 \tag{17}$$

if we can show $u_1(x, t) - u_0(x, t) \leq 0$. Let

$$v_1(x, t) = u_0(x, t) - u_1(x, t),$$

then $v_1(x, t)$ satisfies the differential equation

$$v_{1_{xx}}(x, t) = v_{1_t}(x, t)$$

with the boundary conditions:

$$v_{1_x}(0, t) = 0 \qquad \text{and} \qquad v_1[x_1(t), t] = u_0[x_1(t), t] \geq 0.$$

By Lemma 3 a solution exists to this problem; and since $v_{1_x}(0, t) = 0$ we can reflect the solution about $x = 0$ thus making $v_1(x, t)$ satisfy the alternate boundary conditions:

$$v_1[x_1(t), t] = u_0[x_1(t), t] \geq 0 \qquad \text{and} \qquad v_1[-x_1(t), t] = u_0[x_1(t), t] \geq 0.$$

By applying Lemma 1 with these boundary conditions, $v_1(x, t)$ must assume its maximum and minimum along $x_1(t)$; and since $v_1(x, t) \not\equiv C$, $v_1(x, t) \geq 0$. By the expression (17), $x_2(t)$ must lie to the right of $x_1(t)$; and by considering

$$x_2(t) - x_0(t) = - \int_0^{x_1(t)} u_1(x, t)\, dx,$$

$x_2(t)$ must also lie to the left of $x_0(t) = t$. In fact, by replacing $x_2(t)$ by $x_n(t)$ in the preceding argument, all $x_n(t)$, for $n = 2, 3, 4, \cdots$, must lie between $x_0(t)$ and $x_1(t)$. By continuing this process

$$x_3(t) - x_2(t) = - \int_0^{x_1(t)} [u_2(x, t) - u_1(x, t)]\, dx - \int_{x_1(t)}^{x_2(t)} u_2(x, t)\, dx \leq 0 \tag{18}$$

since

$$v_2(x, t) = u_2(x, t) - u_1(x, t) \geq 0;$$

and

$$x_3(t) - x_1(t) = -\int_0^{x_2(t)} [u_2(x, t) - u_0(x, t)]\, dx + \int_{x_2(t)}^{t} u_0(x, t)\, dx \geq 0. \tag{19}$$

The expressions (18) and (19) show that $x_3(t)$ lies to the left of $x_2(t)$ and to the right of $x_1(t)$. By further continuing the process it is seen that

$$x_{2k-1}(t) \leq x_{2k}(t) \leq x_{2(k-1)}(t)$$

and

$$x_{2k-1}(t) \leq x_{2k+1}(t) \leq x_{2k}(t)$$

for $k = 1, 2, 3, \cdots$. That is, each curve $x_i(t)$, $i = 2, 3, 4, \cdots$, lies in the region between the preceding two curves, $x_{i-1}(t)$ and $x_{i-2}(t)$.

To establish the existence of the $\text{Lim}_{n\to\infty} [x_n(t)]$, consider the difference, as defined by Eq. (13) of

$$x_{n+1}(t) - x_n(t) = -\int_0^{x_n(t)} u_n(x, t)\, dx + \int_0^{x_{n-1}(t)} u_{n-1}(x, t)\, dx. \tag{20}$$

If $x_n(t)$ lies to the right of $x_{n-1}(t)$, then Eq. (20) becomes

$$x_{n+1}(t) - x_n(t) = \int_0^{x_{n-1}(t)} [u_{n-1}(x, t) - u_n(x, t)]\, dx + \int_{x_n(t)}^{x_{n-1}(t)} u_n(x, t)\, dx, \tag{21}$$

and if $x_n(t)$ lies to the left of $x_{n-1}(t)$, Eq. (20) becomes

$$x_{n+1}(t) - x_n(t) = \int_0^{x_n(t)} [u_{n-1}(x, t) - u_n(x, t)]\, dx + \int_{x_n(t)}^{x_{n-1}(t)} u_{n-1}(x, t)\, dx. \tag{22}$$

To estimate the absolute value of the difference in Eq. (21), choose a time, say $t = T$, and by using the expression (15), Eq. (21) becomes

$$\begin{aligned} | x_{n+1}(T) - x_n(T) | = \bigg| \int_0^{x_{n-1}(T)} &[u_{n-1}(x, T) - u_n(x, T)]\, dx \\ &+ \int_{x_n(T)}^{x_{n-1}(T)} u_n(x, T)\, dx \bigg|. \end{aligned} \tag{23}$$

Since $u_n(x, T) < T$ (for $0 \leq u_n(x, T) \leq u_n(0, T) \leq T$), by substituting T for $u_n(x, T)$ in the second term of the right hand side of Eq. (23), the left hand side satisfies the following expression

$$\begin{aligned} | x_{n+1}(T) - x_n(T) | \leq \int_0^{x_{n-1}(T)} &| [u_n(x, T) - u_{n-1}(x, T)] | \cdot | dx | \\ &+ T \, | x_n(T) - x_{n-1}(T) |. \end{aligned} \tag{24}$$

To estimate the integral in the inequality (24), form

$$v(x, t) = u_n(x, t) - u_{n-1}(x, t)$$

which satisfies

$$v_{xx}(x, t) = v_t(x, t)$$

with the boundary conditions

$$v_x(0, t) = u_{n_x}(0, t) - u_{n-1_x}(0, t) = 0$$

and

$$v[x_{n-1}(t), t] = u_n[x_{n-1}(t), t] \geq 0.$$

By Lemma 3 a solution exists and since $v_x(0, t) = 0$, reflect the solution about $x = 0$, thus making $v(x, t)$ satisfy the alternate boundary conditions

$$v[x_{n-1}(t), t] = u_n[x_{n-1}(t), t] \geq 0$$

$$v[-x_{n-1}(t), t] = u_n[x_{n-1}(t), t] \geq 0.$$

Now, by applying Lemma 1, $v(x, t)$ must assume its maximum along $x_{n-1}(t)$ since $v(x, t) \not\equiv C$. But, by the inequality (15), along $x = x_{n-1}(t)$,

$$u_n[x_{n-1}(t), t] - u_{n-1}[x_{n-1}(t), t] = u_n[x_{n-1}(t), t] \leq x_n(t) - x_{n-1}(t).$$

Therefore

$$| u_n(x, T) - u_{n-1}(x, T) | \leq | x_n(T) - x_{n-1}(T) |. \tag{25}$$

The inequality (24) now becomes

$$| x_{n+1}(T) - x_n(T) | \leq | x_n(T) - x_{n-1}(T) | \int_0^{x_{n-1}(T)} | dx | + T | x_n(T) - x_{n-1}(T) |$$

$$\leq 2T | x_n(T) - x_{n-1}(T) |$$

since $x_{n-1}(T) < T$. Now choose $T_0 = \frac{1}{4}$, then for $t = T \leq \frac{1}{4}$

$$| x_{n+1}(t) - x_n(t) | \leq \tfrac{1}{2} | x_n(t) - x_{n-1}(t) |. \tag{26}$$

The result (26) may be deduced in a similar manner from Eq. (22). The inequality (26) establishes the existence of the $\mathrm{Lim}_{n\to\infty} [x_n(t)] = x(t)$ for $t \leq \frac{1}{4}$.

Since for $t \leq \frac{1}{4}$ one has already shown that $| x_{n+1}(t) - x_n(t) |$ may be made less than ϵ for $n > N(\epsilon)$, and since by the expression (25)

$$| u_n(x, t) - u_{n-1}(x, t) | \leq | x_n(t) - x_{n-1}(t) |,$$

the $\mathrm{Lim}_{n\to\infty} [u_n(x, t)] = u^*(x, t)$ exists for $t \leq \frac{1}{4}$.

Since it is clear that the limits of the iterations satisfy the integral equation (12), it remains to show that $u^*(x, t)$ satisfies the differential equation (1) with the boundary conditions of Lemma 2. To do this let $u(x, t)$ be defined as the solution of Lemma 2 for $x = x(t)$ where $x(t) = \mathrm{Lim}_{n\to\infty} [x_n(t)]$. Set

$$x^*(t) = t - \int_0^{x(t)} u(x, t)\, dx,$$

and form the difference

$$x^*(t) - x_n(t) = \int_0^{x_{n-1}(t)} [u_{n-1}(x, t) - u(x, t)]\, dx - \int_{x_{n-1}(t)}^{x(t)} u(x, t)\, dx. \tag{27}$$

Next, consider the limit of Eq. (27) as $n \to \infty$, but choose the sequence through which n varies so that $x_{n-1}(t)$ always lies to the left of $x(t)$. Then

$$\operatorname*{Lim}_{n\to\infty} [x^*(t) - x_n(t)] = \operatorname*{Lim}_{n\to\infty} \left\{ \int_0^{x_{n-1}(t)} [u_{n-1}(x, t) - u(x, t)]\, dx \right\};$$

so set

$$-v_{n-1}(x, t) = u_{n-1}(x, t) - u(x, t)$$

which satisfies the equation

$$v_{n-1_{xx}}(x, t) = v_{n-1_t}(x, t)$$

with the boundary conditions

$$v_{n-1_x}(0, t) = 0 \qquad \text{and} \qquad v_{n-1}[x_{n-1}(t), t] = u[x_{n-1}(t), t] = f(t).$$

Therefore $v_{n-1}(x, t)$ satisfies Lemma 3. Since $v_{n-1_x}(0, t) = 0$, reflect $v_{n-1}(x, t)$ about $x = 0$, then $v_{n-1}(x, t)$ still satisfies the same differential equation with the alternate boundary conditions

$$v_{n-1}[x_{n-1}(t), t] = f(t) \qquad \text{and} \qquad v_{n-1}[-x_{n-1}(t), t] = f(t).$$

By Lemma 1 either $v_{n-1}(x, t) \equiv 0$ since $v_{n-1}(0, 0) = 0$ which is what we desire, or the maximum or minimum of $v_{n-1}(x, t)$ lies on $x = x_{n-1}(t)$. If the second condition is true, let n tend to infinity in the prescribed manner, then

$$\operatorname*{Lim}_{n\to\infty} \{v_{n-1}[x_{n-1}(t), t]\} = \operatorname*{Lim}_{n\to\infty} \{-u[x_{n-1}(t), t]\} = -u[x(t), t] = 0.$$

This implies that $x^*(t) = x(t)$ which implies that $u^*(x, t) = u(x, t)$ or that $u^*(x, t)$ satisfies the differential equation (1).

In verifying the solution of the problem, one must show that $dx(t)/dt$ exists where $x(t) = \operatorname{Lim}_{n\to\infty} [x_n(t)]$. This is easily shown by the definition of a derivative and with the aid of Eq. (12):

$$\frac{dx(t)}{dt} = \operatorname*{Lim}_{\Delta t\to 0} \left[\frac{x(t + \Delta t) - x(t)}{\Delta t} \right] \tag{28}$$

or

$$\frac{x(t + \Delta t) - x(t)}{t} = \frac{t + \Delta t - t}{t} - \frac{\int_0^{x(t+\Delta t)} u(x, t + \Delta t)\, dx - \int_0^{x(t)} u(x, t)\, dx}{\Delta t}$$

$$= 1 - \int_0^{x(t)} \left[\frac{x(u, t + \Delta t) - u(x, t)}{\Delta t} \right] dx - \int_{x(t)}^{x(t+\Delta t)} \frac{u(x, t + \Delta t)}{\Delta t}\, dx \tag{28'}$$

$$= 1 - \int_0^{x(t)} u_t[x, t + \theta(x)\Delta t]\, dx - \frac{u[\phi(\Delta t), t + \Delta t]}{\Delta t} \int_{x(t)}^{x(t+\Delta t)} dx$$

where $0 \leq \theta(x) \leq 1$ and $x(t) \leq \phi(\Delta t) \leq x(t + \Delta t)$. $u[\phi(\Delta t), t + \Delta t]$ is a mean of the values u takes as it varies from $u[x(t), t + \Delta t]$ to $u[x(t + \Delta t), t + \Delta t]$. Using the last expression above, the left hand side of Eq. (28′) may be written as

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} \frac{1 - \int_0^{x(t)} u_t[x, t + \theta(x)\Delta t]\, dx}{1 + u[\phi(\Delta t), t + \Delta t]}. \tag{29}$$

The integral on the left hand side of Eq. (29) may be evaluated as follows:

$$\int_0^{x(t)} u_t[x, t + \theta(x)\Delta t]\, dx = \int_0^{x(t)} u_{xx}[x, t + \theta(x)\Delta t]\, dx. \tag{30}$$

Since $u_{xx}(x, t)$ is continuous and bounded for all finite $t \geq 0$ and $0 \leq x \leq x(t)$ we may write Eq. (30) as

$$\int_0^{x(t)} u_t[x, t + \theta(x)\Delta t]\, dx = \int_0^{x(t)} [u_{xx}(x, t) + \epsilon]\, dx = u_x[x(t), t] - u_x(0, t) + \epsilon x(t).$$

By using the continuity of $u_{xx}(x, t)$ and by considering a finite region $0 \leq x \leq x(t)$, $0 \leq t < T^*$, the term $\epsilon x(t)$ may be neglected since $x(t) < T^*$. Since $u(x, t)$ is continuous in the same region as $u_{xx}(x, t)$, the

$$\lim_{\Delta t \to 0} \{u[\phi(\Delta t), t + \Delta t]\} = u[x(t), t] = 0.$$

Therefore, we may let $\Delta t \to 0$ on the right hand side of Eq. (29) to obtain

$$\frac{dx(t)}{dt} = -u_x[x(t), t]. \tag{31}$$

To show the uniqueness of $x = x(t) = \lim_{n\to\infty} [x_n(t)]$, assume two solutions of our problem, $U(x, t)$ producing the curve $x = X(t)$ and $V(x, t)$ producing the curve $x = Y(t)$. If $X(t) \equiv Y(t)$ then it is already known that $U(x, t) \equiv V(x, t)$;[5] so assume $X(t) > Y(t)$ for $t > 0$. From Eq. (12) we have

$$X(t) = t - \int_0^{X(t)} U(x, t)\, dx$$

$$Y(t) = t - \int_0^{Y(t)} V(x, t)\, dx.$$

Form the difference

$$X(t) - Y(t) = \int_0^{Y(t)} [V(x, t) - U(x, t)]\, dx - \int_{Y(t)}^{X(t)} U(x, t)\, dx. \tag{32}$$

Since by Eq. (15), $U(x, t) \geq 0$, set $W(x, t) = V(x, t) - U(x, t)$ which satisfies the equation

$$W_{xx}(x, t) = W_t(x, t) \tag{33}$$

with the boundary conditions

$$W_x(0, t) = 0 \qquad \text{and} \qquad W[Y(t), t] = -U[Y(t), t] \leq 0.$$

[5] L. Bieberbach, *Differentialgleichungen,* Dover Publications, New York, 1944, pp. 391-392.

Since $W_x(0, t) = 0$ reflect this solution about $x = 0$ so that $W(x, t)$ still satisfies Eq. (33) but with the alternate boundary conditions

$$W[-Y(t), t] = -U[Y(t), t] \leq 0$$

and

$$W[Y(t), t] = -U[Y(t), t] \leq 0 \quad .$$

Applying Lemma 1 to $W(x, t)$ we see that since it cannot satisfy $W(x, t) \equiv C$, its maximum value is at (0, 0) and

$$W(x, t) \leq 0.$$

By forming the difference.

$$X(t) - Y(t) = \int_0^{Y(t)} W(x, t)\, dx - \int_{Y(t)}^{X(t)} U(x, t)\, dx \leq 0, \tag{34}$$

Eq. (34) contradicts our assumption that $X(t) > Y(t)$. Since X can be replaced by Y and U by V in the above argument to give a contradiction on the assumption that $Y(t) > X(t)$, then $X(t) = Y(t)$. The uniqueness of our solution is shown under the assumption that $X(t)$ and $Y(t)$ do not intersect infinitely often as $t \to 0$.

The author wishes to thank Dr. L. Norenberg and Dr. Y. W. Chen for their suggestions and especially to thank Dr. E. Isaacson for proofreading this note and offering improvements in methods and organization.