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A NOTE ON THE EXISTENCE OF POSITIVE SOLUTIONS OF ONE-DIMENSIONAL p-LAPLACIAN BOUNDARY VALUE PROBLEMS*

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Abstract. This paper is concerned with the existence of positive solutions of a multi-point boundary value problem for higher-order differential equation with one-dimensional p-Laplacian. Examples are presented to illustrate the main results. The result in this paper generalizes those in existing papers.

Keywords: one-dimension p-Laplacian differential equation, nonlocal boundary value problem, positive solution, fixed-point theorem

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1. Introduction

Recently, there have been many papers concerning the existence of positive solutions of BVPs for differential equations with or without p-Laplacian; we refer the readers to [1]–[3], [6]–[7], [9]–[10], [12]–[14], [16]–[19], [21]–[22], and [25].

In [20], Ma studied the existence of positive solutions of the following three-point boundary value problem

$$\begin{cases} x''(t) = -a(t)f(x(t)), \ \ 0 < t < 1, \\ x(0) = 0 = x(1) = \alpha x(\eta), \end{cases} \tag{1}$$

where $\eta \in (0,1)$, $\alpha \geqslant 0$, $a \in C[0,1]$ is nonnegative and there exists at least one point $t_0 \in [0,1]$ such that $a(t_0) > 0$, and $f \in C[0,\infty)$ nonnegative. Under some

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assumptions imposed on f (f is superlinear or sublinear), it was proved that the BVP (1) has positive solutions if

$$\alpha\eta < 1. \tag{$*$}$$

In [18], Ma studied the following multi-point boundary value problem

$$\begin{cases} x''(t) + a(t)f(x(t)) = 0, \ \ 0 \leqslant t \leqslant 1, \\ x(0) = x(1) - \sum_{i=1}^{m} \beta_i x(\xi_i) = 0, \end{cases} \tag{2}$$

where $0 < \xi_i < 1$, $\beta_i \geqslant 0$ with

$$\sum_{i=1}^{m} \beta_i \xi_i < 1, \tag{$**$}$$

a and f are nonnegative continuous functions, and there is $t_0 \in [\xi_m, 1]$ so that $a(t_0) > 0$. Let

$$\lim_{x\to 0} \frac{f(x)}{x} = l, \quad \lim_{x\to\infty} \frac{f(x)}{x} = L.$$

He proved that if $l = 0$, $L = \infty$ or $l = \infty$, $L = 0$, then the BVP (2) has at least one *positive solution*. In [25], Zhang and Sun also studied the existence of *positive solutions* of the BVP (2) under some conditions on the first eigenvalue of the relevant linear operator and f.

In [15], Liu studied the following four-point boundary value problem

$$\begin{cases} x''(t) + f(t, x(t)) = 0, \ \ 0 \leqslant t \leqslant 1, \\ x(0) - \alpha x(\xi) = x(1) - \beta x(\eta) = 0, \end{cases} \tag{3}$$

where $0 < \xi, \eta < 1$, $\alpha, \beta \geqslant 0$, and f is a nonnegative continuous function. By imposing assumptions on f, Liu established existence results for at least one or two *positive solutions* of the BVP (3) provided

$$0 < \alpha(1-\xi) < 1, \quad \beta\eta < 1, \quad \alpha\xi(1-\beta) + (1-\alpha)(1-\beta\eta) > 0. \tag{$***$}$$

In [26], Zhang and Wang studied the following multi-point boundary value problem

$$\begin{cases} x''(t) = -f(t, x(t)), \ \ 0 \leqslant t \leqslant 1, \\ x(0) - \sum_{i=1}^{m} \alpha_i x(\xi_i) = x(1) - \sum_{i=1}^{m} \beta_i x(\xi_i) = 0, \end{cases} \tag{4}$$

where $0 < \xi_i < \ldots < \xi_m < 1$, $\alpha_i, \beta_i \in [0, \infty)$ with

$$(****) \qquad \sum_{i=1}^{m} \alpha_i < 1, \quad \sum_{i=1}^{m} \beta_i < 1.$$

Under certain conditions on f, they established some existence results for *positive solutions* of the BVP (4).

In the paper [15], the multi-point boundary value problem for second-order ordinary differential equation

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t), \ \ 0 < t < 1, \\ x(0) - \displaystyle\sum_{i=1}^{m} \alpha_i x(\xi_i) = 0, \\ x(1) - \displaystyle\sum_{j=1}^{m} \beta_j x(\eta_j) = 0 \end{cases}$$

was considered, where $0 < \xi_i, \eta_i < 1$, $\alpha_i, \beta_j \in \mathbb{R}$, $m \geqslant 2$, $e \in L^1[0,1]$, and f is a Carathéodory function. With the help of coincidence degree theory, it was proved that the above BVP has at least one *solution* under the assumptions

$$\sum_{i=1}^{n} \alpha_i = 1, \quad \sum_{i=1}^{m} \beta_i = 1$$

and some other conditions imposed on f, namely,

(A_1) there exist functions α, β, γ, ϱ, and $\theta \in [0,1)$ such that

$$|f(t,x,y)| \leqslant \varrho(t) + \alpha(t)|x| + \beta(t)|y| + \gamma(t)|x|^{\theta}, \ \ (x,y) \in \mathbb{R}^2, \ \ t \in [0,1]$$

or

$$|f(t,x,y)| \leqslant \varrho(t) + \alpha(t)|x| + \beta(t)|y| + \gamma(t)|y|^{\theta}, \ \ (x,y) \in \mathbb{R}^2, \ \ t \in [0,1];$$

(A_2) there exists a constant $A > 0$ such that for each $x \in D(L)$, if $|x(t)| > A$ or $|x'(t)| > A$ for all $t \in [0,1]$, then

$$\sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} (\xi_i - s)(f(t, x(t), x'(t)) + e(t)) \, \mathrm{d}t \neq 0$$

or

$$\int_0^1 (1-s)(f(t,x(t),x'(t)) + e(t)) \, \mathrm{d}t - \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)(f(t, x(t), x'(t)) + e(t)) \, \mathrm{d}t \neq 0;$$

(A$_3$) there exists a constant $B > 0$ such that for $a, b \in \mathbb{R}$, if $|a| > B$ or $|b| > B$, then either

$$\sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} (\xi_i - s)(f(t, a+bt, b) + e(t)) \,\mathrm{d}t + \int_0^1 (1-s)(f(t, a+bt, b) + e(t)) \,\mathrm{d}t - \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)(f(t, a+bt, b) + e(t)) \,\mathrm{d}t > 0$$

or

$$\sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} (\xi_i - s)(f(t, a+bt, b) + e(t)) \,\mathrm{d}t + \int_0^1 (1-s)(f(t, a+bt, b) + e(t)) \,\mathrm{d}t - \sum_{i=1}^{m} \beta_i \int_0^{\eta_i} (\eta_i - s)(f(t, a+bt, b) + e(t)) \,\mathrm{d}t < 0;$$

(A$_4$) $\|\alpha\|_1 + \|\beta\|_1 < \frac{1}{3}$.

One notes that it is easy to prove that the corresponding Green's functions of the above mentioned BVPs are positive under the assumptions $(*)$, $(**)$ or $(***)$. This leads the authors to get positive solutions of the corresponding BVPs by using fixed-point theorems in cones in Banach spaces. One can also see from $(*)$, $(**)$, and $(***)$ that the assumptions guaranteeing the positivity of Green's function become more complicated. There is no paper concerned with the existence of positive solutions of the above BVP when $\alpha_i \geqslant 0$, $\beta_i \geqslant 0$ with $\sum\limits_{i=1}^{n} \alpha_i = \sum\limits_{i=1}^{m} \beta_i = 1$.

In [3], Bai tried to investigate the following multi-point boundary value problem

$$\begin{cases} [\varphi(x'(t))]' + a(t) f(t, x(t)) = 0, \quad 0 \leqslant t \leqslant 1, \\ x(0) = x(1) - \sum\limits_{i=1}^{m} \beta_i x(\xi_i) = 0, \end{cases} \tag{5}$$

where $\varphi(x) = |x|^{p-2}x$ for $x \neq 0$ and $\varphi(0) = 0$ with $p > 1$, a is continuous and nonnegative and there is $t_0 \in [\xi_m, 1]$ so that $a(t_0) > 0$, f is a continuous nonnegative function, $0 < \xi_i < \ldots < \xi_m < 1$, $\beta_i \geqslant 0$ and $(**)$ holds. However, the results in [3] are wrong, see [23], [24].

In the BVP (5), the presence of p-Laplacian and multi-point make it very complicated to prove the positivity of Green's functions.

Recently, in the papers [8], [11], [23], and [24], the existence of multiple positive solutions of the BVP (5) under the assumption

$$(*****)\qquad \sum_{i=1}^{m}\beta_i < 1$$

and the following more general BVP

$$(6)\qquad \begin{cases} [\varphi(x'(t))]' + a(t)f(t,x(t),x'(t)) = 0, \quad 0 \leqslant t \leqslant 1,\\ x(0) - \displaystyle\sum_{i=1}^{m}\alpha_i x(\xi_i) = 0,\\ x(1) - \displaystyle\sum_{i=1}^{m}\beta_i x(\xi_i) = 0 \end{cases}$$

under the assumptions $(****)$ were studied, where a and f are positive continuous functions.

For the BVP (5), it is easy to see that $(*****)$ is weaker than $(**)$ when $\varphi(x) = x$ and for the BVP (6), $(****)$ is weaker than $(***)$ when $\varphi(x) = x$ and $m = 1$ if $\xi = \eta$ or $m = 2$ and $\xi_1 = \min\{\xi,\eta\}$ and $\xi_2 = \max\{\xi,\eta\}$ if $\xi \neq \eta$. A problem appears naturally, whether there exist weaker conditions than $(****)$ imposed on ξ_i, α_i, β_i $(i = 1,\ldots,m)$ such that the solutions of the BVP (6) are positive. The methods in [8], [11], [23], [24] are based upon transforming the BVP (6) into the integral equation

$$x(t) = B_x + \int_0^t \varphi^{-1}\Big(A_x - \int_0^s a(u)f(u,x(u),x'(u))\,\mathrm{d}u\Big)\,\mathrm{d}s,$$

where A_x and B_x satisfy

$$\begin{aligned} &\frac{1}{1-\sum_{i=1}^m\alpha_i}\sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\varphi^{-1}\Big(A_x - \int_0^s a(u)f(u,x(u),x'(u))\,\mathrm{d}u\Big)\,\mathrm{d}s\\ &\qquad - \frac{1}{1-\sum_{i=1}^m\beta_i}\sum_{i=1}^{m}\beta_i\int_0^{\xi_i}\varphi^{-1}\Big(A_x - \int_0^s a(u)f(u,x(u),x'(u))\,\mathrm{d}u\Big)\,\mathrm{d}s\\ &\qquad + \frac{1}{1-\sum_{i=1}^m\beta_i}\int_0^{1}\varphi^{-1}\Big(A_x - \int_0^s a(u)f(u,x(u),x'(u))\,\mathrm{d}u\Big)\,\mathrm{d}s = 0,\\ &B_x = \frac{1}{1-\sum_{i=1}^m\alpha_i}\sum_{i=1}^{m}\alpha_i\int_0^{\xi_i}\varphi^{-1}\Big(A_x - \int_0^s a(u)f(u,x(u),x'(u))\,\mathrm{d}u\Big)\,\mathrm{d}s. \end{aligned}$$

Then fixed-point theorems are used to get positive solutions of the BVP (6). One may see that the methods in the above mentioned papers can not be applied even to the case when $\sum\limits_{i=1}^{m}\alpha_i = 1$ or $\sum\limits_{i=1}^{m}\beta_i = 1$.

Motivated by the papers mentioned above, this paper is concerned with the following more general BVP for higher order differential equation with p-Laplacian

$$(7)\qquad \begin{cases} [\varphi(x^{(n-1)}(t))]' + f(t, x(t), \ldots, x^{(n-1)}(t)) = 0, \ \ t \in (0,1), \\ x^{(i)}(0) = 0, \ \ i = 0, \ldots, n-3, \\ x^{(n-2)}(0) = \displaystyle\sum_{i=1}^{m} \alpha_i x^{(n-2)}(\xi_i), \\ x^{(n-2)}(1) = \displaystyle\sum_{i=1}^{m} \beta_i x^{(n-2)}(\xi_i), \end{cases}$$

where $n \geqslant 2$, $0 < \xi_i < \ldots < \xi_m < 1$, $\alpha_i, \beta_i \in [0, \infty)$, f is a continuous function, φ is the one-dimensional p-Laplacian with $\varphi(x) = |x|^{p-2}x$ for $x \neq 0$ and $\varphi(0) = 0$, its inverse function is denoted by φ^{-1}. We will establish existence results for positive solutions of the BVP (7) under the following assumptions:

$$(******)\qquad \sum_{i=1}^{m} \alpha_i \leqslant 1, \ \sum_{i=1}^{m} \beta_i < 1, \quad \text{or} \quad \sum_{i=1}^{m} \alpha_i < 1, \ \sum_{i=1}^{m} \beta_i \leqslant 1,$$
$$\text{or} \quad \sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \beta_i = 1.$$

It is easy to see that (7) contains (1), (2), (4), (5), and (6) as special cases; and when, in the BVP (7), we choose $\varphi(x) = x$, $n = 2$, replace $f(t, x, x', \ldots, x^{(n-1)})$ by $f(t, x)$, and set $\xi_1 = \xi$, $\xi_2 = \eta$, $\alpha_1 = \alpha$, $\alpha_2 = 0$ and $\beta_1 = 0$, $\beta_2 = \beta$, we get (3). One sees that (3) is a special case of (7). Condition (******) is weaker than each one of (****). The methods used are a modification of those in [23] and the result in this paper is different from those in [3]–[5], [8], [11], [13]–[15], [18], [20]–[21], [23]–[26].

R e m a r k. The condition (******) is the best possible one. Consider the BVP

$$\begin{cases} x''(t) = -2, \ \ 0 < t < 1, \\ x(0) = \dfrac{1}{2} x(13/30) + \dfrac{1}{2} x(9/10), \\ x(1) = \delta x(9/10), \end{cases}$$

where $\delta > 0$ is a constant. Corresponding to the BVP (7), we have $\varphi(x) = x$, $\xi_1 = 13/30$, $\xi_2 = 9/10$, $\alpha_1 = 1/2, \alpha_2 = 1/2$, $\beta_1 = 0$, $\beta_2 = \delta$, $f(t, x, y) = -2 < 0$. One sees that $\alpha_1 + \alpha_2 = 1$ and $\beta_1 + \beta_2 = \delta$. If $\delta \neq 1$, it is easy to see that the above BVP has a unique solution

$$x(t) = -t^2 + \frac{449}{600} t + \frac{\frac{151}{600} - \frac{273}{2000}\delta}{1 - \delta}.$$

Then

$$x(0) = \frac{\frac{151}{600} - \frac{273}{2000}\delta}{1-\delta}, \quad x(1) = -\frac{151}{600} + \frac{\frac{151}{600} - \frac{273}{2000}\delta}{1-\delta}.$$

One sees that x is a positive solution if $\delta < 1$ and a non-positive solution if $\delta > 1$ and small enough. If $\delta = 1$, the problem has no solution.

The remainder of the paper is divided into two parts. In Section 2, the main results are presented, and Examples are given to illustrate the main theorems in Section 3.

2. Main results

In this section, we present the main result. This will be done by using the following fixed-point theorems.

Let X and Y be Banach spaces, $L\colon D(L)(\subset X) \to Y$ a Fredholm operator of index zero, and $P\colon X \to X$, $Q\colon Y \to Y$ projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \quad \operatorname{Ker} Q = \operatorname{Im} L, \quad X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L|_{D(L)\cap \operatorname{Ker} P}\colon D(L) \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible; we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, $D(L) \cap \overline{\Omega} \neq \emptyset$, the map $N\colon X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N\colon \overline{\Omega} \to X$ is compact.

Theorem M1 ([6]). *Let L be a Fredholm operator of index zero and let N be L-compact on Ω. Assume that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x,\lambda) \in [(D(L) \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;*
(ii) *$Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$;*
(iii) *$\deg(\wedge QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, where $\wedge\colon \operatorname{Ker} L \to Y/\operatorname{Im} L$ is an isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$.

Lemma M2 ([6]). *Let X and Y be real Banach spaces. Suppose $L\colon D(L) \subset X \to Y$ is a Fredholm operator of index zero with $\operatorname{Ker} L = \{0\}$, and $N\colon X \to Y$ is L-compact on any open bounded subset of X. If $0 \in \Omega \subset X$ is an open bounded subset and $Lx \neq \lambda Nx$ for all $x \in D(L) \cap \partial\Omega$ and $\lambda \in [0,1]$, then there is at least one $x \in \Omega$ so that $Lx = Nx$.*

Choose $X = C^{(n-2)}[0,1] \times C^0[0,1]$, $Y = C^0[0,1] \times C^0[0,1]$. It is easy to see that X is a Banach space with the norm

$$\|(x_1,x_2)\| = \max\{\|x_1^{(i)}\|_\infty = \max_{t\in[0,1]} |x_1^{(i)}(t)| \colon\ i = 0,\ldots,n-2,\ \|x_2\|_\infty = \max_{t\in[0,1]} |x_2(t)|\},$$

Y is a Banach space with the norm

$$\|(u,v)\| = \max\{\|u\|_\infty = \max_{t\in[0,1]} |u(t)|,\ \|v\|_\infty = \max_{t\in[0,1]} |v(t)|\}.$$

Define the operators

$$\begin{aligned} L(x_1,x_2) &= (x_1^{(n-1)}, x_2'),\ (x_1,x_2) \in X \cap D(L),\\ N(x_1,x_2) &= (\varphi^{-1}(x_2), -f(t,x_1,\ldots,x^{(n-2)},\varphi^{-1}(x_2)),\ (x_1,x_2) \in X, \end{aligned}$$

where

$$\begin{aligned} D(L) = \Big\{&(x_1,x_2) \in C^{(n-1)}[0,1] \times C^1[0,1] \colon\ x_1^{(n-2)}(0) = \sum_{i=1}^m \alpha_i x_1^{(n-2)}(\xi_i),\\ &x_1^{(n-2)}(1) = \sum_{i=1}^m \beta_i x_1^{(n-2)}(\xi_i),\ x_1^{(i)}(0) = 0,\ i = 0,\ldots,n-3\Big\}. \end{aligned}$$

It is easy to see that if $L(x_1,x_2) = N(x_1,x_2)$, then x_1 is a solution of the BVP (6). We consider the operator equation $L(x_1,x_2) = \lambda N(x_1,x_2)$ for some $\lambda \in (0,1)$. Then

$$\text{(8)}\qquad \begin{cases} x_1^{(n-1)}(t) = \lambda\varphi^{-1}(x_2(t)),\ \ t \in [0,1],\\ x_2'(t) = -\lambda f(t, x_1(t),\ldots,x_1^{(n-2)}(t),\varphi^{-1}(x_2(t))),\ t \in [0,1],\\ x_1^{(n-2)}(0) = \sum\limits_{i=1}^m \alpha_i x_1^{(n-2)}(\xi_i),\\ x_1^{(n-2)}(1) = \sum\limits_{i=1}^m \beta_i x_1^{(n-2)}(\xi_i),\\ x_1^{(i)}(0) = 0,\ \ i = 0,\ldots,n-3. \end{cases}$$

Suppose that

(H₁) $f\colon [0,1] \times [0,\infty) \times \mathbb{R}^{n-1} \to [0,\infty)$ is continuous with $f(t,0,\ldots,0) \not\equiv 0$ on each sub-interval of [0,1];

(H₂) $\alpha_i \geqslant 0$, $\beta_i \geqslant 0$ satisfy

$$\sum_{i=1}^m \alpha_i \leqslant 1,\quad \sum_{i=1}^m \beta_i < 1, \qquad \text{or} \qquad \sum_{i=1}^m \alpha_i < 1,\quad \sum_{i=1}^m \beta_i \leqslant 1,$$
$$\text{or}\ \ \sum_{i=1}^m \alpha_i = \sum_{i=1}^m \beta_i = 1.$$

Lemma 1. *Suppose that* (H_1) *and* (H_2) *hold and* (x_1, x_2) *is a solution of* (8). *Then there is* $\xi \in [0,1]$ *so that* $x_1^{(n-1)}(\xi) = 0$.

P r o o f. *Case 1.* $\sum\limits_{i=1}^{m} \alpha_i \leqslant 1$, $\sum\limits_{i=1}^{m} \beta_i < 1$.

In this case, if $x_1^{(n-1)}(t) > 0$ for all $t \in [0,1]$, then

$$x_1^{(n-2)}(0) = \sum_{i=1}^{m} \alpha_i x_1^{(n-2)}(\xi_i) > \sum_{i=1}^{m} \alpha_i x_1^{(n-2)}(0).$$

If $\sum\limits_{i=1}^{m} \alpha_i = 1$, then $x_1^{(n-2)}(0) > x_1^{(n-2)}(0)$, a contradiction. If $0 < \sum\limits_{i=1}^{m} \alpha_i < 1$, then $x_1^{(n-2)}(0) > 0$, and it follows that $x_1^{(n-2)}(1) > 0$. So

$$x_1^{(n-2)}(1) = \sum_{i=1}^{m} \beta_i x_1^{(n-2)}(\xi_i) < \sum_{i=1}^{m} \beta_i x_1^{(n-2)}(1) \leqslant x_1^{(n-2)}(1),$$

a contradiction. If $\sum\limits_{i=1}^{m} \alpha_i = 0$, then $x_1^{(n-2)}(0) = 0$ and $x_1^{(n-2)}(1) > x_1^{(n-2)}(0) = 0$ for all $t \in [0,1]$. Therefore,

$$x_1^{(n-2)}(1) = \sum_{i=1}^{m} \beta_i x_1^{(n-2)}(\xi_i) < \sum_{i=1}^{m} \beta_i x_1^{(n-2)}(1) \leqslant x_1^{(n-2)}(1),$$

leads to a contradiction. Hence, $x_1^{(n-1)}(t) > 0$ for all $t \in [0,1]$ is impossible.

If $x_1^{(n-1)}(t) < 0$ for all $t \in [0,1]$, then

$$x_1^{(n-2)}(1) = \sum_{i=1}^{m} \beta_i x_1^{(n-2)}(\xi_i) > \sum_{i=1}^{m} \beta_i x_1^{(n-2)}(1),$$

thus $x_1^{(n-2)}(1) > 0$, and it follows that $x_1^{(n-2)}(0) > 0$. So

$$x_1^{(n-2)}(0) = \sum_{i=1}^{m} \alpha_i x_1^{(n-2)}(\xi_i) < \sum_{i=1}^{m} \alpha_i x_1^{(n-2)}(0) \leqslant x_1^{(n-2)}(0)$$

yields a contradiction, too. It follows from above discussion that there exists $\xi \in [0,1]$ such that $x_1^{(n-1)}(\xi) = 0$.

Case 2. $\sum\limits_{i=1}^{m} \alpha_i < 1$, $\sum\limits_{i=1}^{m} \beta_i \leqslant 1$.

The proof is similar to that of Case 1.

Case 3. $\sum\limits_{i=1}^{m} \alpha_i = \sum\limits_{i=1}^{m} \beta_i = 1$.

The proof is similar to that of Case 1. $\square$

Lemma 2. *Suppose that* (H_1) *and* (H_2) *hold. If* (x_1, x_2) *is a solution of* (8), *then* $x_1(t) > 0$ *for all* $t \in (0,1)$.

P r o o f. It follows from (8) that

$$[\varphi(x_1^{(n-1)}(t))]' = -\lambda\varphi(\lambda) f\Big(t, x_1(t), \ldots, x_1^{(n-2)}(t), \frac{x_1^{(n-1)}(t)}{\lambda}\Big).$$

By Lemma 1 there is $\xi \in [0,1]$ so that $x_1^{(n-1)}(\xi) = 0$. Denote

$$F(t) = \lambda\varphi(\lambda) f\Big(t, x_1(t), \ldots, x_1^{(n-2)}(t), \frac{x_1^{(n-1)}(t)}{\lambda}\Big).$$

One sees that $F(t) \geqslant 0$ for all $t \in [0,1]$ and

$$x_1^{(n-2)}(t) = \begin{cases} \displaystyle\int_t^1 \varphi^{-1}\Big(\int_\xi^s F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s + x_1^{(n-2)}(1), & t \in [\xi, 1], \\ \displaystyle\int_0^t \varphi^{-1}\Big(\int_s^\xi F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s + x_1^{(n-2)}(0), & t \in [0, \xi]. \end{cases}$$

Without loss of generality, suppose

$$0 < \xi_1 < \ldots < \xi_{i_0} \leqslant \xi < \xi_{i_0+1} < \ldots < \xi_m < 1.$$

Since $[\varphi(x_1^{(n-1)}(t))]' \leqslant 0$, we get that $x_1^{(n-1)}(t)$ is decreasing on $[0,1]$. Hence, $x_1^{(n-1)}(t) \geqslant 0$ for $t \in [0,\xi]$ and $x_1^{(n-1)}(t) \leqslant 0$ on $[\xi, 1]$. Thus $x_1^{(n-2)}(t)$ is increasing on $[0,\xi]$ and decreasing on $[\xi,1]$.

On the other hand, one sees from (8) that

$$\begin{aligned} x_1^{(n-2)}(0) &= \sum_{i=1}^{i_0} \alpha_i x_1^{(n-2)}(\xi_i) + \sum_{i=i_0+1}^{m} \alpha_i x_1^{(n-2)}(\xi_i) \\ &= \sum_{i=1}^{i_0} \alpha_i \int_0^{\xi_i} \varphi^{-1}\Big(\int_s^\xi F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s + x_1^{(n-2)}(0)\sum_{i=1}^{i_0}\alpha_i \\ &\quad + \sum_{i=i_0+1}^{m} \alpha_i \int_{\xi_i}^1 \varphi^{-1}\Big(\int_\xi^s F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s + x_1^{(n-2)}(1)\sum_{i=i_0+1}^{m}\alpha_i, \\ x_1^{(n-2)}(1) &= \sum_{i=1}^{i_0} \beta_i \int_0^{\xi_i} \varphi^{-1}\Big(\int_s^\xi F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s + x_1^{(n-2)}(0)\sum_{i=1}^{i_0}\beta_i \\ &\quad + \sum_{i=i_0+1}^{m} \beta_i \int_{\xi_i}^1 \varphi^{-1}\Big(\int_\xi^s F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s + x_1^{(n-2)}(1)\sum_{i=i_0+1}^{m}\beta_i. \end{aligned}$$

It follows that

$$\Big(1-\sum_{i=1}^{i_0}\alpha_i\Big)x_1^{(n-2)}(0)-\sum_{i=i_0+1}^{m}\alpha_i x_1^{(n-2)}(1)$$
$$=\sum_{i=1}^{i_0}\alpha_i\int_0^{\xi_i}\varphi^{-1}\Big(\int_s^{\xi}F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s+\sum_{i=i_0+1}^{m}\alpha_i\int_{\xi_i}^{1}\varphi^{-1}\Big(\int_{\xi}^{s}F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s,$$

and

$$-\sum_{i=1}^{i_0}\beta_i x_1^{(n-2)}(0)+\Big(1-\sum_{i=i_0+1}^{n}\beta_i\Big)x_1^{(n-2)}(1)$$
$$=\sum_{i=1}^{i_0}\beta_i\int_0^{\xi_i}\varphi^{-1}\Big(\int_s^{\xi}F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s+\sum_{i=i_0+1}^{m}\beta_i\int_{\xi_i}^{1}\varphi^{-1}\Big(\int_{\xi}^{s}F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s.$$

One sees that

$$\begin{aligned}\Delta&=\begin{vmatrix}1-\sum\limits_{i=1}^{i_0}\alpha_i & \sum\limits_{i=i_0}^{m}\alpha_i\\ -\sum\limits_{i=1}^{i_0}\beta_i & 1-\sum\limits_{i=i_0+1}^{m}\beta_i\end{vmatrix}\\
&=\Big(1-\sum_{i=1}^{i_0}\alpha_i\Big)\Big(1-\sum_{i=i_0+1}^{m}\beta_i\Big)+\Big(\sum_{i=i_0}^{m}\alpha_i\Big)\Big(\sum_{i=1}^{i_0}\beta_i\Big)\\
&\geqslant\Big(1-\sum_{i=1}^{i_0}\alpha_i\Big)\Big(1-\sum_{i=i_0+1}^{m}\beta_i\Big).\end{aligned}$$

To prove that $\Delta>0$, we consider two cases.

Case 1. $\sum\limits_{i=1}^{m}\alpha_i\leqslant 1$, $\sum\limits_{i=1}^{m}\beta_i<1$.

In this case it is easy to see that $\sum\limits_{i=i_0+1}^{m}\beta_i<1$. Note that $x_1^{(n-2)}(t)$ is increasing on $[0,\xi]$ and decreasing on $[\xi,1]$, so if $\sum\limits_{i=1}^{i_0}\alpha_i=1$, then

$$x_1^{(n-2)}(0)=\sum_{i=1}^{m}\alpha_i x_1^{(n-2)}(\xi_i)>\sum_{i=1}^{m}\alpha_i x_1^{(n-2)}(0)=\sum_{i=1}^{i_0}\alpha_i x_1^{(n-2)}(0)=x_1^{(n-2)}(0)$$

leads to a contradiction. Hence, $\sum\limits_{i=1}^{i_0}\alpha_i<1$ and thus, $\Delta>0$.

Case 2. $\sum\limits_{i=1}^{m} \alpha_i < 1$, $\sum\limits_{i=1}^{m} \beta_i \leqslant 1$. Similar to Case 1, we get that $\Delta > 0$.

Case 3. $\sum\limits_{i=1}^{m} \alpha_i = \sum\limits_{i=1}^{m} \beta_i = 1$. Similar to Case 1, we get that $\Delta > 0$.

Furthermore, we have

$$\Delta_1 = \begin{vmatrix} 1 - \sum\limits_{i=1}^{i_0} \alpha_i & \begin{array}{c} \sum\limits_{i=1}^{i_0} \alpha_i \int_0^{\xi_i} \varphi^{-1}\Big(\int_s^{\xi} F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s \\ + \sum\limits_{i=i_0+1}^{n} \alpha_i \int_{\xi_i}^{1} \varphi^{-1}\Big(\int_{\xi}^{s} F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s \end{array} \\ -\sum\limits_{i=1}^{i_0} \beta_i & \begin{array}{c} \sum\limits_{i=1}^{i_0} \beta_i \int_0^{\xi_i} \varphi^{-1}\Big(\int_s^{\xi} F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s \\ + \sum\limits_{i=i_0+1}^{m} \beta_i \int_{\xi_i}^{1} \varphi^{-1}\Big(\int_{\xi}^{s} F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s \end{array} \end{vmatrix} \geqslant 0,$$

$$\Delta_2 = \begin{vmatrix} \begin{array}{c} \sum\limits_{i=1}^{i_0} \alpha_i \int_0^{\xi_i} \varphi^{-1}\Big(\int_s^{\xi} F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s \\ + \sum\limits_{i=i_0+1}^{m} \alpha_i \int_{\xi_i}^{1} \varphi^{-1}\Big(\int_{\xi}^{s} F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s \end{array} & -\sum\limits_{i=i_0+1}^{m} \alpha_i \\ \begin{array}{c} \sum\limits_{i=1}^{i_0} \beta_i \int_0^{\xi_i} \varphi^{-1}\Big(\int_s^{\xi} F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s \\ + \sum\limits_{i=i_0+1}^{m} \beta_i \int_{\xi_i}^{1} \varphi^{-1}\Big(\int_{\xi}^{s} F(u)\,\mathrm{d}u\Big)\,\mathrm{d}s \end{array} & 1 - \sum\limits_{i=i_0+1}^{m} \beta_i \end{vmatrix} \geqslant 0,$$

whence $x_1^{(n-2)}(0) = \Delta_1/\Delta \geqslant 0$, $x_1^{(n-2)}(1) = \Delta_2/\Delta \geqslant 0$. Since (H_1) implies that

$$[\varphi(x_1^{(n-1)}(t))]' = -\lambda\varphi(\lambda) f\Big(t, x_1(t), \ldots, x_1^{(n-2)}(t), \frac{x_1^{(n-1)}(t)}{\lambda}\Big) \leqslant 0,$$

then $x_1^{(n-1)}(t)$ is decreasing, so $x_1^{(n-2)}(t) \geqslant \min\{x_1^{(n-2)}(0), x_1^{(n-2)}(1)\} \geqslant 0$. It follows that $x_1^{(n-2)}(t) \geqslant 0$ for all $t \in [0,1]$. Since $x_1^{(i)}(0) = 0$ $(i = 0, \ldots, n-3)$, we get that $x_1(t) \geqslant 0$ for all $t \in (0,1)$.

We claim that $x_1(t) > 0$ for all $t \in (0,1)$. In fact, if there exists $t_0 \in (0,1)$ such that $x_1(t_0) = 0$, then $x_1(t) \equiv 0$ for $t \in [0, t_0]$ or $[t_0, 1]$. Then (H_1) and (8) imply that $f(t, 0, \ldots, 0) \equiv 0$ for $t \in [0, t_0]$ or $[t_0, 1]$, which is a contradiction. Hence, $x_1(t) > 0$ for all $t \in (0,1)$.

Furthermore, we suppose

(H$_3$) there are continuous nonnegative functions a, b_i, and c so that

$$|f(t,x_0,\dots,x_{n-2},x_{n-1})| \leqslant a(t)+\sum_{i=0}^{n-2} b_i(t)\varphi(|x_i|)+c(t)\varphi(|x_{n-1}|)$$

holds for $(t,x_0,\dots,x_{n-1})\in[0,1]\times\mathbb{R}^n$.

(H$_4$) If $\sum\limits_{i=1}^{m}\alpha_i<1$, $\sum\limits_{i=1}^{m}\beta_i\leqslant 1$, the following inequality holds:

$$\varphi\Big(1+\frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m\alpha_i}\Big)\Big[\sum_{i=0}^{n-3}\varphi\Big(\frac{1}{(n-2-i)!}\Big)\int_0^1 b_i(s)\,\mathrm{d}s+\int_0^1 b_{n-2}(s)\,\mathrm{d}s\Big] + \int_0^1 c(s)\,\mathrm{d}s<1.$$

(H$_5$) If $\sum\limits_{i=1}^{m}\alpha_i\leqslant 1$, $\sum\limits_{i=1}^{m}\beta_i< 1$, the following inequality holds:

$$\varphi\Big(1+\frac{\sum_{i=1}^m \beta_i(1-\xi_i)}{1-\sum_{i=1}^m\beta_i}\Big)\Big[\sum_{i=0}^{n-3}\varphi\Big(\frac{1}{(n-2-i)!}\Big)\int_0^1 b_i(s)\,\mathrm{d}s+\int_0^1 b_{n-2}(s)\,\mathrm{d}s\Big] + \int_0^1 c(s)\,\mathrm{d}s<1.$$

□

Theorem L1. *Suppose that* $\alpha_i\geqslant 0$, $\beta_i\geqslant 0$, $\sum\limits_{i=1}^{m}\alpha_i<1$, $\sum\limits_{i=1}^{m}\beta_i\leqslant 1$, *and* (H$_1$), (H$_3$) *and* (H$_4$) *hold. Then the BVP* (7) *has at least one positive solution.*

Proof. Consider the system (8). It follows from Lemma 1 that there is $\xi\in[0,1]$ so that $x_2(\xi)=0$. Then

$$\begin{aligned}|x_2(t)| &= \Big|-\lambda\int_\xi^t f(s,x_1(s),\dots,x_1^{(n-2)}(s),\varphi^{-1}(x_2(s)))\,\mathrm{d}s\Big|\\ &\leqslant \int_0^1 |f(s,x_1(s),\dots,x_1^{(n-2)}(s),\varphi^{-1}(x_2(s)))|\,\mathrm{d}s\\ &\leqslant \int_0^1\Big(a(s)+\sum_{i=0}^{n-2} b_i(s)\varphi(|x^{(i)}(s)|)+c(s)|x_2(s)|\Big)\,\mathrm{d}s,\end{aligned}$$

$$
\begin{aligned}
|x_1^{(n-2)}(0)| &\leqslant \frac{1}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i |x_1^{(n-2)}(0) - x_1^{(n-2)}(\xi_i)| \\
&\leqslant \frac{1}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \xi_i |x_1^{(n-1)}(\theta_i)|, \quad \theta_i \in [0,\xi_i], \\
&\leqslant \left(\frac{1}{1-\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \xi_i\right) \varphi^{-1}(\|x_2\|_\infty).
\end{aligned}
$$

Thus,

$$
|x_1^{(n-2)}(t)| \leqslant |x_1^{(n-2)}(0)| + \left|\int_0^t x_1^{(n-1)}(s)\,\mathrm{d}s\right| \leqslant \left(1 + \frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\right)\varphi^{-1}(\|x_2\|_\infty).
$$

So for $i = 0,\dots,n-3$, we get

$$
\begin{aligned}
|x_1^{(i)}(t)| &\leqslant \left| x^{(i)}(0) + \int_0^t \frac{(t-s)^{n-3-i}}{(n-3-i)!} x^{(n-2)}(s)\,\mathrm{d}s \right| \\
&\leqslant \int_0^t \frac{(t-s)^{n-3-i}}{(n-3-i)!}\,\mathrm{d}s \|x_1^{(n-2)}\|_\infty \\
&\leqslant \frac{1}{(n-2-i)!} \|x_1^{(n-2)}\|_\infty \\
&\leqslant \frac{1}{(n-2-i)!} \left(1 + \frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\right)\varphi^{-1}(\|x_2\|_\infty).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|x_2(t)| \leqslant &\int_0^1 a(s)\,\mathrm{d}s \\
&+ \sum_{i=0}^{n-3} \int_0^1 b_i(s)\,\mathrm{d}s \varphi\Big(\frac{1}{(n-2-i)!}\Big) \varphi\left(1 + \frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\right)\|x_2\|_\infty \\
&+ \int_0^1 b_{n-2}(s)\,\mathrm{d}s\, \varphi\left(1 + \frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\right)\|x_2\|_\infty + \int_0^1 c(s)\,\mathrm{d}s \|x_2\|_\infty.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_2\|_\infty \leqslant &\int_0^1 a(s)\,\mathrm{d}s \\
&+ \sum_{i=0}^{n-3} \int_0^1 b_i(s)\,\mathrm{d}s \varphi\Big(\frac{1}{(n-2-i)!}\Big) \varphi\left(1 + \frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\right)\|x_2\|_\infty \\
&+ \int_0^1 b_{n-2}(s)\,\mathrm{d}s\, \varphi\left(1 + \frac{\sum_{i=1}^m \alpha_i\xi_i}{1-\sum_{i=1}^m \alpha_i}\right)\|x_2\|_\infty + \int_0^1 c(s)\,\mathrm{d}s \|x_2\|_\infty.
\end{aligned}
$$

Thus

$$\Big[1-\sum_{i=0}^{n-3}\varphi\Big(\frac{1}{(n-2-i)!}\Big)\int_0^1 b_i(s)\,\mathrm{d}s\,\varphi\Big(1+\frac{\sum_{i=1}^m\alpha_i\xi_i}{1-\sum_{i=1}^m\alpha_i}\Big)$$
$$-\int_0^1 b_{n-2}(s)\,\mathrm{d}s\,\varphi\Big(1+\frac{\sum_{i=1}^m\alpha_i\xi_i}{1-\sum_{i=1}^m\alpha_i}\Big)-\int_0^1 c(s)\,\mathrm{d}s\Big]\|x_2\|_\infty\leqslant\int_0^1 a(s)\,\mathrm{d}s.$$

It follows from (H_4) that there is a constant $M>0$ such that $\|x_2\|_\infty\leqslant M$. Since $|x_1^{(i)}(t)|\leqslant(n-2-i)!^{-1}\|x_1^{(n-2)}\|_\infty$ and $|x_1^{(n-2)}(t)|\leqslant\Big(1+\sum\limits_{i=1}^m\alpha_i\xi_i\Big/\Big(1-\sum\limits_{i=1}^m\alpha_i\Big)\Big)\times\varphi^{-1}(\|x_2\|_\infty)$, there are constants $M_i>0$ such that $\|x_1^{(i)}\|_\infty\leqslant M_i$ for all $i=0,\dots,n-2$. Hence,

$$\|(x_1,x_2)\|\leqslant\max\{M,M_i\colon\ i=0,\dots,n-2\}.$$

Let

$$\Omega_0=\{(x_1,x_2)\colon\ L(x_1,x_2)=\lambda N(x_1,x_2)\text{ for some }\lambda\in[0,1]\}.$$

Then Ω_0 is bounded. Let $\Omega\supseteq\Omega_0$ be a bounded open subset of X. Then $L(x_1,x_2)\neq\lambda N(x_1,x_2)$ for all $((x_1,x_2),\lambda)\in[D(L)\cap\partial\Omega]\times[0,1]$. Since (H_1) and (H_2) imply that L is a Fredholm operator of index zero with $\operatorname{Ker}L=\{0\}$, $N\colon X\to Y$ is L-compact on any open bounded subset of X. It follows from Lemma M2 that $L(x_1,x_2)=N(x_1,x_2)$ has at least one solution $x=(x_1,x_2)$. So x_1 is a solutions of (7). It follows from Lemma 2 that $x_1(t)>0$ for all $t\in(0,1)$. Hence, x_1 is a positive solution of (7). □

Theorem L2. *Suppose that* $\alpha_i\geqslant0$, $\beta_i\geqslant0$, $\sum\limits_{i=1}^m\alpha_i\leqslant1$, $\sum\limits_{i=1}^m\beta_i<1$, *and* (H_1), (H_3), *and* (H_5) *hold. Then the BVP* (7) *has at least one positive solution.*

P r o o f. Similar to that of Theorem L1 and omitted. □

We introduce some further assumptions.

(H_6) There exists constant $A>0$ such that for each $(x_1,x_2)\in D(L)$, if $|x_1^{(n-2)}(t)|>A$ for all $t\in[0,1]$, then

$$\int_0^1 f(s,x_1(s),\dots,x_1^{(n-2)}(s),\varphi^{-1}(x_2(s)))\,\mathrm{d}s$$
$$-\sum_{i=1}^m\beta_i\int_0^{\xi_i} f(s,x_1(s),\dots,x_1^{(n-2)}(s),\varphi^{-1}(x_2(s)))\,\mathrm{d}s\neq0.$$

(H_7) $1>\sum\limits_{i=0}^{n-3}\int_0^1 b_i(s)\,\mathrm{d}s\,\varphi(1/(n-2-i)!)+\int_0^1 b_{n-2}(s)\,\mathrm{d}s+\int_0^1 c(s)\,\mathrm{d}s.$

(H_8) There exists a constant $B > 0$ such that

$$-\int_0^1 f\Big(s, \frac{s^{n-2}}{(n-2)!}a, \ldots, a, 0\Big)\,\mathrm{d}s + \sum_{i=1}^m \beta_i \int_0^{\xi_i} f\Big(s, \frac{s^{n-2}}{(n-2)!}a, \ldots, a, 0\Big)\,\mathrm{d}s \neq 0 \quad \text{for all } |a| > B.$$

(H_9) For $\lambda \in (0,1), a \in \mathbb{R}$, let

$$H(t,a) = f\Big(t, \frac{t^{n-2}}{(n-2)!}a, \frac{t^{n-3}}{(n-3)!}a, \ldots, a, -\frac{\lambda}{1-\lambda}a\Big).$$

Then there exists a constant $C > 0$ such that

$$a\Big(-\int_0^1 H(s,a)\,\mathrm{d}s + \sum_{i=1}^m \beta_i \int_0^{\xi_i} H(s,a)\,\mathrm{d}s\Big) < 0 \quad \text{for all } |a| > C.$$

Theorem L3. *Suppose that* $\alpha_i \geqslant 0$, $\beta_i \geqslant 0$, $\sum\limits_{i=1}^m \alpha_i = \sum\limits_{i=1}^m \beta_i = 1$, *and* ($\mathrm{H}_1$), ($\mathrm{H}_3$), ($\mathrm{H}_6$)–($\mathrm{H}_9$) *hold. Then the BVP* (7) *has at least one positive solution.*

Proof. In this case, we apply Lemma M1 to get positive solutions of (7). It is easy to show that

(i) $\operatorname{Ker} L = \{((t^{n-2}/(n-2)!)a, b) \in D(L)\colon a, b \in \mathbb{R}\}$;

(ii) $\operatorname{Im} L = \Big\{(u,v) \in Y\colon \sum\limits_{i=1}^m \alpha_i \int_0^{\xi_i} u(s)\,\mathrm{d}s = 0, \int_0^1 v(s)\,\mathrm{d}s = \sum\limits_{i=1}^m \beta_i \int_0^{\xi_i} v(s)\,\mathrm{d}s\Big\}$;

(iii) L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$.

In fact, for $(x_1, x_2) \in \operatorname{Ker} L$, we get

$$\begin{cases} x_1^{(n-1)}(t) = 0, \quad t \in [0,1], \\ x_2'(t) = 0, \quad t \in [0,1], \\ x_1^{(n-2)}(0) = \sum\limits_{i=1}^m \alpha_i x_1^{(n-2)}(\xi_i), \\ x_1^{(n-2)}(1) = \sum\limits_{i=1}^m \beta_i x_1^{(n-2)}(\xi_i), \\ x_1^{(i)}(0) = 0, \quad i = 0, \ldots, n-3. \end{cases}$$

It is easy to get (i). For $(u,v) \in \operatorname{Im} L$, we get that there exists $(x_1,x_2) \in D(L)$ such that

$$\begin{cases} x_1^{(n-1)}(t) = u(t), \quad t \in [0,1], \\ x_2'(t) = v(t), \quad t \in [0,1], \\ x_1^{(n-2)}(0) = \sum\limits_{i=1}^{m} \alpha_i x_1^{(n-2)}(\xi_i), \\ x_1^{(n-2)}(1) = \sum\limits_{i=1}^{m} \beta_i x_1^{(n-2)}(\xi_i), \\ x_1^{(i)}(0) = 0, \quad i = 0,\dots,n-3. \end{cases}$$

Then we get that

$$\sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} u(s)\,\mathrm{d}s = 0, \qquad \int_0^1 v(s)\,\mathrm{d}s = \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} v(s)\,\mathrm{d}s.$$

Let $P(x_1,x_2) = ((t^{n-2}/(n-2)!)x_1^{(n-2)}(0), x_2(0))$ for each $(x_1,x_2) \in X$ and

$$Q(u,v) = \left(\frac{\sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} u(s)\,\mathrm{d}s}{\sum_{i=1}^{m} \alpha_i \xi_i}, \frac{\int_0^1 v(s)\,\mathrm{d}s - \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} v(s)\,\mathrm{d}s}{1 - \sum_{i=1}^{m} \beta_i \xi_i} \right), \quad (u,v) \in Y.$$

It is easy to show that

$$\operatorname{Im} P = \operatorname{Ker} L, \quad \operatorname{Ker} Q = \operatorname{Im} L, \quad X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

The isomorphism $\wedge\colon \operatorname{Ker} L \to Y/\operatorname{Im} L$ is defined by

$$\wedge\Big(\frac{t^{n-2}}{(n-2)!} a, b \Big) = (a,b), \quad (a,b) \in \mathbb{R}.$$

The generalized inverse $K_L\colon \operatorname{Im} L \to D(L) \cap \operatorname{Ker} P$ can be defined by

$$K_L(u,v) = \left(\int_0^t \frac{(t-s)^{n-2}}{(n-2)!} u(s)\,\mathrm{d}s, \int_0^t v(s)\,\mathrm{d}s \right), \quad (u,v) \in \operatorname{Im} L.$$

It follows that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$, since f is continuous.

First, prove that

$$\Omega_1 = \{(x_1,x_2) \in D(L) \setminus \operatorname{Ker} L\colon\ L(x_1,x_2) = \lambda N(x_1,x_2) \ \text{for some } \lambda \in (0,1)\}$$

is bounded. For $(x_1,x_2)\in\Omega_1$, we get (8). Since $\alpha_i\geqslant 0$, $\beta_i\geqslant 0$, $\sum\limits_{i=1}^m\alpha_i=\sum\limits_{i=1}^m\beta_i=1$, and (H_1) holds, we get from Lemma 1 that there is $\xi\in[0,1]$ so that $x_2(\xi)=0$. It follows from (8) that

$$\begin{aligned}\int_0^1 & f(t,x_1(t),\ \ldots,x_1^{(n-2)}(t),{}^{(n-2)}\varphi^{-1}(x_2(t))\,\mathrm{d}s\\ &-\sum_{i=1}^m\beta_i\int_0^{\xi_i} f(t,x_1(t),\ldots,x_1^{(n-2)}(t),{}^{(n-2)}\varphi^{-1}(x_2(t))\,\mathrm{d}s=0.\end{aligned}$$

Then (H_6) implies that there exists $\eta\in[0,1]$ such that $|x_1^{(n-2)}(\eta)|\leqslant A$. Thus,

$$|x_1^{(n-2)}(t)|=\left|x_1^{(n-2)}(\eta)+\int_\eta^t x^{(n-1)}(s)\,\mathrm{d}s\right|\leqslant A+\int_0^1|x_1^{(n-1)}(s)|\,\mathrm{d}s\leqslant A+\varphi^{-1}(\|x_2\|).$$

Then for $i=0,\ldots,n-3$, one has that

$$\begin{aligned}|x_1^{(i)}(t)|&\leqslant\left|x^{(i)}(0)+\int_0^t\frac{(t-s)^{n-3-i}}{(n-3-i)!}x^{(n-2)}(s)\,\mathrm{d}s\right|\\ &\leqslant\int_0^t\frac{(t-s)^{n-3-i}}{(n-3-i)!}\,\mathrm{d}s\|x_1^{(n-2)}\|_\infty\\ &\leqslant\frac{1}{(n-2-i)!}\|x_1^{(n-2)}\|_\infty\\ &\leqslant\frac{1}{(n-2-i)!}(A+\varphi^{-1}(\|x_2\|_\infty).\end{aligned}$$

Similarly to the proof of Theorem L1, we then get

$$\begin{aligned}|x_2(t)|\leqslant&\int_0^1 a(s)\,\mathrm{d}s+\sum_{i=0}^{n-3}\int_0^1 b_i(s)\,\mathrm{d}s\varphi\Big(\frac{1}{(n-2-i)!}\Big)\varphi(A+\varphi^{-1}(\|x_2\|_\infty))\\ &+\int_0^1 b_{n-2}(s)\,\mathrm{d}s\,\varphi(A+\varphi^{-1}(\|x_2\|_\infty))+\int_0^1 c(s)\,\mathrm{d}s\|x_2\|_\infty.\end{aligned}$$

Thus

$$\begin{aligned}\|x_2\|_\infty\leqslant&\int_0^1 a(s)\,\mathrm{d}s+\sum_{i=0}^{n-3}\int_0^1 b_i(s)\,\mathrm{d}s\,\varphi\Big(\frac{1}{(n-2-i)!}\Big)\varphi(A+\varphi^{-1}(\|x_2\|_\infty))\\ &+\int_0^1 b_{n-2}(s)\,\mathrm{d}s\,\varphi(A+\varphi^{-1}(\|x_2\|_\infty))+\int_0^1 c(s)\,\mathrm{d}s\|x_2\|_\infty.\end{aligned}$$

Since (H_7) holds, by comparing the degree of $\|x_2\|_\infty$ on both sides of the above inequality, we get that there exists a constant $M>0$ such that $\|x_2\|_\infty\leqslant M$. Similar to the proof of Theorem L1, we get that Ω_1 is bounded.

Second, we prove that $\Omega_2 = \{(x_1, x_2) \in \operatorname{Ker} L\colon\ \lambda \wedge (x_1, x_2) + (1-\lambda) QN(x_1, x_2) = 0,\ \lambda \in [0,1]\}$ is bounded.

Let $(x_1, x_2) = ((t^{n-2}/(n-2)!)a, b) \in \Omega_2$, denote

$$G(s,a,b) = f(t, (t^{n-2}/(n-2)!)a, \ldots, a, \varphi^{-1}(b)).$$

Then

$$\lambda(a,b) + (1-\lambda)\bigg(\frac{\varphi^{-1}(b)\sum_{i=1}^m \alpha_i\xi_i}{\sum_{i=1}^m \alpha_i\xi_i}, \frac{-\int_0^1 G(s,a,b)\,\mathrm{d}s + \sum_{i=1}^m \beta_i \int_0^{\xi_i} G(s,a,b)\,\mathrm{d}s}{1-\sum_{i=1}^m \beta_i\xi_i}\bigg) = 0.$$

We get

$$\begin{cases} \lambda a + (1-\lambda)\varphi^{-1}(b) = 0, \\ \lambda b + (1-\lambda)\dfrac{-\int_0^1 G(s,a,b)\,\mathrm{d}s + \sum_{i=1}^m \beta_i \int_0^{\xi_i} G(s,a,b)\,\mathrm{d}s}{1-\sum_{i=1}^m \beta_i\xi_i} = 0. \end{cases}$$

If $\lambda = 0$, then $b = 0$ and thus,

$$-\int_0^1 f\Big(s, \frac{s^{n-2}}{(n-2)!}a, \ldots, a, 0\Big)\,\mathrm{d}s + \sum_{i=1}^m \beta_i \int_0^{\xi_i} f\Big(s, \frac{s^{n-2}}{(n-2)!}a, \ldots, a, 0\Big)\,\mathrm{d}s = 0.$$

It follows from (H_8) that $|a| \leqslant B$.

If $\lambda = 1$, then $a = b = 0$.

If $\lambda \in (0,1)$, it follows that

$$\frac{a}{b} = \frac{\varphi^{-1}(b)\big(1-\sum_{i=1}^m b_i\xi_i\big)}{-\int_0^1 G(s,a,b)\,\mathrm{d}s + \sum_{i=1}^m \beta_i \int_0^{\xi_i} G(s,a,b)\,\mathrm{d}s}.$$

Then

$$b\varphi^{-1}(b)\Big(1-\sum_{i=1}^m b_i\xi_i\Big) = a\Big(-\int_0^1 G(s,a,b)\,\mathrm{d}s + \sum_{i=1}^m \beta_i \int_0^{\xi_i} G(s,a,b)\,\mathrm{d}s\Big).$$

From the definition of $H(t,a)$, one has

$$b\varphi^{-1}(b)\Big(1-\sum_{i=1}^m b_i\xi_i\Big) = a\Big(-\int_0^1 H(s,a)\,\mathrm{d}s + \sum_{i=1}^m \beta_i \int_0^{\xi_i} H(s,a)\,\mathrm{d}s\Big).$$

If $|a| > C$, then we get from (H_9) that

$$0 \leqslant b\varphi^{-1}(b)\Big(1-\sum_{i=1}^m b_i\xi_i\Big) = a\Big(-\int_0^1 H(s,a)\,\mathrm{d}s + \sum_{i=1}^m \beta_i \int_0^{\xi_i} H(s,a)\,\mathrm{d}s\Big) < 0,$$

a contradiction. Hence, $|a| \leqslant C$. Then

$$b\varphi^{-1}(b) = \frac{a\big(-\int_0^1 G(s,a,b)\,\mathrm{d}s + \sum_{i=1}^m \beta_i \int_0^{\xi_i} G(s,a,b)\,\mathrm{d}s\big)}{1 - \sum_{i=1}^m b_i\xi_i}$$

implies that there exists a constant $D > 0$ such that $|b| \leqslant D$.

From the above discussion, we get that there exists $M_1 > 0$ such that $\|(x_1,x_2)\| \leqslant M_1$. Thus, Ω_2 is bounded.

Now, we prove that $\Omega_3 = \{(x_1,x_2) \in \operatorname{Ker} L\colon\ Nx \in \operatorname{Im} L\}$ is bounded.

Let $(x_1,x_2) = ((t^{n-2}/(n-2)!)a, b) \in \Omega_3$. We have

$$\Big(\varphi^{-1}(b), -f\Big(t, \frac{t^{n-2}}{(n-2)!}a, \ldots, a, \varphi^{-1}(b)\Big)\Big) \in \operatorname{Im} L.$$

Then

$$\sum_{i=1}^m \alpha_i \int_0^{\xi_i} \varphi^{-1}(b)\,\mathrm{d}s = 0, \qquad \int_0^1 G(s,a,b)\,\mathrm{d}s = \sum_{i=1}^m \beta_i \int_0^{\xi_i} G(s,a,b)\,\mathrm{d}s.$$

It follows that $b = 0$ and

$$\int_0^1 f\Big(t, \frac{t^{n-2}}{(n-2)!}a, \ldots, a, 0\Big)\,\mathrm{d}s = \sum_{i=1}^m \beta_i \int_0^{\xi_i} f\Big(t, \frac{t^{n-2}}{(n-2)!}a, \ldots, a, 0\Big)\,\mathrm{d}s.$$

It follows from (H_8) that $|a| \leqslant B$. Thus Ω_3 is bounded.

Set Ω to be an open bounded subset of X such that $\Omega \supset \bigcup_{i=1}^{3} \overline{\Omega_i}$. We know that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By the definition of Ω, we have $\Omega \supset \overline{\Omega_1}$ and $\Omega \supset \overline{\Omega_2}$, thus $L(x_1,x_2) \neq \lambda N(x_1,x_2)$ for $x \in (D(L)/\operatorname{Ker} L) \cap \partial\Omega$ and $\lambda \in (0,1)$; $N(x_1,x_2) \notin \operatorname{Im} L$ for $(x_1,x_2) \in \operatorname{Ker} L \cap \partial\Omega$.

In fact, let $H((x_1,x_2),\lambda) = \pm\lambda(x_1,x_2) + (1-\lambda)QN(x_1,x_2)$. According to the definition of Ω, we know that $\Omega \supset \overline{\Omega_3}$, thus $H((x_1,x_2),\lambda) \neq 0$ for $(x_1,x_2) \in \partial\Omega \cap \operatorname{Ker} L$, thus by the homotopy property of the degree,

$$\begin{aligned}\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= \deg(H(\cdot,0), \Omega \cap \operatorname{Ker} L, 0)\\ &= \deg(H(\cdot,1), \Omega \cap \operatorname{Ker} L, 0) = \deg(\pm\wedge, \Omega \cap \operatorname{Ker} L, 0) \neq 0.\end{aligned}$$

Thus by Lemma M1, $L(x_1,x_2) = N(x_1,x_2)$ has at least one solution in $D(L) \cap \overline{\Omega}$, then x_1 is a solution of the BVP (7). It follows from Lemma 2 that x_1 is a positive solution of (7). The proof is completed. $\square$

3. Examples

Now, we present some examples to illustrate the main results.

Example 1. Consider the following BVP

$$
\begin{cases} x''(t) + c(t)|x'(t)| + b(t)|x(t)| + r(t) = 0, \quad t \in (0,1), \\ x(0) = x(\frac{1}{2}), \\ x(1) = \frac{1}{2}x(\frac{2}{3}), \end{cases} \tag{9}
$$

where b and c are nonnegative continuous functions on [0,1] and $r \in C[0,1]$ with $r(t) \not\equiv 0$ on each subinterval of [0,1].

Corresponding to the BVP (7), we find that $\varphi(x) = x$, $n = 2$, $f(t,x,y) = c(t)|y| + b(t)|x| + r(t)$, and $\xi_1 = \frac{1}{2}$, $\xi_2 = \frac{2}{3}$, $\alpha_1 = 1, \alpha_2 = 0$, $\beta_1 = 0$, $\beta_2 = \frac{1}{2}$.

It is easy to see that (H_1) holds and

$$
|f(t,x,y)| \leqslant c(t)|y| + b(t)|x| + |r(t)|
$$

for all $t \in [0,1]$ and $x, y \in \mathbb{R}$, which implies that (H_3) holds. Also, $\alpha_1 + \alpha_2 = 1$ and $\beta_1 + \beta_2 < 1$.

We find from Theorem L2 that if (H_5) holds, i.e. $\frac{4}{3}\int_0^1 b(s)\,\mathrm{d}s + \int_0^1 c(s)\,\mathrm{d}s < 1$, then (9) has at least one positive solution for each $r \in C[0,1]$ with $r(t) \not\equiv 0$ on each subinterval of [0,1].

Example 2. Consider the BVP

$$
\begin{cases} (\varphi_3(y''))' + a(t)\varphi_3(|y|) + b(t)\varphi_3(|y'|) + c(t)\varphi_3(|y''|) + r(t) = 0, \quad t \in (0,1), \\ u(0) = \frac{1}{2}x(\frac{1}{2}), \\ u(1) = \frac{2}{3}x(\frac{1}{4}) + \frac{1}{3}x(\frac{1}{2}), \end{cases} \tag{10}
$$

where $\varphi_3(x) = |x|x$, and a, b, c, and r are nonnegative continuous functions.

Corresponding to the BVP (7), we find that $\varphi(x) = |x|x$, $n = 3$,

$$
f(t,x,y,z) = c(t)\varphi_3(|z|) + b(t)\varphi_3(|y|) + a(t)\varphi_3(|x|) + r(t),
$$

and $\xi_1 = \frac{1}{4}$, $\xi_2 = \frac{1}{2}$, $\alpha_1 = 0$, $\alpha_2 = \frac{1}{2}$, $\beta_1 = \frac{2}{3}$, $\beta_2 = \frac{1}{3}$.

It is easy to see that (H_1) and (H_3) hold and $\alpha_1 + \alpha_2 < 1$ and $\beta_1 + \beta_2 = 1$.

Then by an application of Theorem L1, (10) has at least one positive solution if

$$
\varphi_3\Big(\frac{3}{2}\Big)\Big(\int_0^1 a(s)\,\mathrm{d}s + \frac{3}{2}\int_0^1 b(s)\,\mathrm{d}s\Big) + \int_0^1 c(s)\,\mathrm{d}s < 1
$$

for each $r \in C[0,1]$ with $r(t) \not\equiv 0$ on each subinterval of [0,1].

R e m a r k 1. The BVP (9) and the BVP (10) cannot be solved by the results obtained in [14], since f depends on x'. It is easy to find that (9) and (10) cannot be solved by the theorems in [8], [11], [23], [24], since $\sum\limits_{i=1}^{2}\alpha_i = 1$ in (9) and $\sum\limits_{i=1}^{2}\beta_i = 1$ in (10).

E x a m p l e 3. Consider the following BVP

$$\begin{cases} x''(t) + b(t)|\sin x'(t)| + c(t)x(t) + r(t) = 0, \quad t \in (0,1),\\ x(0) = x(\frac{1}{2}),\\ x(1) = x(\frac{1}{2}),\end{cases}$$

where r, b, and c are nonnegative continuous functions with $\int_{1/2}^{1} c(s)\,\mathrm{d}s > 0$ and $r(t) \not\equiv 0$ on each sub-interval of [0,1].

Corresponding to the BVP (7), we find that $\varphi(x) = x$, $n = 2$ and $f(t,x,y) = b(t)|\sin y| + c(t)x + r(t)$, $\xi_1 = \frac{1}{2}$, $\alpha_1 = 1$, $\beta_1 = 1$.

One sees that $f\colon\ [0,1]\times[0,\infty)\times\mathbb{R}\to[0,\infty)$ is continuous with $f(t,0,\ldots,0)\not\equiv 0$ on each sub-interval of [0,1]. Thus (H_1) holds. It is easy to see that (H_3) holds with $\alpha_1 = \beta_1 = 1$.

It is easy to see that there exists a constant $A > 0$ such that for each $(x_1,x_2)\in D(L)$, if $|x_1(t)| > A$ for all $t\in[0,1]$, then

$$\begin{aligned}\int_0^1 & f(s,x_1(s),x_2(s))\,\mathrm{d}s - \int_0^{1/2} f(s,x_1(s),x_2(s))\,\mathrm{d}s\\ &= \int_{1/2}^1 f(s,x_1(s),x_2(s))\,\mathrm{d}s\\ &= \int_{1/2}^1 (b(s)|\sin x_2(s)| + c(s)x_1(s) + r(s))\,\mathrm{d}s\\ &\begin{cases} > \int_{1/2}^1 (b(s)|\sin x_2(s)| + r(s))\,\mathrm{d}s + A\int_{1/2}^1 c(s)\,\mathrm{d}s > 0 & \text{if } x(s) > A\\ < \int_{1/2}^1 (b(s)|\sin x_2(s)| + r(s))\,\mathrm{d}s - A\int_{1/2}^1 c(s)\,\mathrm{d}s < 0 & \text{if } x(s) < -A\end{cases}\end{aligned}$$

Hence, (H_6) holds. One sees that

$$-\int_0^1 f(s,a,0)\,\mathrm{d}s + \int_0^{1/2} f(s,a,0)\,\mathrm{d}s = -\int_{1/2}^1 (c(s)a + r(s))\,\mathrm{d}s \neq 0$$

implying that there exists a constant $B > 0$ such that

$$-\int_0^1 f(s,a,0)\,\mathrm{d}s + \int_0^{1/2} f(s,a,0)\,\mathrm{d}s \neq 0 \quad \text{for all } |a| > B.$$

Thus (H_8) holds. For $\lambda \in (0,1)$, $a \in \mathbb{R}$, one sees that

$$H(t,a) = f\Big(t, a, -\frac{\lambda}{1-\lambda}a\Big).$$

It is easy to find that

$$\begin{aligned} a\Big(-\int_0^1 H(s,a)\,\mathrm{d}s + \sum_{i=1}^m \beta_i \int_0^{\xi_i} H(s,a)\,\mathrm{d}s\Big) \\ &= -2a\int_{1/2}^1 \Big(b(s)\Big|\sin\Big(-\frac{\lambda}{1-\lambda}a\Big)\Big| + c(s)a + r(s)\Big)\,\mathrm{d}s \\ &= -2a^2\int_{1/2}^1 c(s)\,\mathrm{d}s - 2a\int_{1/2}^1 \Big(b(s)\Big|\sin\frac{\lambda}{1-\lambda}a\Big| + r(s)\Big)\,\mathrm{d}s. \end{aligned}$$

Hence there exists a constant $C > 0$ such that

$$a\Big(-\int_0^1 H(s,a)\,\mathrm{d}s + \sum_{i=1}^m \beta_i \int_0^{\xi_i} H(s,a)\,\mathrm{d}s\Big) < 0 \quad \text{for all } |a| > C.$$

Thus (H_9) holds. By an application of Theorem L3, the BVP (11) has at least one positive solution if $\int_0^1 b(s)\,\mathrm{d}s + \int_0^1 c(s)\,\mathrm{d}s < 1$, i.e. ($\mathrm{H}_7$) holds, for each $r \in C[0,1]$ with $r(t) \not\equiv 0$ on each subinterval of [0,1].

Remark 2. The BVP (11) cannot be covered by the results obtained in [14], [15] since f depends on x'. It is easy to find that (11) cannot be solved by the theorems in [11], [15], [23], [24], since $\sum_{i=1}^2 \alpha_i = \sum_{i=1}^2 \beta_i = 1$ in (11).

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References

[1] *R. P. Agarwal, D. O'Regan, P. J. Y. Wong*: Positive Solutions of Differential, Difference and Integral Equations. Kluwer Academic, Dordrecht, 1999.

[2] *C. Bai, J. Fang*: Existence of multiple positive solutions for nonlinear m-point boundary value problems. Appl. Math. Comput. *140* (2003), 297–305.

[3] *C. Bai, J. Fang*: Existence of multiple positive solutions for nonlinear multi-point boundary value problems. J. Math. Anal. Appl. *281* (2003), 76–85.

[4] *M. A. delPino, M. Elgueta, R. F. Manásevich*: A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0) = u(T) = 0$, $p > 1$. J. Differ. Equ. *80* (1989), 1–13.

[5] *M. A. delPino, R. F. Manásevich*: Multiple solutions for the p-Laplacian under global nonresonance. Proc. Am. Math. Soc. *112* (1991), 131–138.

[6] *R. E. Gaines, J. L. Mawhin*: Coincidence Degree and Nonlinear Differential Equations. Lecture Notes in Math., Vol. 568. Springer, Berlin, 1977.
[7] *Y. Guo, W. Ge*: Three positive solutions for the one dimension p-Laplacian. J. Math. Anal. Appl. *286* (2003), 491–508.
[8] *D. Ji, M. Feng, W. Ge*: Multiple positive solutions for multipoint boundary value problems with sign changing nonlinearity. Appl. Math. Comput. *196* (2008), 511–520.
[9] *G. L. Karakostas*: Triple positive solutions for the Φ-Laplacian when Φ is a sup-multiplicative-like function. Electron. Diff. Equ. *69* (2004), 1–13.
[10] *G. L. Karakostas*: Positive solutions for the Φ-Laplacian when Φ is a sup-multiplicative-like function. Electron. Diff. Equ. *68* (2004), 1–12.
[11] *N. Kosmatov*: Symmetric solutions of a multi-point boundary value problem. J. Math. Anal. Appl. *309* (2005), 25–36.
[12] *K. D. Lan*: Multiple positive solutions of semilinear differential equations with singularities. J. Lond. Math. Soc., Ser. II *63* (2001), 690–704.
[13] *W.-C. Lian, F. Wong*: Existence of positive solutions for higher-order generalized p-Laplacian BVPs. Appl. Math. Lett. *13* (2000), 35–43.
[14] *R. Liang, J. Peng, J. Shen*: Positive solutions to a generalized second order three-point boundary value problem. Appl. Math. Comput. *196* (2008), 931–940.
[15] *B. Liu*: Positive solutions of a nonlinear four-point boundary value problems. Appl. Math. Comput. *155* (2004), 179–203.
[16] *Y. Liu, W. Ge*: Multiple positive solutions to a three-point boundary value problem with p-Laplacian. J. Math. Anal. Appl. *277* (2003), 293–302.
[17] *H. Lü, D. O'Regan, C. Zhong*: Multiple positive solutions for the one dimensional singular p-Laplacian. Appl. Math. Comput. *133* (2002), 407–422.
[18] *R. Ma*: Existence of positive solutions for superlinear semipositone m-point boundary-value problems. Proc. Edinb. Math. Soc., II. Ser. *46* (2003), 279–292.
[19] *R. Ma*: Multiple nonnegative solutions of second-order systems of boundary value problems. Nonlinear Anal., Theory Methods Appl. *42* (2000), 1003–1010.
[20] *R. Ma*: Nodal solutions for a second-order m-point boundary value problem. Czech. Math. J. *56* (2006), 1243–1263.
[21] *R. Ma, N. Castaneda*: Existence of solutions of nonlinear m-point boundary value problems. J. Math. Anal. Appl. *256* (2001), 556–567.
[22] *R. Ma, B. Thompson*: Positive solutions for nonlinear m-point eigenvalue problems. J. Math. Anal. Appl. *297* (2004), 24–37.
[23] *Y. Wang, W. Zhao, W. Ge*: Multiple positive solutions for boundary value problems of second order delay differential equations with one-dimensional p-Laplacian. J. Math. Anal. Appl. *326* (2007), 641–654.
[24] *J. R. L. Webb*: Positive solutions of some three point boundary value problems via fixed point index theory. Nonlinear Anal., Theory Methods Appl. *47* (2001), 4319–4332.
[25] *G. Zhang, J. Sun*: Positive solutions of m-point boundary value problems. J. Math. Anal. Appl. *291* (2004), 406–418.
[26] *Z. Zhang, J. Wang*: On existence and multiplicity of positive solutions to singular multi-point boundary value problems. J. Math. Anal. Appl. *295* (2004), 502–512.

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