A NOTE ON THE FEKETE–SZEGÖ PROBLEM FOR CLOSE-TO-CONVEX FUNCTIONS WITH RESPECT TO CONVEX FUNCTIONS

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Abstract. We discuss the sharpness of the bound of the Fekete–Szegö functional for close-to-convex functions with respect to convex functions. We also briefly consider other related developments involving the Fekete–Szegö functional \(|a_3 - \lambda a_2^2|\) for \(0 \leq \lambda \leq 1\) as well as the corresponding Hankel determinant for the Taylor–Maclaurin coefficients \(\{a_n\}_{n \in \mathbb{N}-\{1\}}\) of normalized univalent functions in the open unit disk \(\mathbb{D}\), \(\mathbb{N}\) being the set of positive integers.

1. Introduction

A classical problem in geometric function theory of complex analysis, which was settled by Fekete and Szegö [4], is to find for each \(\lambda \in [0,1]\) the maximum value of the coefficient functional \(\Phi_{\lambda}(f)\) given by

\[
\Phi_{\lambda}(f) := |a_3 - \lambda a_2^2|
\]

over the class \(\mathcal{S}\) of univalent functions \(f\) in the open unit disk

\[
\mathbb{D} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}
\]

of the following normalized form (see, for details, [5][22][24]):

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}).
\]

By applying the Loewner method, Fekete and Szegö [4] proved that

\[
\max_{f \in \mathcal{S}} \Phi_{\lambda}(f) = \begin{cases} 
1 + 2 \exp \left(-\frac{2\lambda}{3}\right) & (0 \leq \lambda < 1) \\
1 & (\lambda = 1).
\end{cases}
\]

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For various compact subclasses $F$ of the class $A$ of all analytic functions $f$ in $D$ of the form (1.2), as well as with $\lambda$ being an arbitrary real or complex number, many authors computed
\begin{equation}
\max_{f \in F} \Phi_\lambda(f)
\end{equation}
or calculated the upper bound of (1.3) (see, e.g., [2,8,11,21]).

Let $S^*$ denote the class of starlike functions, that is, $f \in A$ and $\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0$ ($z \in D$).

Given $\delta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $g \in S^*$, let $C_\delta(g)$ denote the class of functions called close-to-convex with argument $\delta$ with respect to $g$, that is, the class of all functions $f \in A$ such that
\begin{equation}
\text{Re}\left(\frac{e^{i\delta}zf'(z)}{g(z)}\right) > 0 \quad (z \in D).
\end{equation}

We also suppose that, given $g \in S^*$, $C(g) := \bigcup_{g \in S^*} C_\delta(g)$ and that, given $\delta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $C_\delta := \bigcup_{g \in S^*} C_\delta(g)$. Let
\[ C := \bigcup_{\delta \in (-\frac{\pi}{2}, \frac{\pi}{2})} \bigcup_{g \in S^*} C_\delta(g) \]
denote the class of close-to-convex functions (see, for details, [20, pp. 184–185], [6,10]).

For the whole class $C$, the sharp bound of the Fekete–Szegö coefficient functional $\Phi_\lambda$ for $\lambda \in [0,1]$, given by (1.1), was calculated by Koepf [13] who extended the earlier result for the class $C_0$ and for $\lambda \in \mathbb{R}$ due to Keogh and Merkes [11], namely, it holds
\[ \max_{f \in C} \Phi_\lambda(f) = \max_{f \in C_0} \Phi_\lambda(f) = \begin{cases} |3 - 4\lambda| & (\lambda \in (-\infty, \frac{1}{4}] \cup [1, \infty)) \\ \frac{1}{4} + \frac{1}{\lambda} & (\lambda \in \left[\frac{1}{4}, \frac{1}{2}\right]) \\ 1 & (\lambda \in \left[\frac{1}{2}, 1\right]). \end{cases} \]

For various subclasses of the class of close-to-convex functions, the problem to estimate the coefficient functional $\Phi_\lambda$ is continued in several subsequent works (see, for details, [9,12,14,16]). Some interesting and important subclasses of the class $C$ are the classes $C_\delta^c$ and $C^c$, which are defined below.

Let $S^c$ denote the class of convex functions, that is, $f \in S^c$ if
\[ f \in A \quad \text{and} \quad \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in D). \]

Since $S^c \subsetneq S^*$, the class $C_\delta^c := \bigcup_{g \in S^c} C_\delta(g)$ is a proper subclass of the class $C_\delta$ and the class
\[ C^c := \bigcup_{\delta \in (-\frac{\pi}{2}, \frac{\pi}{2})} \bigcup_{g \in S^c} C_\delta(g) \]
is a proper subclass of the class $C$. 

The class $C_{c0}$ was defined by Abdel-Gawad and Thomas \cite{1}. The class $C_{c}$ of close-to-convex functions with respect to convex functions was introduced by Srivastava, Mishra and Das \cite{23}. In both of these cited papers, the authors (Abdel-Gawad and Thomas \cite{1} and Srivastava, Mishra and Das \cite{23}) considered the coefficient functional $\Phi_\lambda$ with $\lambda \in [0,1]$ also. In fact, in Srivastava, Mishra and Das \cite{23} extended, for the class $C_{c}$, the earlier result of Abdel-Gawad and Thomas \cite{1} for the class $C_{c0}$. However, in each of the above-cited papers, the proof for the sharpness of the bound in (1.3) for $\lambda \in \left(\frac{2}{3}, 1\right]$ was proposed incorrectly as $5/6$.

This note is motivated essentially by the earlier papers \cite{1} and \cite{23}. The main purpose of our investigation here is to discuss such sharpness results for the bound in (1.3). We also provide a rather brief consideration of other related developments involving the Fekete–Szegö functional $|a_3 - \lambda a_2^2| (0 \leq \lambda \leq 1)$ in (1.1) as well as the corresponding Hankel determinant for the Taylor–Maclaurin coefficients $\{a_n\}_{n \in \mathbb{N} \setminus \{1\}}$ of normalized univalent functions of the form (1.2).

2. Main Observation

As we remarked in Section 1, in both of the afore cited papers \cite{1,23}, the upper bounds of the Fekete–Szegö coefficient functional $\Phi_\lambda$ $(0 \leq \lambda \leq 1)$ for the classes $C_{c0}$ and $C_{c}$, were computed. In fact, Theorems 5 and 6 of Srivastava, Mishra and Das \cite{23} state that the following sharp inequality

\begin{equation}
\max_{f \in C_{c}} \Phi_\lambda (f) \leq \frac{5}{6} \quad (\lambda \in \left[\frac{2}{3}, 1\right])
\end{equation}

holds true and that this result is the same as in \cite{1} for the class $C_{c0}$ (a part of Theorem 3). However, the assertion that the extremal function, for which the equality in (2.1) is satisfied when $\lambda \in \left(\frac{2}{3}, 1\right]$, belongs to $C_{c}$ is incorrect. Indeed, here in this section, we note that the above-cited papers \cite{1,23} contain a statement to the effect that the equality in (2.1) is attained by a function $f \in A$ given by

\begin{equation}
\frac{zf'(z)}{h(z)} = \frac{1 + \omega(z)}{1 - \omega(z)} \quad (z \in \mathbb{D}),
\end{equation}

where $h \in S_{c}$ is of the form

\begin{equation}
h(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in \mathbb{D}; \ b_2 = b_3 := 1)
\end{equation}

and $\omega$ is a function of the form

\begin{equation}
\omega(z) = \sum_{n=1}^{\infty} \beta_n z^n \quad (z \in \mathbb{D})
\end{equation}

with

\begin{equation}
\beta_1 := \frac{2 - 3\lambda}{6\lambda} \pm \frac{\sqrt{6\lambda - 4}}{6\lambda} \quad \text{and} \quad \beta_2 := 1 - \beta_1^2.
\end{equation}

Unfortunately, however, $\omega$ is not a Schwarz function for $\lambda \in \left(\frac{2}{3}, 1\right]$. We recall here that a Schwarz function means an analytic self-mapping of $\mathbb{D}$ with $\omega(0) := 0$. Let us
denote the class of Schwarz functions by $\mathcal{B}_0$. In order to see that $\omega \notin \mathcal{B}_0$, we verify (by straightforward computation) that, for $\lambda \in (\frac{2}{3}, 1]$, the following inequality:

$$|\beta_2| \leq 1 - |\beta_1|^2$$

is false, so a necessary condition for $\omega$ to be in $\mathcal{B}_0$ (see, for example, [5] Vol. II, p. 78) does not hold true. Alternatively, in order to get a contradiction, we suppose that $\omega$ with its coefficients in (2.5) is a Schwarz function. Thus, clearly, (2.4) holds true. Hence we find from (2.5) that $1 - |\beta_1|^2 \geq |\beta_2| = |1 - |\beta_1|^2|$. Thus we have $|1 - \beta_1^2| = 1 - |\beta_1|^2$ and, therefore, $\beta_1 = |\beta_1|$ or $\beta_1 = -|\beta_1|$. This means that $\beta_1$ is a real number, which by (2.5) is possible only for $\lambda = \frac{2}{3}$. Consequently, for $\lambda \in (\frac{2}{3}, 1]$, the function $\omega$ with its coefficients in (2.5) does not belong to $\mathcal{B}_0$. So, in light of (2.2), it does not follow that $f$ is in $C^\infty$ or in $C_0^\infty$.

Equivalently, let

$$p(z) := \frac{1 + \omega(z)}{1 - \omega(z)} \quad (z \in \mathbb{D}),$$

where $\omega$ is as given above. Then

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{D}),$$

where, in view of (2.7), (2.1) and (2.5), we have $c_1 = 2\beta_1$ and $c_2 = 2(\beta_2 + \beta_1^2) = 2$. We observe further that, for $\lambda \in (\frac{2}{3}, 1]$, the function $p$ does not belong to the Carathéodory class. We recall here that the Carathéodory class, denoted as $\mathcal{P}$, contains analytic functions $p$ of the form (2.4) with a positive real part. In order to see that $p \notin \mathcal{P}$, we verify for $\lambda \in (\frac{2}{3}, 1]$ that the inequality $|c_2 - c_1^2/2| \leq 2 - |c_1|^2/2$, is false, which happens to be a necessary condition for $p$ to be in the class $\mathcal{P}$ (see, for example, [22] p. 166).

3. Concluding remarks and further developments

By means of Theorem 3 of Abdel-Gawad and Thomas [1], Theorems 1 to 4 of Srivastava, Mishra and Das [23], and in light of our observation in Section 2, we arrive at the following result.

**Theorem 1.** Each of the following assertions holds true:

$$\max_{f \in C^\infty} \Phi_\lambda(f) = \max_{f \in C_0^\infty} \Phi_\lambda(f) = \begin{cases} \frac{5}{6} - \frac{2}{3} & (\lambda \in [0, \frac{2}{3}]) \\ \frac{4}{3} + \frac{2}{3\lambda^2} & (\lambda \in [\frac{2}{3}, 1]) \end{cases}$$

$$\max_{f \in C^\infty} \Phi_\lambda(f) \leq \frac{5}{6} \quad (\lambda \in (\frac{2}{3}, 1]).$$

**Remark 1.** The sharpness of the inequality in (3.2) for the classes $C^\infty$ and $C_0^\infty$ is an open problem.

We now note that, by Loewner Theorem (see, for example, [5] Vol. I, p. 1127)), the function $h \in \mathcal{S}_0$ of the form (2.3) (with $b_2 = b_3 := 1$) is uniquely determined, that is, $h(z) = \frac{1}{1 - z} = \sum_{n=1}^{\infty} z^n$ ($z \in \mathbb{D}$). Then (1.4) with $g := h$ is of the form

$$\text{Re}(e^{i\delta}(1 - z)f'(z)) > 0 \quad (z \in \mathbb{D})$$
and defines the class $C_\ell(h)$, and further the class $C(h)$. For the first time, the inequality in (3.3), treated as the univalence criterion, was distinguished explicitly in [20] p. 185]. For the class $C(h)$, the upper bound of the Fekete–Szegö coefficient functional $\Phi_\lambda$ for $\lambda \in \mathbb{R}$ was recently obtained in [14], where the following result was proven.

**Theorem 2.** It is asserted that

$$\max_{f \in C(h)} \Phi_\lambda(f) \leq \left\{ \begin{array}{ll}
\frac{1}{2} - \frac{3}{2} \lambda & (\lambda \in (-\infty, \frac{2}{9}] \cup [\frac{1}{4}, \infty)) \\
\frac{1}{2} - \frac{3}{2 \lambda} + \frac{1}{2} & (\lambda \in [\frac{2}{9}, \frac{10}{9}]).
\end{array} \right.$$  

For each $\lambda \in (-\infty, \frac{2}{9}] \cup [\frac{1}{4}, \infty)$, the inequality is sharp and the equality in (3.4) is attained by a function in $C_0(h)$.

**Remark 2.** For $\lambda \in (-\infty, \frac{2}{9}] \cup [\frac{1}{4}, \infty)$, we can rewrite (3.4) as the following corollary.

**Corollary 1.** The following assertion holds true:

$$\max_{f \in C(h)} \Phi_\lambda(f) = \left\{ \begin{array}{ll}
\frac{1}{2} - \frac{3}{2} \lambda & (\lambda \in (-\infty, \frac{2}{9}] \cup [\frac{1}{4}, \infty)) \\
\frac{1}{2} + \frac{1}{\lambda} & (\lambda \in [\frac{2}{9}, \frac{10}{9}]).
\end{array} \right.$$  

**Remark 3.** For $\lambda \in [0, \frac{2}{9}]$, the result (3.5) asserted by Corollary 1 coincides with (3.1). Thus, naturally, Theorem 1 and Theorem 2 yield Corollary 2 below.

**Corollary 2.** Each of the following assertions holds true:

$$\max_{f \in C(h)} \Phi_\lambda(f) = \max_{f \in C_0} \Phi_\lambda(f) = \max_{f \in C_0} \Phi_\lambda(f) = \left( \lambda \in [0, \frac{2}{9}] \right),$$

$$\max_{f \in C(h)} \Phi_\lambda(f) \leq \frac{9\lambda^2 - 30\lambda + 26}{6(4 - 3\lambda)} \leq \frac{5}{6} \quad (\lambda \in (\frac{2}{9}, 1]).$$

**Remark 4.** The maximum of $\Phi_\lambda$ for $\lambda \in [0, \frac{2}{9}]$, over the class $C_0$ of close-to-convex functions with respect to convex functions and over its subclass $C(h)$ of close-to-convex functions with respect to convex function $h$, are identical.

**Remark 5.** The sharpness of the inequality in (3.4) for $\lambda \in (\frac{2}{9}, \frac{10}{9})$ is an open problem.

**Remark 6.** We reiterate the fact that the Fekete–Szegö coefficient functional $|a_3 - \lambda a_2^2|$ is well known for its rich history in geometric function theory. Its origin was in the disproof by Fekete and Szegö [4] of the 1933 conjecture of Littlewood and Paley that the coefficients of odd univalent functions are bounded by unity (see, for details, [4]). The $\lambda$-generalized Fekete–Szegö coefficient functional $|a_3 - \lambda a_2^2|$ has since received great attention, particularly in connection with many subclasses of the class $S$ of normalized analytic and univalent functions. On the other hand, in the year 1976, Noonan and Thomas [17] defined the $q$th Hankel determinant of
the function $f$ in (1.2) by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (n, q \in \mathbb{N}; \; a_1 := 1).$$

The determinant $H_q(n)$ has also been considered by several other authors. For example, Noor [18] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions $f$ given by (1.1) with bounded boundary. In particular, sharp upper bounds on $H_2(2)$ were obtained in the recent works [7,18] for different classes of functions.

We note, in particular, that

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2 \quad \text{and} \quad H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

The Hankel determinant $H_2(1) = a_3 - a_2^2$ is the classical Fekete–Szegö coefficient functional. The upper bounds of $H_2(2)$ for some specific analytic function classes were discussed quite recently by Deniz et al. [3] (see also [19]).

References

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