ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 6, Number 1, Winter 1976

A NOTE ON THE FRINK METRIZATION THEOREM

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1. **Introduction.** In this note we show how the Frink Metrization Theorem [2, Theorem 3] may be used to give an extremely easy proof of the Nagata "double sequence" metrization theorem [3, Theorem VI.2]. Nagata used his "double sequence" theorem to give a very elegant proof of the Nagata-Smirnov Metrization Theorem [3, Theorem VI.3], as well as several other well-known metrization theorems. In fact, Nagata's proof proves more than the classical statement of the Nagata-Smirnov Theorem itself, (see Theorem 3 below). Using Theorem 3, we shall give a simple proof of a recent metrization theorem of D. Burke and D. Lutzer.

2. **Theorems.** For an elegant proof of the Frink Metrization Theorem, we refer the reader to Mrs. Frink's original paper [2, Theorem 3].

THEOREM 1. (A. H. Frink). *A T_1-space X is metrizable if and only if for every $x \in X$, there exists a neighborhood basis $\{V_n(x) : n = 1, 2, \cdots\}$ such that if $V_n(x)$ is given, then there exists $m = m(x, n)$ such that $V_m(y) \cap V_m(x) \neq \emptyset$ implies that $V_m(y) \subset V_n(x)$.*

The celebrated Nagata "double sequence" metrization theorem [3, Theorem VI.2] is an easy consequence of Theorem 1.

THEOREM 2. (J. Nagata). *A T_1-space X is metrizable if and only if for each $x \in X$, there exist two sequences of neighborhoods of x, $\{U_n(x) : n = 1, 2, \cdots\}$ and $\{H_n(x) : n = 1, 2, \cdots\}$ such that the following three conditions hold:*

(i) *$\{U_n(x) : n = 1, 2, \cdots\}$ is a neighborhood base at x.*
(ii) *$y \notin U_n(x)$ implies that $H_n(y) \cap H_n(x) = \emptyset$.*
(iii) *$y \in H_n(x)$ implies that $H_n(y) \subset U_n(x)$.*

PROOF. The "only if" part of the theorem is clear. Therefore, assume that conditions (i), (ii) and (iii) hold. Without loss of generality, we may assume that $U_{n+1}(x) \subset U_n(x)$ for all n and x. Define $V_n(x) = H_1(x) \cap \cdots \cap H_n(x)$ for all $x \in X$ and all natural numbers n. The sequences $\{U_n(x)\}$ and $\{V_n(x)\}$ still satisfy conditions (i), (ii) and (iii).

Received by the editors on June 16, 1974.
AMS (MOS) subject classifications (1970): Primary 54E35.

Given $V_n(x)$, by (i) there exists $m > n$ with $U_m(x) \subset V_n(x)$; similarly, there exists $k > m$ with $U_k(x) \subset V_m(x)$. Suppose that $V_k(y) \cap V_k(x) \neq \emptyset$. By (ii), $y \in U_k(x)$; since $U_k(x) \subset V_m(x)$ we have $y \in V_m(x)$. By (iii), $V_m(y) \subset \bigcup_m(x)$, which implies that $V_k(y) \subset V_n(x)$. The metrizability of X now follows by Theorem 1, completing the proof.

Using Theorem 2, Nagata gave an elegant proof of the Nagata-Smirnov Metrization Theorem [3, Theorem VI.3]. Actually, however, Nagata's proof established the stronger result below; for completeness we give a sketch of Nagata's proof, the details of which may be found in [3, page 194].

THEOREM 3. (J. Nagata). *A necessary and sufficient condition that a regular space X be metrizable is that the following two conditions be satisfied*:

(1) *The space X has an open basis which may be represented as a sequence $G_1, G_2, \cdots$ of closure preserving collections*;

(2) $\{V_n(x) : n = 1, 2, \cdots\}$ *is a neighborhood basis for each $x \in X$ where $V_n(x) = X$ if $x \notin G$ for every $G \in G_n$ and otherwise* $V_n(x) = \bigcap\{G : x \in G \in G_n\}$.

PROOF. The "necessity" of the condition is clear from Stone's theorem that every metric space is paracompact [4]. To prove the converse, let $W_n(x) = X - \bigcup\{c\ell(V) : x \notin c\ell(V)$ and $V \in G_n\}$. If $x \in U \subset c\ell(U) \subset V_n(x)$ for some $U \in G_m$, then define $U_{nm}(x) = V_n(x)$ and $H_{nm}(x) = U \cap W_m(x)$; otherwise, define $U_{nm}(x) = X$ and $H_{nm}(x) = V_n(x) \cap W_m(x)$. One now verifies in a straightforward way that the sequences $\{U_{nm}(x) : n = 1, 2, \cdots; m = 1, 2, \cdots\}$ and $\{H_{nm}(x) : n = 1, 2, \cdots; m = 1, 2, \cdots\}$ satisfy the conditions (i), (ii), and (iii) of Theorem 2, completing the proof that X is metrizable.

A collection $\{G_a : a \in A\}$ is said to be *hereditarily closure preserving* provided that if $H_a \subset G_a$ for every $a \in A$, then the collection $\{H_a : a \in A\}$ is closure preserving. In [1], D. Burke and D. Lutzer generalized the Nagata-Smirnov Theorem by showing that a regular space is metrizable if and only if it has an hereditarily closure preserving open basis. The Burke-Lutzer proof in non-trivial; however, using the essential idea of Lemma 4 of [1], the Burke-Lutzer Theorem is an easy consequence of Theorem 3.

THEOREM 4. (D. Burke and D. Lutzer). *A regular space X is metrizable if and only if X has a σ-hereditarily closure preserving open basis.*

PROOF. The "only if" part is an easy consequence of Stone's theorem that every metric space is paracompact [4]. To prove the converse, let X have a σ-hereditarily closure preserving open basis $B = \bigcup\{G_n : n = 1, 2, \cdots\}$ where each G_n is an hereditarily closure preserving collection. For $x \in X$, let $V_n(x) = X$ if $x \notin G$ for all $G \in G_n$ and let $V_n(x) = \bigcap\{G : x \in G \in G_n\}$ otherwise. If $\{x\}$ is open, then clearly $\{V_n(x) : n = 1, 2, \cdots\}$ is a neighborhood basis at x. Therefore, suppose $\{x\}$ is not open. Since X has a σ-closure preserving open basis, the singleton set $\{x\}$ is a G_δ. Suppose that G_n is not point-finite at x. Then there exist infinitely many members of G_n which contain x, say $H_1, H_2, \cdots$. Choose a strictly decreasing sequence $P_1, P_2, \cdots$ of open supersets of x with $P_n \subset H_n$ for all n such that $\{x\} = \bigcap P_n$. Let $E_n = P_n - P_{n+1}$ for $n = 1, 2, \cdots$. Since $E_n \subset H_n$ for all n, the family $\{E_n : n = 1, 2, \cdots\}$ is closure preserving. But this is a contradiction since $x \notin c\ell(E_n)$ for all n and $x \in c\ell(\bigcup \{E_n : n = 1, 2, \cdots,\})$. It follows that G_n is point-finite at x, that is, $V_n(x)$ is open. The metrizability of X now follows by Theorem 3, completing the proof.

REFERENCES

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