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A Note on the Geometry of RW Space-Times

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Abstract: A conformally flat GRW space-time is a perfect fluid RW space-time. In this note, we investigated the influence of many differential curvature conditions, such as the existence of recurrent and semi-symmetric curvature tensors. In each case, the form of the Ricci curvature tensor, the energy–momentum tensor, the energy density, the pressure of the fluid, and the equation of state are determined and interpreted. For example, it is demonstrated that a Ricci semi-symmetric RW space-time reduces to Einstein space-time or a Ricci recurrent RW space-time, and the perfect fluid space-time is referred to as Yang pure space-time or dark matter era.

Keywords: RW space-times; Ricci semi-symmetric; Ricci recurrent; projective curvature tensor

MSC: 53C25; 83F05

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1. Introduction

One of the most significant areas of research in both mathematics and physics is the geometry of generalized Robertson–Walker (or GRW) space-times. A warped product manifold with a one-dimensional base manifold serves as the representation of a GRW space-time. The term Friedmann–Lemaitre–Robertson–Walker metrics, which accurately captures the contributions of different scientists to this issue, is currently used in physics for Robertson–Walker-type metrics. There are many exciting decomposition theorems on Lorentzian manifolds. The author of [1] described a particularly remarkable decomposition of a Lorentzian manifold to a GRW space-time. The existence of a time-like concircular vector field is sufficient for a Lorentzian manifold to be a GRW space-time. This condition becomes weaker as follows in the presence of another condition [2]. If a unit time-like torse-forming vector field ω^i that is an eigenvector of the Ricci tensor S_{ij} exists on a Lorentzian manifold M, then M is a GRW space-time. By a unit time-like torse-forming, we mean that there is a scalar function φ on M such that

$$\nabla_k \omega_j = \varphi\left(\omega_k \omega_j + g_{kj}\right), \tag{1}$$

$$\omega^i \omega_i = -1. \tag{2}$$

The factor φ coincides with the Hubble's parameter H on a GRW space-time M. How rapidly the universe is expanding is determined by Hubble's parameter H (for a description of H and further information, see [3]). This torse-forming vector field is also an eigenvector of the Ricci tensor S_{ij}, that is, $\omega^i S_{ij} = \psi \omega_j$ where ψ is the corresponding eigenvalue of ω_j [1,2,4]. In [5], a GRW space-time Ricci tensor has been established to be

$$S_{ij} = \frac{S-\psi}{n-1} g_{ij} + \frac{S-n\psi}{n-1} \omega_i \omega_j + (n-2)\omega^k \omega^l C_{kijl} \tag{3}$$

where C_{kijl} is the Weyl conformal curvature tensor and S is the scalar curvature. The classical Robertson–Walker (or RW) space-time is a conformally flat GRW space-time, which allows the Ricci curvature form to change as

$$S_{ij} = \frac{S-\psi}{n-1} g_{ij} + \frac{S-n\psi}{n-1} \omega_i \omega_j. \tag{4}$$

On the other hand, the Ricci tensor of a perfect fluid space-time has the form

$$S_{ij} = \alpha g_{ij} + \beta \tau_i \tau_j. \tag{5}$$

Accordingto this equation, an RW space-time is a perfect fluid space-time where

$$\alpha = \frac{S-\psi}{n-1}, \beta = \frac{S-n\psi}{n-1}, \tau_i = \omega_i. \tag{6}$$

For further information about perfect fluid space-times and characterization of GRW space-times and RW space-times, the reader is recommended to read [2,4–7]. An algebraic curvature condition is that a space-time is a perfect fluid space-time [8]. Manifolds having this algebraic curvature criterion are known as quasi-Einstein manifolds in differential geometry [9]. However, there are additional types of differential curvature conditions that can be used, such as the existence of recurrent and semi-symmetric curvature tensors. Many alternative differential curvature conditions are examined in this article by using Riemann and Ricci curvature tensors. In each case, the form of the Ricci tensor, energy–momentum tensor, pressure, energy density and equation of state of the perfect fluid is given.

2. Notes on RW Space-Times

It is easy to obtain the scalar curvature of RW space-time, the eigenvalue of the Ricci tensor corresponding to ω and the divergence of the one form ω as

$$S = n\alpha - \beta,\ \psi = \alpha - \beta \tag{7}$$
$$\nabla^j \omega_j = (n-1)\varphi \tag{8}$$

It should be observed that the form (5) on an RW space-time has a perfect fluid structure that is unique up to a sign. For this, we assume that there exists a vector field v that is time-like and

$$S_{ij} = \bar{\alpha} g_{ij} + \bar{\beta} v_i v_j.$$

Then,

$$\begin{aligned} \omega^i S_{ij} &= \bar{\alpha}\omega_j + \bar{\beta}\left(\omega^i v_i\right) v_j \\ (\psi - \bar{\alpha})\omega_j &= \bar{\beta}\left(\omega^i v_i\right) v_j. \end{aligned}$$

Since any two time-like vectors can not be orthogonal to each other, $\psi - \bar{\alpha} = \bar{\beta} = 0$; that is, M is Einstein, or $\omega_j = \pm v_j$.

Einstein's field equations without cosmological constant are

$$S_{ij} - \frac{S}{2} g_{ij} = kT_{ij}$$

where T_{ij} is the energy–momentum tensor, and k is the gravitational constant. Thus,

$$\begin{aligned} \alpha g_{ij} + \beta \omega_i \omega_j - \frac{S}{2} g_{ij} &= kT_{ij} \\ \left(\alpha - \frac{S}{2}\right) g_{ij} + \beta \omega_i \omega_j &= kT_{ij}. \end{aligned} \tag{9}$$

However, the energy–momentum tensor for a perfect fluid space-time with velocity vector field ω is given by

$$T_{ij} = pg_{ij} + (p+\mu)\omega_i\omega_j \tag{10}$$

where p and μ are pressure and energy density, respectively [10]. Equations (9) and (10) show that

$$kp = \alpha - \frac{S}{2} = \frac{S-\psi}{n-1} - \frac{S}{2} = \frac{(3-n)S-2\psi}{2(n-1)} \tag{11}$$

$$k(p+\mu) = \beta = \frac{S-n\psi}{n-1} \tag{12}$$

$$k\mu = \frac{S-n\psi}{n-1} - \frac{(3-n)S-2\psi}{2(n-1)} = \frac{S-2\psi}{2}. \tag{13}$$

3. Ricci Curvature Conditions on RW Space-Times

3.1. Semi-Symmetric Ricci Curvature

A space-time is called Ricci semi-symmetric if [11]

$$\nabla_l\nabla_k S_{ij} - \nabla_k\nabla_l S_{ij} = 0.$$

Taking the covariant derivative of Equation (5) twice, one obtains

$$\begin{aligned}
\nabla_l\nabla_k S_{ij} - \nabla_k\nabla_l S_{ij} &= \nabla_l\nabla_k\big(\alpha g_{ij} + \beta\omega_i\omega_j\big) - \nabla_k\nabla_l\big(\alpha g_{ij} + \beta\omega_i\omega_j\big) \\
&= \beta\nabla_l\nabla_k\big(\omega_i\omega_j\big) - \beta\nabla_k\nabla_l\big(\omega_i\omega_j\big) \\
&= \beta\big[\omega_i(\nabla_l\nabla_k - \nabla_k\nabla_l)\omega_j + \omega_j(\nabla_l\nabla_k - \nabla_k\nabla_l)\omega_i\big].
\end{aligned} \tag{14}$$

It is clear that this equation implies that an RW space-time is Ricci semi-symmetric if and only if either $\beta = 0$ or $\nabla_l\nabla_k\omega_j = \nabla_k\nabla_l\omega_j$. Let us consider the first condition. It is noted that $\beta = 0$ implies that an RW space-time is Einstein. The converse is also true. Assume that the space-time is Einstein, then

$$\begin{aligned}
\frac{S}{n}g_{ij} &= \alpha g_{ij} + (n\alpha - S)\omega_i\omega_j, \\
\frac{S}{n}\omega_j &= (\alpha - n\alpha + S)\omega_j, \\
\frac{S}{n} &= \alpha - n\alpha + S,
\end{aligned}$$

that is, $\alpha = \frac{S}{n}$. Equation (5) yields $\beta = n\alpha - S$ and consequently $\beta = 0$. The second condition is equivalent to $\omega_h S^h_{ilk} = 0$

Theorem 1. *An RW space-time M is Ricci semi-symmetric if and only if M is Einstein or* $\omega_h S^h_{ilk} = 0$.

Now, assume that $\beta = 0$. Then an RW space-time is Einstein and the eigenvalue is $\psi = \frac{S}{n}$. Let us rewrite the Ricci tensor and the energy–momentum tensor for a perfect fluid space-time in the case $\beta = 0$ as

$$S_{ij} = \frac{S}{n}g_{ij} \tag{15}$$

$$T_{ij} = pg_{ij} + (p+\mu)\omega_i\omega_j \tag{16}$$

where

$$\alpha = \psi = \frac{S}{n}. \tag{17}$$

The equation of state in this case

$$
\begin{aligned}
k(p+\mu) &= 0, & (18)\\
kp &= -\left(\frac{n-2}{2n}\right)S, & (19)\\
k\mu &= \left(\frac{n-2}{2n}\right)S. & (20)\\
kT_{ij} &= -\left(\frac{n-2}{2n}\right)Sg_{ij}. & (21)
\end{aligned}
$$

That is, the perfect fluid is referred to as the dark energy. On the other hand, if $\nabla_l \nabla_k \omega_j = \nabla_k \nabla_l \omega_j$, then

$$\omega_h S^h_{ilk} = 0.$$

This equation yields

$$\omega_h S^h_k = 0.$$

Using Equation (5), it is

$$0 = (\alpha - \beta)\omega_j$$

and consequently $\psi = 0, \alpha = \beta$, and in this case, it is

$$S_{ij} = \frac{S}{n-1}\left(g_{ij} + \omega_i \omega_j\right). \tag{22}$$

Equations (9) and (10) show that

$$
\begin{aligned}
kT_{ij} &= -\frac{(n-3)}{2(n-1)}Sg_{ij} + \frac{S}{n-1}\omega_i\omega_j, & (23)\\
kp &= -\frac{(n-3)}{2(n-1)}S, & (24)\\
k\mu &= \frac{S}{2}, & (25)\\
k(p+\mu) &= \frac{S}{n-1}. & (26)
\end{aligned}
$$

Theorem 2. *Let M be a Ricci semi-symmetric RW space-time. Then, M satisfies one of the following:*

1. *($\beta = 0$) M is Einstein. The Ricci tensor and the equation of state take the form of Equation (15) and Equations (18)–(21). The perfect fluid is referred to as dark matter era.*
2. *($\alpha = \beta$) The Ricci tensor, the energy–momentum tensor, and the equation of state take the form of Equations (22)–(26).*

Remark 1. *Notably, an RW space-time is a perfect fluid space-time. Dark matter era refers to perfect fluid space with the equation of state $p + \mu = 0$ [12]. However, so far, according to [13], a four-dimensional perfect fluid space-time with $p + \mu \neq 0$ is RW space-time if and only if it is a Yang pure space-time. These space-times are identified by a Ricci tensor, which is a Codazzi tensor [13].*

Corollary 1. *A four-dimensional Ricci semi-symmetric RW space-time is a Yang pure space-time given that $\beta \neq 0$.*

3.2. Generalized Recurrent Ricci Curvature

A space-time M is called generalized Ricci recurrent if there are two $1-$form a and b such that

$$(\nabla_X S)(Y,Z) = a(X)S(Y,Z) + b(X)g(Y,Z) \tag{27}$$

where $X, Y, Z \in \mathfrak{X}(M)$ and a, b are called the corresponding recurrence $1-$forms. In local coordinates, one may write

$$\nabla_l S_{ij} = a_l S_{ij} + b_l g_{ij}. \tag{28}$$

Two contractions of this equation by g^{ij} and g^{li} yield

$$\nabla_l S = S a_l + n b_l \tag{29}$$

$$\frac{1}{2}\nabla_j S = a_l S^l_j + b_j. \tag{30}$$

A third contraction with ω^i infers

$$\omega^i \nabla_l S_{ij} = a_l \left(\omega^i S_{ij}\right) + b_l \omega_j. \tag{31}$$

Since ω^i is an eigenvector of the Ricci tensor, it is

$$\nabla_l \left(\omega^i S_{ij}\right) - S_{ij} \nabla_l \omega^i = (\psi a_l + b_l)\omega_j \tag{32}$$

$$\nabla_l (\psi \omega_j) - S_{ij} \nabla_l \omega^i = (\psi a_l + b_l)\omega_j. \tag{33}$$

Now, we may insert the definition of the Ricci tensor as

$$\nabla_l (\psi \omega_j) - (\alpha g_{ij} + \beta \omega_i \omega_j) \nabla_l \omega^i = (\psi a_l + b_l)\omega_j \tag{34}$$

$$\nabla_l (\psi \omega_j) - \alpha \nabla_l \omega_j = (\psi a_l + b_l)\omega_j \tag{35}$$

$$(\nabla_l \psi)\omega_j + (\psi - \alpha)\nabla_l \omega_j = (\psi a_l + b_l)\omega_j. \tag{36}$$

By multiplying both sides by ω^j, one obtains

$$(\nabla_l \psi) = \psi a_l + b_l. \tag{37}$$

Back substitution in Equation (36) results in

$$(\psi - \alpha)\nabla_l \omega_j = 0. \tag{38}$$

Thus, we have two cases, namely, $\psi - \alpha = 0$ and $\nabla_l \omega_j = 0$. The first case $\psi - \alpha = 0$ implies that $\beta = 0$ and so M is Einstein, and the Ricci tensor and the equation of state take the form of Equation (15) and Equations (18)–(21). To consider the second case $\nabla_l \omega_j = 0$, it is clear that $\varphi = 0$. In this case, the perfect fluid is called static. One may use the fact that

$$\psi = (n-1)\left(\varphi^2 + \dot{\varphi}\right)$$

where $\dot{\varphi} = \omega^k \nabla_k \varphi$ to obtain $\psi = 0$, that is $\alpha = \beta$. In this case, the Ricci tensor, the energy–momentum tensor, and the equation of state take the form

$$\begin{aligned}
S_{ij} &= \frac{S}{n-1}(g_{ij} + \omega_i \omega_j), \\
kT_{ij} &= \frac{S}{2n-2}(-(n-3)g_{ij} + 2\omega_i \omega_j), \\
k(p+\mu) &= \frac{S}{n-1}, \\
kp &= -\frac{(n-3)S}{2(n-1)}, \\
k\mu &= \frac{S}{2}.
\end{aligned}$$

The covariant derivative of the Ricci tensor is now given by

$$\nabla_l S_{ij} = \frac{\nabla_l S}{n-1}\left(g_{ij} + \omega_i \omega_j\right).$$

Using the defining property of the generalized Ricci recurrent tensor, it is

$$a_l S_{ij} + b_l g_{ij} = \frac{\nabla_l S}{n-1}\left(g_{ij} + \omega_i \omega_j\right).$$

Now, the definition of the Ricci tensor yields

$$a_l \frac{S}{n-1}\left(g_{ij} + \omega_i \omega_j\right) + b_l g_{ij} = \frac{\nabla_l S}{n-1}\left(g_{ij} + \omega_i \omega_j\right).$$

One may simplify this equation as

$$\begin{aligned} 0 &= \left(\frac{\nabla_l S}{n-1} - a_l \frac{S}{n-1} - b_l\right) g_{ij} + \left(\frac{\nabla_l S}{n-1} - a_l \frac{S}{n-1}\right) \omega_i \omega_j \\ 0 &= (\nabla_l S - S a_l - (n-1) b_l) g_{ij} + (\nabla_l S - S a_l) \omega_i \omega_j. \end{aligned}$$

Different contractions of this equation infer

$$\begin{aligned} b_l &= 0, \\ 0 &= \nabla_l S - S a_l - n b_l. \\ S a_l &= \nabla_l S \end{aligned}$$

The defining equation of the generalized Ricci recurrent RW space-time reduces to

$$\nabla_l S_{ij} = a_l S_{ij}$$

For a non-zero scalar curvature S, it is

$$\nabla_l S_{ij} = \nabla_l (\ln S) S_{ij}.$$

Theorem 3. *Let M be a generalized Ricci recurrent RW space-time. Then M reduces to be Einstein or to a Ricci recurrent RW space-time of the form*

$$\nabla_l S_{ij} = \nabla_l (\ln S) S_{ij}.$$

Moreover, M satisfies one of the following:

1. *M is Einstein. The Ricci tensor and the equation of state take the form of Equations (15) and (18)–(21).*
2. *M is a static perfect fluid, and the Ricci tensor, the energy–momentum tensor, and the equation of state take the form of Equations (22)–(26).*

A space-time M is called Ricci recurrent if there is a $1-$form a such that

$$(\nabla_X S)(Y, Z) = a(X) S(Y, Z)$$

where X, Y, Z are vector fields on M and a is called the recurrence $1-$form. In local coordinates, one may write

$$\nabla_l S_{ij} = a_l S_{ij}.$$

It should be noted that a Ricci recurrent space-time is a generalized Ricci recurrent space-time. Let M be a Ricci recurrent RW space-time. Then M reduces to be Einstein or to a Ricci recurrent RW space-time of the form

$$\nabla_l S_{ij} = \nabla_l (\ln S) S_{ij}.$$

Moreover, M is either Einstein and Equations (15)–(21) hold or M is a static perfect fluid and the Ricci tensor, the energy–momentum tensor, and the equation of state take the form of Equations (22)–(26).

A space-time M is called Ricci symmetric if [14]

$$(\nabla_X S)(Y, Z) = 0$$

where X, Y, Z are vector fields on M. In local coordinates, one may write

$$\nabla_l S_{ij} = 0.$$

It should be noted that a Ricci symmetric space-time is a Ricci recurrent space-time. In the Ricci flat case, it is easy to show that only one case of the above result will hold, namely, M is Einstein.

Corollary 2. *Let M be a Ricci symmetric RW space-times. Then M reduces to be Einstein, and the Ricci tensor and the equation of state take the form of Equations (15) and (18)–(21).*

3.3. Codazzi Type of Ricci Tensor

The RW space-time is of Codazzi type of Ricci tensor if

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z) \tag{39}$$

where X, Y, Z are vector fields on M. In local coordinates, it is

$$\nabla_k S_{ij} = \nabla_i S_{kj}. \tag{40}$$

To obtain contraction of this equation by $\omega^i \omega^k$, let us first evaluate both sides as

$$\begin{aligned} \omega^i \omega^k \nabla_k S_{ij} &= \omega^i \left(\dot{\alpha} g_{ij} + \dot{\beta} \omega_i \omega_j + \beta \dot{\omega}_i \omega_j + \beta \omega_i \dot{\omega}_j \right) \\ &= \dot{\alpha} \omega_j - \dot{\beta} \omega_j - \beta \dot{\omega}_j \\ &= \dot{\psi} \omega_j - \beta \dot{\omega}_j \end{aligned} \tag{41}$$

and

$$\begin{aligned} \omega^i \omega^k \nabla_i S_{kj} &= \omega^i \nabla_i \left(\omega^k S_{kj} \right) - \omega^i S_{kj} \nabla_i \omega^k \\ &= \omega^i \nabla_i (\psi \omega_j) - \omega^i S_{kj} \nabla_i \omega^k \\ &= \dot{\psi} \omega_j + \psi \dot{\omega}_j - \omega^i \varphi S_{kj} \left(\delta_i^k + \omega_i \omega^k \right) \\ &= \dot{\psi} \omega_j + \psi \dot{\omega}_j - \omega^i \varphi (S_{ij} + \psi \omega_i \omega_j) \\ &= \dot{\psi} \omega_j + \psi \dot{\omega}_j - \omega^i \varphi (\psi \omega_j - \psi \omega_j) \\ &= \dot{\psi} \omega_j + \psi \dot{\omega}_j \end{aligned} \tag{42}$$

The above equations imply

$$\begin{aligned} \dot{\psi} \omega_j - \beta \dot{\omega}_j &= \dot{\psi} \omega_j + \psi \dot{\omega}_j \\ (\psi + \beta) \dot{\omega}_j &= 0 \end{aligned}$$

Therefore, $\psi + \beta = 0$ or $\dot{\omega}_j = 0$. The first case infers $\alpha = 0$ $\psi = -\beta = S$ and consequently

$$S_{ij} = -S\omega_i\omega_j. \tag{43}$$

Space-times with this form of Ricci curvature are called Ricci simple space-times [15]. The energy–momentum tensor, the pressure and energy density are consequently given by

$$\begin{aligned} kT_{ij} &= -\frac{S}{2}g_{ij} - \frac{3S}{2}\omega_i\omega_j, & (44)\\ kp &= -\frac{S}{2}, & (45)\\ k\mu &= -S, & (46)\\ k(p+\mu) &= -\frac{3S}{2}. & (47) \end{aligned}$$

The second condition implies that the fluid acceleration is zero and the velocity vector field is geodesic.

Theorem 4. *Let M be an RW space-time admitting a Codazzi type of Ricci tensor. Then, the velocity vector field is geodesic or M is Ricci simple and*

$$\begin{aligned} S_{ij} &= -S\omega_i\omega_j, & (48)\\ kT_{ij} &= -\frac{S}{2}g_{ij} - \frac{3S}{2}\omega_i\omega_j, & (49)\\ k(p+\mu) &= -\frac{3S}{2}, & (50)\\ kp &= -\frac{S}{2}, & (51)\\ k\mu &= -S. & (52) \end{aligned}$$

4. Riemann Curvature Tensor on RW Space-Times

The Riemann curvature tensor of an RW space-time is completely determined by the vector ω as follows. It is clear that the conformal curvature tensor is null and so

$$\begin{aligned} 0 &= C_{jklm} \\ &= S_{jklm} + \frac{1}{n-2}\Big[g_{jm}S_{kl} - g_{km}S_{jl} + g_{kl}S_{jm} - g_{jl}S_{km}\Big] \\ &\quad - \frac{S}{(n-1)(n-2)}\Big[g_{jm}g_{kl} - g_{km}g_{jl}\Big]. \end{aligned} \tag{53}$$

Now, the Riemann curvature tensor has the form

$$\begin{aligned} S_{jklm} &= \frac{S}{(n-1)(n-2)}\Big[g_{jm}g_{kl} - g_{km}g_{jl}\Big] \\ &\quad - \frac{1}{n-2}\Big[g_{jm}S_{kl} - g_{km}S_{jl} + g_{kl}S_{jm} - g_{jl}S_{km}\Big] \end{aligned} \tag{54}$$

Using the form of the Ricci curvature tensor, one obtains

$$\begin{aligned} S_{jklm} &= \frac{S - 2(n-1)\alpha}{(n-1)(n-2)}\Big[g_{jm}g_{kl} - g_{km}g_{jl}\Big] \\ &\quad - \frac{\beta}{n-2}\Big[g_{jm}\omega_k\omega_l - g_{km}\omega_j\omega_l + g_{kl}\omega_j\omega_m - g_{jl}\omega_k\omega_m\Big]. \end{aligned} \tag{55}$$

It is clear that

$$\begin{aligned}\nabla_r(\omega_k\omega_l) &= \nabla_r(\omega_k)\omega_l + \omega_k\nabla_r(\omega_l)\\ &= \varphi(g_{rk} + \omega_r\omega_k)\omega_l + \varphi(g_{rl} + \omega_r\omega_l)\omega_k\\ &= \varphi(g_{rk}\omega_l + g_{rl}\omega_k + 2\omega_r\omega_k\omega_l)\end{aligned}$$

After lengthy computations using this equation, the covariant derivative of the Riemann curvature tensor may be finally rewritten as

$$\begin{aligned}\nabla_r S_{jklm} &= \frac{\nabla_r S - 2(n-1)\nabla_r\alpha}{(n-1)(n-2)}\left[g_{jm}g_{kl} - g_{km}g_{jl}\right]\\ &-\frac{\nabla_r\beta}{n-2}\left[g_{jm}\omega_k\omega_l - g_{km}\omega_j\omega_l + g_{kl}\omega_j\omega_m - g_{jl}\omega_k\omega_m\right]\\ &-\frac{\varphi\beta}{n-2}\left[g_{jm}g_{rk}\omega_l + g_{jm}g_{rl}\omega_k + 2g_{jm}\omega_r\omega_k\omega_l - g_{km}g_{rj}\omega_l\right]\\ &-\frac{\varphi\beta}{n-2}\left[-g_{km}g_{rl}\omega_j - 2g_{km}\omega_r\omega_j\omega_l + g_{kl}g_{rj}\omega_m + g_{kl}g_{rm}\omega_j\right]\\ &-\frac{\varphi\beta}{n-2}\left[2g_{kl}\omega_r\omega_j\omega_m - g_{jl}g_{rk}\omega_m - g_{jl}g_{rm}\omega_k - 2g_{jl}\omega_r\omega_k\omega_m\right] \qquad (56)\end{aligned}$$

4.1. Locally Symmetric RW Space-Time

Assume that an RW space-time is symmetric, that is, $\nabla_r S_{jklm} = 0$ [16], and consequently, the scalar curvature is constant and $n\nabla\alpha = \nabla\beta$. Thus,

$$\begin{aligned}0 &= \frac{-2\nabla_r\alpha}{(n-2)}\left[g_{jm}g_{kl} - g_{km}g_{jl}\right]\\ &-\frac{n\nabla_r\alpha}{n-2}\left[g_{jm}\omega_k\omega_l - g_{km}\omega_j\omega_l + g_{kl}\omega_j\omega_m - g_{jl}\omega_k\omega_m\right]\\ &-\frac{\varphi\beta}{n-2}\left[g_{jm}g_{rk}\omega_l + g_{jm}g_{rl}\omega_k + 2g_{jm}\omega_r\omega_k\omega_l - g_{km}g_{rj}\omega_l\right]\\ &-\frac{\varphi\beta}{n-2}\left[-g_{km}g_{rl}\omega_j - 2g_{km}\omega_r\omega_j\omega_l + g_{kl}g_{rj}\omega_m + g_{kl}g_{rm}\omega_j\right]\\ &-\frac{\varphi\beta}{n-2}\left[2g_{kl}\omega_r\omega_j\omega_m - g_{jl}g_{rk}\omega_m - g_{jl}g_{rm}\omega_k - 2g_{jl}\omega_r\omega_k\omega_m\right].\end{aligned}$$

By multiplying this equation by g^{jl}, it is

$$0 = \nabla_r\alpha[n\omega_k\omega_m + g_{km}] + \varphi\beta[g_{rk}\omega_m + g_{rm}\omega_k + 2\omega_r\omega_k\omega_m]. \qquad (57)$$

A last contraction with $\omega^k\omega^m$ gives us $\nabla_r\alpha = 0$. Back substitution in the above equation yields

$$\begin{aligned}0 &= \varphi\beta[g_{rk}\omega_m + g_{rm}\omega_k + 2\omega_r\omega_k\omega_m] &(58)\\ &= \varphi\beta[\omega_m\nabla_r u_k + \omega_k\nabla_r u_m] &(59)\\ &= \varphi\beta[\omega_m\nabla_r\omega_k + \omega_k\nabla_r\omega_m] &(60)\end{aligned}$$

From this equation, it is easy to show that either $\beta = 0$ or $\nabla_r\omega_m = 0$. The first case implies that the space-time is Einstein, and the second case infers the space-time is static. In the first case, the Riemann curvature tensor becomes

$$\begin{aligned}S_{jklm} &= \frac{S - 2(n-1)\alpha}{(n-1)(n-2)}\left[g_{jm}g_{kl} - g_{km}g_{jl}\right] &(61)\\ &= \frac{S}{n(n-1)}\left[g_{km}g_{jl} - g_{jm}g_{kl}\right]. &(62)\end{aligned}$$

Therefore, RW space-time has constant curvature. A simple contraction of this equation implies $\alpha = 0$. Now, the manifold is Ricci flat and consequently is flat.

Theorem 5. *Let M be a locally symmetric RW space-time. Then,*

1. *M has a constant curvature. The Riemann tensor, the Ricci tensor, and the equation of state take the form*

$$\begin{aligned} S_{jklm} &= \frac{S}{n(n-1)}\Big[g_{km}g_{jl} - g_{jm}g_{kl}\Big]. \\ S_{ij} &= \frac{S}{n}g_{ij}, \\ kT_{ij} &= -\left(\frac{n-2}{2n}\right)Sg_{ij}, \\ kp &= -k\mu = -\left(\frac{n-2}{2n}\right)S, \\ k(p+\mu) &= 0. \end{aligned}$$

2. *M is a static space-time.*

4.2. Recurrent RW Space-Times

A space-time is called recurrent if there is one form a such that

$$(\nabla_U S)(X,Y,Z,W) = a(U)S(X,Y,Z,W).$$

In local coordinates, it is
$$\nabla_r S_{jklm} = a_r S_{jklm}.$$

Thus, an RW space-time is recurrent if

$$\begin{aligned} a_r S_{jklm} &= \nabla_r S_{jklm} \\ &= \frac{\nabla_r S - (n-1)\nabla_r \alpha}{(n-1)(n-2)}\Big[g_{jm}g_{kl} - g_{km}g_{jl}\Big] \\ &\quad -\frac{\nabla_r \beta}{n-2}\Big[g_{jm}\omega_k\omega_l - g_{km}\omega_j\omega_l + g_{kl}\omega_j\omega_m - g_{jl}\omega_k\omega_m\Big] \\ &\quad -\frac{\varphi\beta}{n-2}\big[g_{jm}g_{rk}\omega_l + g_{jm}g_{rl}\omega_k + 2g_{jm}\omega_r\omega_k\omega_l - g_{km}g_{rj}\omega_l\big] \\ &\quad -\frac{\varphi\beta}{n-2}\big[-g_{km}g_{rl}\omega_j - 2g_{km}\omega_r\omega_j\omega_l + g_{kl}g_{rj}\omega_m + g_{kl}g_{rm}\omega_j\big] \\ &\quad -\frac{\varphi\beta}{n-2}\Big[2g_{kl}\omega_r\omega_j\omega_m - g_{jl}g_{rk}\omega_m - g_{jl}g_{rm}\omega_k - 2g_{jl}\omega_r\omega_k\omega_m\Big] \end{aligned} \tag{63}$$

Using the calculations in the above subsection, one obtains

$$a_r S_{km} = -\frac{\nabla_r \alpha}{n-2}g_{km} + \nabla_r\beta\omega_k\omega_m + \varphi\beta[g_{rk}\omega_m + g_{rm}\omega_k + 2\omega_r\omega_k\omega_m].$$

Using two contractions with g^{km} and $\omega^k\omega^m$, this equation infers

$$\begin{aligned} a_r S &= -\frac{n\nabla_r \alpha}{n-2} - \nabla_r\beta \\ a_r\psi\omega_k &= -\frac{\nabla_r \alpha}{n-2}\omega_k - \nabla_r\beta\omega_k + \varphi\beta[-g_{rk} - \omega_r\omega_k] \\ a_r\psi &= -\frac{\nabla_r \alpha}{n-2} - \nabla_r\beta \end{aligned}$$

The subtraction of these two equations implies

$$\begin{aligned} a_r(S-\psi) &= -\frac{n-1}{n-2}\nabla_r\alpha \\ a_r(n-1)\alpha &= -\frac{n-1}{n-2}\nabla_r\alpha \\ a_r\alpha &= -\frac{1}{n-2}\nabla_r\alpha. \end{aligned}$$

Theorem 6. *Let M be a recurrent RW space-time. Then, M is Ricci simple or the recurrence form is given by*

$$a_r = -\frac{1}{n-2}\frac{1}{\alpha}\nabla_r\alpha.$$

4.3. Harmonic RW Space-Time

A contraction of $\nabla_r S_{jklm}$ with g^{rj} infers

$$\begin{aligned} \nabla^j S_{jklm} &= \frac{\nabla_m S-(n-1)\nabla_m\alpha}{(n-1)(n-2)}g_{kl} - \frac{\nabla_l S-(n-1)\nabla_l\alpha}{(n-1)(n-2)}g_{km} \\ &\quad -\frac{\nabla_m\beta}{n-2}\omega_k\omega_l + \frac{\dot\beta}{n-2}g_{km}\omega_l - \frac{\dot\beta}{n-2}g_{kl}\omega_m + \frac{\nabla_l\beta}{n-2}\omega_k\omega_m \\ &\quad -\frac{\varphi\beta}{n-2}[g_{km}\omega_l + g_{lm}\omega_k + 2\omega_m\omega_k\omega_l - n g_{km}\omega_l] \\ &\quad -\frac{\varphi\beta}{n-2}[-g_{km}\omega_l + 2g_{km}\omega_l + n g_{kl}\omega_m + g_{kl}\omega_m] \\ &\quad -\frac{\varphi\beta}{n-2}[-2g_{kl}\omega_m - g_{lk}\omega_m - g_{lm}\omega_k - 2\omega_l\omega_k\omega_m] \end{aligned} \tag{64}$$

Thus, the divergence of the Riemann tensor is give by

$$\begin{aligned} \nabla^j S_{jklm} &= \frac{1}{(n-1)(n-2)}((\nabla_m\psi)g_{kl} - (\nabla_l\psi)g_{km}) \\ &\quad +\left(\frac{\dot\beta}{n-2}+\varphi\beta\right)(g_{km}\omega_l - g_{kl}\omega_m) \\ &\quad -\frac{1}{n-2}((\nabla_m\beta)\omega_k\omega_l - (\nabla_l\beta)\omega_k\omega_m). \end{aligned} \tag{65}$$

Assume that M is harmonic, that is,

$$\begin{aligned} 0 &= \nabla^j S_{jklm} \\ &= \frac{1}{(n-1)(n-2)}((\nabla_m\psi)g_{kl} - (\nabla_l\psi)g_{km}) \\ &\quad +\left(\frac{\dot\beta}{n-2}+\varphi\beta\right)(g_{km}\omega_l - g_{kl}\omega_m) \\ &\quad -\frac{1}{n-2}((\nabla_m\beta)\omega_k\omega_l - (\nabla_l\beta)\omega_k\omega_m). \end{aligned} \tag{66}$$

Therefore, one obtains

$$
\begin{aligned}
0 &= \frac{1}{n-2}(\nabla_m\psi + \nabla_m\beta) + \big(-\dot{\beta} + \varphi\beta(1-n)\big)\omega_m \\
0 &= \frac{n\nabla_m\alpha}{n-2} - \big(\dot{\beta} + (n-1)\varphi\beta\big)\omega_m \\
0 &= \frac{n\dot{\alpha}}{n-2} + \big(\dot{\beta} + (n-1)\varphi\beta\big) \\
0 &= n\dot{\alpha} + (n-2)\big(\dot{\beta} + (n-1)\varphi\beta\big)
\end{aligned} \tag{67}
$$

However, a harmonic RW space-time has a divergence free Ricci tensor, that is,

$$
\begin{aligned}
0 &= \nabla^j S_{jk} \\
0 &= \omega^k \nabla^j S_{jk} = \nabla^j\Big(\omega^k S_{jk}\Big) - S_{jk}\nabla^j\omega^k \\
&= \nabla^j\big(\psi\omega_j\big) - \varphi S_{jk}\Big(g^{jk} + \omega^j\omega^k\Big) \\
&= \psi\nabla^j\big(\omega_j\big) + \omega_j\nabla^j(\psi) - \varphi(S-\psi) \\
&= \psi\varphi(n-1) + \dot{\psi} - \varphi(n\alpha - \beta - \alpha + \beta) \\
&= \dot{\psi} - (n-1)\varphi\alpha + \psi\varphi(n-1) \\
&= \dot{\psi} - (n-1)\varphi\beta.
\end{aligned} \tag{68}
$$

Thus, $\dot{\psi} = (n-1)\varphi\beta$. Equation (68) now becomes

$$
\begin{aligned}
0 &= n\dot{\alpha} + (n-2)\big(\dot{\beta} + \dot{\psi}\big) \\
&= (2n-2)\dot{\alpha}.
\end{aligned}
$$

Hence, $\dot{\alpha} = 0$, $\dot{\beta} = -\dot{\psi} = -(n-1)\varphi\beta$ and Equation (67) reduce to

$$
\begin{aligned}
0 = \;& \frac{1}{(n-1)}((\nabla_m\psi)g_{kl} - (\nabla_l\psi)g_{km}) \\
& - \frac{1}{n-2}\varphi\beta(g_{km}\omega_l - g_{kl}\omega_m) \\
& - ((\nabla_m\beta)\omega_k\omega_l - (\nabla_l\beta)\omega_k\omega_m).
\end{aligned} \tag{69}
$$

A contraction by g^{kl} implies

$$
\begin{aligned}
0 &= \nabla_m\psi + \frac{n-1}{n-2}\varphi\beta\omega_m + \nabla_m\beta + \dot{\beta}\omega_m \\
&= \nabla_m\alpha + \left(1 - \frac{1}{n-2}\right)\dot{\beta}\omega_m \\
&= \nabla_m\alpha + \frac{n-3}{n-2}\dot{\beta}\omega_m
\end{aligned} \tag{70}
$$

Again, transfecting this equation by ω^m yields

$$
0 = \dot{\alpha} - \frac{n-3}{n-2}\dot{\beta} = -\frac{n-3}{n-2}\dot{\beta} = \frac{n-3}{n-2}(n-1)\varphi\beta.
$$

Therefore, $\beta = 0$ or $\varphi = 0$.

Theorem 7. *Let M be a harmonic RW space-time. Then, M is Einstein or M is a static space-time.*

5. Conclusions

A conformally flat GRW space-time satisfies an algebraic curvature condition; namely, it is a perfect fluid RW space-time. The existence of one of the differential curvature conditions (i.e., semi-symmetric Ricci curvature, generalized recurrent Ricci curvature tensor, recurrent Ricci curvature tensor, parallel Ricci curvature tensor, Codazzi Ricci tensor, locally symmetric, and harmonic Riemann curvature tensor) implies the RW space-time has a constant curvature or is a static space-time.

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References

1. Chen, B.-Y. A simple characterization of generalized Robertson–Walker spacetimes. *Gen. Relativ. Gravit.* **2014**, *46*, 1833. [CrossRef]
2. Mantica, C.A.; Molinari, L.G. Generalized Robertson—Walker spacetimes—A survey. *Int. Geom. Methods Mod. Phys.* **2017**, *14*, 1730001. [CrossRef]
3. Griffiths, J.B.; Podolsky, J. *Exact Space-Times in Einstein's General Relativity*; Cambridge University Press: Cambridge, UK, 2009.
4. Mantica, C.A.; Molinari, L.G.; Suh, Y.; Shenawy, S. Perfect-Fluid, Generalized Robertson-Walker Space-times, and Gray's Decomposition. *J. Math. Phys.* **2019**, *60*, 052506. [CrossRef]
5. Mantica, C.A.; Molinari, L.G. On the Weyl and Ricci tensors of Generalized Robertson-Walker space-times. *J. Math. Phys.* **2016**, *57*, 102502. [CrossRef]
6. Mantica, C.A.; De, U.C.; Jin Suh, Y.; Molinari, L.G. Perfect fluid spacetimes with harmonic generalized curvature tensor. *Osaka J. Math.* **2019**, *56*, 173–182.
7. Sanchez, M. On the geometry of generalized Robertson-Walker spacetimes: Geodesics. *Gen. Relativ. Gravit.* **1998**, *30*, 915–932. [CrossRef]
8. Capozziello, S.; Mantica, C.A.; Molinari, L. Cosmological perfect-fluids in f(R) gravity. *Int. J. Geom. Methods Mod. Phys.* **2019**, *16*, 1950008. [CrossRef]
9. Chaki, M.C. On generalized quasi Einstein manifolds. *Publ. Math. Debr.* **2001**, *58*, 683–691. [CrossRef]
10. Ehlers, J.; Geren, P.; Sachs, R.K. Isotropic Solutions of the Einstein-Liouville Equations. *J. Math. Phys.* **1968**, *9*, 1344–1349. [CrossRef]
11. De, U.C.; Suh, Y.J.; Chaubey, S.K. Semi-Symmetric Curvature Properties of Robertson–Walker Spacetimes. *J. Math. Phys. Anal. Geom.* **2022**, *18*, 368–381.
12. Chavanis, P.-H. Cosmology with a stiff matter era. *Phys. Rev. D* **2015**, *92*, 103004. [CrossRef]
13. Guilfoyle, B.S.; Nolan, B.C. Yang's Gravitational Theory. *Gen. Relativ. Gravit.* **1998**, *30*, 473–495. [CrossRef]
14. Guler, S.; Demirbag, S.A. On Ricci symmetric generalized quasi Einstein spacetimes. *Miskolc Math. Notes* **2015**, *16*, 853–868. [CrossRef]
15. Mantica, C.A.; Suh, Y.J.; De, U. A note on generalized Robertson—Walker space-times. *Int. J. Geom. Methods Mod. Phys.* **2016**, *13*, 1650079. [CrossRef]
16. Eriksson, I.; Senovilla, J.M.M. Note on (conformally) semi-symmetric spacetimes. *Class. Quantum Gravity* **2010**, *27*, 027001. [CrossRef]