A note on the Geronimus transformation and Sobolev orthogonal polynomials

Maxim Derevyagin · Francisco Marcellán

Abstract In this note we recast the Geronimus transformation in the framework of polynomials orthogonal with respect to symmetric bilinear forms. We also show that the double Geronimus transformations lead to non-diagonal Sobolev type inner products.

1 Introduction

Let us consider the following problem. Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of monic polynomials orthogonal with respect to a nontrivial probability measure supported on an infinite subset of the real line. The problem consists in finding necessary and

M. Derevyagin (✉)
Department of Mathematics MA 4-2, Technische Universität Berlin,
Strasse des 17. Juni 136, 10623 Berlin, Germany
e-mail: derevyagin.m@gmail.com

F. Marcellán
Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30,
28911 Leganés, Spain
e-mail: pacomarc@ing.uc3m.es

Present Address:
M. Derevyagin
Department of Mathematics, KU Leuven, Celestijnenlaan 200b, box 2400, 3001 Heverlee, Belgium

sufficient conditions for the real numbers $A_n, n = 0, 1, \ldots,$ to make the sequence of monic polynomials

$$P_n(t) + A_n P_{n-1}(t), A_n \neq 0, \quad n = 1, 2, \ldots,$$

orthogonal with respect to some measure supported on the real line. The idea of studying this problem goes back to Shohat's paper [21] concerning quadrature formulas associated to n nodes with a degree of exactness less that $2n - 1$. A few years later after the Shohat's publication a complete and final answer to that problem was given by Geronimus [14]. Thus [14] provided us with a procedure of constructing new families of orthogonal polynomials from the given ones. One can also reduce some families of orthogonal polynomials to the known ones with the help of such a procedure.

Recall that if we have a sequence of monic orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$ then the polynomial transformation

$$P_n(t) \rightarrow P_n(t) + A_n P_{n-1}(t), A_n \neq 0, \quad n = 1, 2, \ldots,$$

that gives a new family of orthogonal polynomials, is said to be the Geronimus transformation [7, 22, 24]. In fact, the Geronimus transformation divides the measure of orthogonality by the spectral parameter minus the point of transformation and adds a mass to it at the point of transformation. See also [18], where the sequence of polynomials associated with such a perturbation in a more general algebraic framework (orthogonality with respect to a linear functional defined in the linear space of polynomials with complex coefficients) is studied.

Besides the measure interpretation, the Geronimus transformation can be also interpreted in terms of Jacobi matrices in the framework of the so called discrete Darboux transformations and it is related to LU and UL factorizations of shifted Jacobi matrices [7]. Although the Geronimus transformation has its origin in mechanical quadrature [21], it has also found many applications in classical analysis, numerical analysis, and physics [7, 22, 23]. In particular, it should be stressed that the Geronimus transformation together with the Christoffel transformation (both called discrete Darboux transformations) give a bridge between orthogonal polynomials and discrete integrable systems [22, 23].

To go deeper in understanding the Geronimus transformation it is somehow natural to consider its iterations. Say, two iterations of the Geronimus transformation lead to the families of orthogonal polynomials defined by

$$P_n(t) \rightarrow P_n(t) + B_n P_{n-1}(t) + C_n P_{n-2}(t), n \geq 1, C_n \neq 0, n \geq 2.$$

Such families have been extensively studied in the literature (see [3, 5, 15], among others). A particular case of the corresponding inverse problem in terms of perturbations of linear functionals has been analyzed in [4]. For more iterations of the Geronimus transformation, see the results contained in [2, 17, 20]. Some particular cases of inverse problems for the cubic case have been analyzed in [19]. On the other hand, in [16] the higher order ordinary linear differential equations associated with polynomials orthogonal with respect to iterated Geronimus transformations of

Laguerre orthogonal polynomials, the so called Krall-Laguerre orthogonal polynomials, are studied in a framework of commutative algebras with orthogonal polynomials as eigenfunctions.

An interesting point in analysis of iterations of the Geronimus transformation is the following. It is well known that the sequence of monic polynomials $\left\{\widetilde{Q}_n^{(\alpha)}\right\}_{n=0}^{\infty}$, which are orthogonal with respect to the Laguerre-Sobolev type inner product

$$[f, g] = \int_0^{\infty} f(t)g(t)t^{\alpha}e^{-t}dt + Mf(0)g(0) + Nf'(0)g'(0) \quad f, g \in \mathcal{P}$$

defined on the linear space $\mathcal{P}$ of polynomials with real coefficients, can be represented in terms of the sequence of classical monic Laguerre polynomials $\{L_n^{(\alpha)}\}_{n=0}^{\infty}$ as follows

$$\widetilde{Q}_n^{(\alpha)}(t) = L_n^{(\alpha+2)}(t) + B_n L_{n-1}^{(\alpha+2)}(t) + C_n L_{n-2}^{(\alpha+2)}(t).$$

Obviously, one cannot get the Laguerre-Sobolev inner product by dividing by t^2 the measure $t^{\alpha+2}e^{-t}dt$ and adding masses to it despite the above formula suggests that the Laguerre-Sobolev type orthogonal polynomials are the two consecutive Geronimus transformations of the classical Laguerre polynomials. This problem brings us to one of the aims of this note.

One of the main ideas of the present paper is to include the Laguerre-Sobolev type orthogonal polynomials and similar Sobolev orthogonal polynomials into the scheme of Darboux transformations. To this end we propose to reconsider the Geronimus transformation in a more general framework related to symmetric bilinear forms.

Recall that a symmetric bilinear form $B(\cdot, \cdot)$ in the linear space $\mathcal{P}$ is a mapping

$$B(\cdot, \cdot) : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$$

that is linear with respect to each of their arguments and has the symmetry property

$$B(f, g) = B(g, f), \quad f, g \in \mathcal{P}.$$

For instance, the form

$$(f, g)_0 = \int_{\mathbb{R}} f(t)g(t)d\mu(t)$$

is symmetric and bilinear. It is not so hard to see that the Gram matrix $\left((t^i, t^j)_0\right)_{i,j=0}^{\infty}$ is a Hankel matrix and is positive definite.

A bilinear form is said to be regular (resp. positive definite) if all leading principal submatrices of its Gram matrix are nonsingular (positive definite). In such cases, the bilinear form generates a sequence of monic orthogonal polynomials in a simple way by using the Gram-Schmidt process. Nonetheless, the main advantage of considering bilinear forms in the context of orthogonality is the ability to include many types of orthogonality such that the corresponding Gram matrix associated with their moments is not a Hankel matrix, e.g. the Sobolev orthogonality (see [6, 8]) and other types of orthogonality related to matrix measures (see [12]) based on the symmetry of a polynomial operator with respect to a bilinear form.

The paper is organized as follows. In Section 2 the classical Geronimus transformation is considered. The structure of the symmetric Jacobi matrix corresponding to the transformed polynomials is discussed in the next section. The double Geronimus

transformation in the framework of bilinear forms is presented in Section 4. The last section gives details of the structure of the symmetric pentadiagonal matrix associated with the recurrence coefficients for the transformed polynomials.

2 The classical Geronimus transformations

In this section we review some of the results of [14] from the point of view of symmetric bilinear forms.

We start with the precise definition of the Geronimus transformation in the framework under consideration.

Definition 2.1 Let us consider a symmetric bilinear form

$$(f, g)_0 = \int_{\mathbb{R}} f(t)g(t)d\mu(t).$$

The Geronimus transformation of $(\cdot, \cdot)_0$ is a symmetric bilinear form $[\cdot, \cdot]_1$ defined on the set $\mathcal{P}$ of real polynomials as follows

$$[tf(t), g(t)]_1 = [f(t), tg(t)]_1 = (f, g)_0 = \int_{\mathbb{R}} f(t)g(t)d\mu(t), \quad f, g \in \mathcal{P}. \quad (2.1)$$

Evidently, this definition doesn't determine $[\cdot, \cdot]_1$ uniquely. However, we can see how the Geronimus transformation looks like.

Proposition 2.2 *Suppose that $d\mu$ has the following representation*

$$d\mu(t) = td\mu_1(t), \quad (2.2)$$

where $d\mu_1$ is a positive measure and it has finite moments. Then the bilinear form $[\cdot, \cdot]_1$ admits the representation

$$[f, g]_1 = \int_0^\infty f(t)g(t)d\mu_1(t) + \left(s_0^* - \int_0^\infty d\mu_1(t)\right) f(0)g(0), \quad f, g \in \mathcal{P}, \quad (2.3)$$

where s_0^* is an arbitrary real number.

Proof It is clear that the value $[1, 1]_1$ can be arbitrary. So, let us denote it by s_0^*, i.e. $s_0^* = [1, 1]_1$. Further, let us compute $[f, g]_1$ for any $f, g \in \mathcal{P}$:

$$
\begin{aligned}
[f, g]_1 &= [f(t) - f(0) + f(0), g(t)]_1 = [f(t) - f(0), g(t)]_1 + [f(0), g(t)]_1 \\
&= [f(t) - f(0), g(t)]_1 + [f(0), g(t) - g(0)]_1 + [f(0), g(0)]_1 \\
&= \left(\frac{f(t) - f(0)}{t}, g(t)\right) + \left(f(0), \frac{g(t) - g(0)}{t}\right) + f(0)g(0)s_0^* \\
&= \int_0^\infty \frac{f(t) - f(0)}{t} g(t)d\mu(t) \\
&\quad + \int_0^\infty f(0)\frac{g(t) - g(0)}{t}d\mu(t) + f(0)g(0)s_0^*.
\end{aligned}
$$

Next, using (2.2) we arrive at

$$[f, g]_1 = \int_0^\infty (f(t) - f(0)) \, g(t) d\mu_1(t)$$
$$+ \int_0^\infty f(0) \, (g(t) - g(0)) \, d\mu_1(t) + f(0)g(0)s_0^*,$$

which can be easily simplified to (2.3). $\qquad\square$

To get an idea about the Geronimus transformation, let us consider one particular example of the initial inner product:

$$(f, g)_0 = \int_0^{+\infty} f(t)g(t)t^\alpha e^{-t} dt, \quad \alpha > 0.$$

Clearly, one of the possible choices for the Geronimus transformation is the following bilinear form

$$[f, g]_1 = \int_0^{+\infty} f(t)g(t)t^{\alpha-1} e^{-t} dt, \quad \alpha > 0,$$

that is the case where $s_0^* = \int_0^\infty t^{\alpha-1} e^{-t} dt$. In this case the forms $(\cdot, \cdot)_0$ and $[\cdot, \cdot]_1$ generate the sequences of monic Laguerre polynomials $\left\{ L_n^{(\alpha)} \right\}_{n=0}^\infty$ and $\left\{ L_n^{(\alpha-1)} \right\}_{n=0}^\infty$, respectively. These polynomials are related as follows

$$L_n^{(\alpha)}(t) + n L_{n-1}^{(\alpha)}(t) = L_n^{(\alpha-1)}(t), \quad n = 0, 1, \ldots.$$

It turns out that a similar relation is also valid for the Geronimus transformation in general.

Theorem 2.3 (cf. [14]) *Let assume that $(\cdot, \cdot)_0$ and $[\cdot, \cdot]_1$ are positive definite and regular bilinear forms, respectively. Let $\{P_n\}_{n=0}^\infty$ be a sequence of monic polynomials orthogonal with respect to $(\cdot, \cdot)_0$. Then a monic polynomial P_n^* of degree n is orthogonal with respect to $[\cdot, \cdot]_1$ if and only if it can be represented as follows*

$$P_n^*(t) = \frac{1}{d_n^*} \begin{vmatrix} P_n(t) & s_0^* P_n(0) + Q_n(0) \\ P_{n-1}(t) & s_0^* P_{n-1}(0) + Q_{n-1}(0) \end{vmatrix}, \tag{2.4}$$

where $d_n^* = s_0^* P_{n-1}(0) + Q_{n-1}(0) \neq 0$. Here, $\{Q_n\}_{n=0}^\infty$ denotes the sequence of monic orthogonal polynomials of the second kind with $\deg Q_n = n - 1$ and defined by $Q_n(x) = \int_0^\infty \frac{P_n(t) - P_n(x)}{t-x} d\mu(t)$.

Proof Since $[\cdot, \cdot]_1$ is regular there exists the corresponding sequence of monic orthogonal polynomials. Suppose that P_n^* is orthogonal, that is,

$$\left[P_n^*(t), t^k \right]_1 = \left[t^k, P_n^*(t) \right]_1 = 0, \quad k = 0, 1, 2, \ldots, n-1.$$

In turn, for the original bilinear form we have

$$\left(P_n^*(t), t^{k-1} \right)_0 = \left[P_n^*(t), t^k \right]_1 = 0, \quad k = 0, 1, 2, \ldots, n-1,$$

which obviously implies that

$$P_n^*(t) = P_n(t) + A_n P_{n-1}(t), \tag{2.5}$$

where A_n is a real number. Next, let us calculate A_n, $n \geq 1$. To this end we are going to use the following equation

$$
\begin{aligned}
0 &= \left[P_n^*(t), 1 \right] = s_0^* P_n^*(0) + \left(\frac{P_n^*(t) - P_n^*(0)}{t}, 1 \right) \\
&= s_0^* P_n^*(0) + \left(\frac{P_n(t) - P_n(0)}{t}, 1 \right) + A_n \left(\frac{P_{n-1}(t) - P_{n-1}(0)}{t}, 1 \right) \\
&= s_0^* P_n^*(0) + \int_0^\infty \frac{P_n(t) - P_n(0)}{t} d\mu(t) + A_n \int_0^\infty \frac{P_{n-1}(t) - P_{n-1}(0)}{t} d\mu(t) \\
&= s_0^* (P_n(0) + A_n P_{n-1}(0)) + Q_n(0) + A_n Q_{n-1}(0) \\
&= s_0^* P_n(0) + Q_n(0) + A_n (s_0^* P_{n-1}(0) + Q_{n-1}(0)), \quad n \geq 1.
\end{aligned}
$$

We see that the equation is equivalent to the orthogonality of the polynomial $P_n + A_n P_{n-1}$ with respect to $[\cdot, \cdot]_1$. Hence, the equation has a unique solution and, so, one has

$$s_0^* P_{n-1}(0) + Q_{n-1}(0) \neq 0.$$

Furthermore, the unique solution of the above equation is

$$A_n = -\frac{s_0^* P_n(0) + Q_n(0)}{s_0^* P_{n-1}(0) + Q_{n-1}(0)}, \tag{2.6}$$

which leads us to formula (2.4). □

Finally, it is worth mentioning that the Geronimus transformation can be also considered in the case when $(\cdot, \cdot)_0$ is regular and is not necessarily positive definite [7]. Moreover, necessary and sufficient conditions for the regularity of $[\cdot, \cdot]_1$ are analyzed in [7, 10, 11].

3 The structure of the transformed Jacobi matrix

It is very well known that, assuming $[\cdot, \cdot]_1$ is positive definite, we can associate with the sequence of monic orthogonal polynomials $\{P_n^*\}_{n=0}^\infty$ a monic tridiagonal Jacobi matrix

$$
J_{mon}^* = \begin{pmatrix} b_0^* & 1 & & \\ \left(c_0^*\right)^2 & b_1^* & 1 & \\ & \left(c_1^*\right)^2 & b_2^* & \ddots \\ & & \ddots & \ddots \end{pmatrix}.
$$

Recall that the entries of J_{mon}^* are defined by the corresponding three-term recurrence relation

$$t P_j^*(t) = P_{j+1}^*(t) + b_j^* P_j^*(t) + \left(c_{j-1}^* \right)^2 P_{j-1}^*(t), \quad j \in \mathbb{Z}_+, \tag{3.1}$$

with the initial conditions

$$P^*_{-1}(t) = 0, \quad P^*_0(t) = 1,$$

where $b^*_j \in \mathbb{R}$ and $c^*_j > 0$, $j \in \mathbb{Z}_+$.

Depending on circumstances it can be also convenient to consider a symmetric tridiagonal Jacobi matrix

$$J^* = \begin{pmatrix} b^*_0 & c^*_0 & & \\ c^*_0 & b^*_1 & c^*_0 & \\ & c^*_1 & b^*_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix} = \Psi^{-1} J^*_{mon} \Psi,$$

where $\Psi = \mathrm{diag}\left(1, c^*_0, c^*_0 c^*_1, c^*_0 c^*_1 c^*_2, \dots\right)$. Indeed, J^* is the matrix of the multiplication operator with respect to the basis of orthonormal polynomials

$$\widehat{P}^*_n(t) = \frac{1}{h^*_n} P^*_n(t), \quad \left(h^*_n\right)^2 = \left[P^*_n, P^*_n\right]_1, \quad h^*_n > 0.$$

In other words, we have the following representation

$$J^* = \left(\left[t\widehat{P}^*_n(t), \widehat{P}^*_m(t)\right]_1\right)^\infty_{n,m=0}.$$

Since J^* corresponds to the Geronimus transformation, it has a special structure, which can be expressed in terms of the coefficients A_n, $n = 1, 2, \dots$, and the free parameter.

Theorem 3.1 (cf. [7]) *Let us assume that $(\cdot, \cdot)_0$, $[\cdot, \cdot]_1$ are positive definite and $\{P_n\}^\infty_{n=0}$ and $\{P^*_n\}^\infty_{n=0}$ are, respectively, the corresponding sequences of monic orthogonal polynomials. Then the matrix J^* admits the following Cholesky decomposition*

$$J^* = LL^\top, \tag{3.2}$$

where the bidiagonal lower triangular matrix L has the form

$$L = \begin{pmatrix} \frac{h_0}{s^*_0} & 0 & & & \\ A_1\frac{h_0}{s^*_0} & \frac{h_1}{\sqrt{A_1}h_0} & 0 & & \\ & \frac{A_2 h_1}{\sqrt{A_1}h_0} & \frac{h_2}{\sqrt{A_2}h_1} & 0 & \\ & & \frac{A_3 h_2}{\sqrt{A_2}h_1} & \frac{h_3}{\sqrt{A_3}h_2} & \ddots \\ & & & \ddots & \ddots \end{pmatrix}. \tag{3.3}$$

Proof We begin by noticing that

$$J^* = \begin{pmatrix} \frac{1}{h^*_0} & 0 & \\ 0 & \frac{1}{h^*_1} & \ddots \\ & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \left[tP^*_0(t), P^*_0(t)\right]_1 & \left[tP^*_0(t), P^*_1(t)\right]_1 & \\ \left[tP^*_1(t), P^*_0(t)\right]_1 & \left[tP^*_1(t), P^*_1(t)\right]_1 & \ddots \\ & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \frac{1}{h^*_0} & 0 & \\ 0 & \frac{1}{h^*_1} & \ddots \\ & \ddots & \ddots \end{pmatrix}. \tag{3.4}$$

Since $\left[t P_n^*(t),\, P_m^*(t)\right]_1 = \left(P_n^*(t),\, P_m^*(t)\right)_0$, the symmetric tridiagonal matrix in the middle of the right hand side of (3.4) reduces to

$$
\left((P_n^*,\, P_m^*)_0\right)_{n,m=0}^{\infty} =
\begin{pmatrix}
(P_0^*,\, P_0^*)_0 & (P_0^*,\, P_1^*)_0 & 0 \\
(P_1^*,\, P_0^*)_0 & (P_1^*,\, P_1^*)_0 & (P_1^*,\, P_2^*)_0 \\
0 & (P_2^*,\, P_1^*)_0 & (P_2^*,\, P_2^*)_0 & \ddots \\
& & & \ddots & \ddots
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
h_0^2 & A_1 h_0^2 & 0 \\
A_1 h_0^2 & h_1^2 + A_1^2 h_0^2 & A_2 h_1^2 \\
0 & A_2 h_1^2 & h_2^2 + A_2^2 h_1^2 & \ddots \\
& & & \ddots & \ddots
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
h_0 & 0 \\
A_1 h_0 & h_1 & 0 \\
& A_2 h_1 & h_2 & \ddots \\
& & & \ddots & \ddots
\end{pmatrix}
\begin{pmatrix}
h_0 & A_1 h_0 \\
0 & h_1 & A_2 h_1 \\
& 0 & h_2 & \ddots \\
& & & \ddots & \ddots
\end{pmatrix}
$$

in view of formula (2.5). Hence it is clear that (3.2) holds with

$$
L =
\begin{pmatrix}
\frac{1}{h_0^*} & 0 \\
0 & \frac{1}{h_1^*} & \ddots \\
& \ddots & \ddots
\end{pmatrix}
\begin{pmatrix}
h_0 & 0 \\
A_0 h_1 & h_1 & 0 \\
& A_2 h_1 & h_2 & \ddots \\
& & & \ddots & \ddots
\end{pmatrix}.
\tag{3.5}
$$

Now, observe that

$$
\begin{aligned}
\left(h_{n+1}^*\right)^2 &= \left[P_{n+1}^*,\, P_{n+1}^*\right]_1 = \left[t P_n^*(t),\, P_{n+1}^*(t)\right]_1 = \left(P_n^*,\, P_{n+1}^*\right)_0 \\
&= (P_n + A_n P_{n-1},\, P_{n+1} + A_{n+1} P_n)_0 \\
&= A_{n+1} h_n^2,
\end{aligned}
$$

which gives

$$
h_{n+1}^* = \sqrt{A_{n+1}}\, h_n^*.
$$

Combining this and $\left(h_0^*\right)^2 = s_0^*$ we get that (3.5) can be easily simplified to (3.3). $\qquad\square$

As a matter of fact, this statement is a trace of the fact that the Geronimus transformation can be interpreted in the matrix language (for details, see [7], as well as [11] for the non-regular case).

In order the paper to be self-contained a direct connection between the matrices J_{mon} and J^*_{mon} associated with the monic orthogonal polynomial sequences $\{P_n\}^\infty_{n=0}$, $\{P^*_n\}^\infty_{n=0}$, respectively, will be stated. Let

$$L_{mon} = \begin{pmatrix} 1 & 0 & & & \\ A_1 & 1 & 0 & & \\ 0 & A_2 & 1 & 0 & \\ 0 & 0 & A_3 & 1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \tag{3.6}$$

be an infinite matrix such that $P^* = L_{mon}P$, where $P^* = \left(P^*_0, P^*_1, \ldots\right)^\top$ and $P = (P_0, P_1, \ldots)^\top$. On the other hand, according to the Christoffel formula (see [9]) or, equivalently, to the fact that

$$\left[t\,P_n(t),\, P^*_m(t)\right]_1 = \left(P_n(t),\, P^*_m(t)\right)_0, \quad m = 0, \ldots, n-1,$$

we get the relation

$$t\,P_n(t) = P^*_{n+1}(t) + F_{n+1}P^*_n(t), \quad F_{n+1} \neq 0, \quad n \geq 0.$$

In matrix terms, we have that $t\,P = U_{mon}P^*$, where

$$U^\top_{mon} = \begin{pmatrix} F_1 & 0 & & & \\ 1 & F_2 & 0 & & \\ 0 & 1 & F_3 & 0 & \\ 0 & 0 & 1 & F_4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \tag{3.7}$$

Thus, we can state the following

Theorem 3.2 *We have that*

$$J_{mon} = U_{mon}L_{mon}, \tag{3.8}$$

$$J^*_{mon} = L_{mon}U_{mon}. \tag{3.9}$$

Proof Notice that

$$t\,P = U_{mon}P^* = U_{mon}L_{mon}P. \tag{3.10}$$

Thus, one gets

$$J_{mon} = U_{mon}L_{mon}. \tag{3.11}$$

On the other hand, we see that

$$t\,P^* = L_{mon}t\,P = L_{mon}U_{mon}P^*. \tag{3.12}$$

As a consequence, we arrive at

$$J^*_{mon} = L_{mon}U_{mon}. \tag{3.13}$$

$\square$

Thus, we have a simple proof of a very well known result (see [7]) in terms of a Darboux transformation with parameter (see also [10, 11] for similar results in the non-regular case).

4 The double Geronimus transformation and the Sobolev orthogonality

In this section we present the double Geronimus transformation in the framework of symmetric bilinear forms. Also, it is shown that this transformation leads to Sobolev inner products and, therefore, to Sobolev orthogonal polynomials.

First, let us clarify what we mean by the double Geronimus transformation.

Definition 4.1 Let us consider a symmetric bilinear form $(\cdot, \cdot)_0$. The double Geronimus transformation of $(\cdot, \cdot)_0$ is a symmetric bilinear form defined on the linear space $\mathcal{P}$ of polynomials with real coefficients as follows

$$[t^2 f(t), g(t)]_2 = [f(t), t^2 g(t)]_2 = (f, g)_0 = \int_{\mathbb{R}} f(t)g(t)d\mu(t), \quad f, g \in \mathcal{P}. \tag{4.1}$$

From (4.1) one can see that the form $[\cdot, \cdot]_2$ is not uniquely defined. In particular, the symmetric matrix (since the form is symmetric)

$$\begin{pmatrix} [1, 1] & [1, t] \\ [t, 1] & [t, t] \end{pmatrix} = \begin{pmatrix} s_0^{**} & s_1^{**} \\ s_1^{**} & s_2^{**} \end{pmatrix}$$

can be chosen arbitrarily. Despite this, one can see the structure of the double Geronimus transformation.

Proposition 4.2 *Suppose that $d\mu$ has the following representation*

$$d\mu(t) = t^2 d\mu_2(t), \tag{4.2}$$

where $d\mu_2$ is a positive measure and it has finite moments. Then the double Geronimus transformation of $(\cdot, \cdot)_0$ admits the representation

$$[f, g]_2 = \int_{\mathbb{R}} f(t)g(t)d\mu_2(t) + \begin{pmatrix} f(0) & f'(0) \end{pmatrix} M \begin{pmatrix} g(0) \\ g'(0) \end{pmatrix}, \tag{4.3}$$

where the symmetric matrix M has the following form

$$M = \begin{pmatrix} s_0^{**} & s_1^{**} \\ s_1^{**} & s_2^{**} \end{pmatrix} - \begin{pmatrix} \int_{\mathbb{R}} d\mu_2(t) & \int_{\mathbb{R}} t d\mu_2(t) \\ \int_{\mathbb{R}} t d\mu_2(t) & \int_{\mathbb{R}} t^2 d\mu_2(t) \end{pmatrix}.$$

Proof The proof is similar to that of Proposition 2.2. First, we see that

$$\begin{aligned} [f, g]_2 &= [f(t) - f(0) - tf'(0), g(t)]_2 + [f(0) + tf'(0), g(t)]_2 \\ &= \left(\frac{f(t) - f(0) - tf'(0)}{t^2}, g(t) \right) + [f(0) + tf'(0), g(t)]_2. \end{aligned}$$

Then, making use of the representation $g(t) = g(t) - g(0) - tg'(0) + g(0) + tg'(0)$ and taking into account (4.2) we get the desired result (4.3). □

Since the symmetric matrix M can be arbitrary, from formula (4.3) one can see that, in general, the double Geronimus transformation $[\cdot, \cdot]_2$ generates Sobolev type inner products. In particular, one recovers the positive diagonal Sobolev type inner products when

$$M = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \lambda_1 \geq 0 \text{ and } \lambda_2 > 0.$$

At the same time, we also get that the double Geronimus transformation adds a matrix mass at the point in some sense. Thus, the corresponding sequence of orthogonal polynomials, which are called in the literature Sobolev type orthogonal polynomials, are not already standard scalar orthogonal polynomials (the scalar Hankel structure of the Gram matrix is destroyed by the perturbation) but not yet essentially matrix orthogonal polynomials although it is convenient to consider them as matrix orthogonal since the Gram matrix is in fact a 2×2 block Hankel matrix (see [1] for some basic properties of Sobolev type orthogonal polynomials).

Now we are in a position to give an explicit formula for the transformed polynomials $\{P_n^{**}\}_{n=0}^\infty$ orthogonal with respect to $[\cdot, \cdot]_2$ in terms of the original polynomials $\{P_n\}_{n=0}^\infty$.

Theorem 4.3 *Let us assume that $(\cdot, \cdot)_0$ and $[\cdot, \cdot]_2$ are both positive definite and regular bilinear forms, respectively. Let $\{P_n\}_{n=0}^\infty$ be a sequence of monic polynomials orthogonal with respect to $(\cdot, \cdot)_0$. Then a monic polynomial P_n^{**} of degree n is orthogonal with respect to $[\cdot, \cdot]_2$ if and only if it can be represented as follows*

$$P_n^{**}(t) = \frac{1}{d_n^{**}} \begin{vmatrix} P_n(t) & R_n'\left(0; s_1^{**}\right) + s_0^{**} P_n(0) & R_n\left(0; s_1^{**}\right) + \left(s_2^{**} - s_0^{**}\right) P_n'(0) \\ P_{n-1}(t) & R_{n-1}'\left(0; s_1^{**}\right) + s_0^{**} P_{n-1}(0) & R_{n-1}\left(0; s_1^{**}\right) + \left(s_2^{**} - s_0^{**}\right) P_{n-1}'(0) \\ P_{n-2}(t) & R_{n-2}'\left(0; s_1^{**}\right) + s_0^{**} P_{n-2}(0) & R_{n-2}\left(0; s_1^{**}\right) + \left(s_2^{**} - s_0^{**}\right) P_{n-2}'(0) \end{vmatrix},$$
(4.4)

where $R_n(t; s) = s P_n(t) + Q_n(t)$, $R_n'(t; s) = s P_n'(t) + Q_n'(t)$, and

$$d_n^{**} = \begin{vmatrix} R_{n-1}'\left(0; s_1^{**}\right) + s_0^{**} P_{n-1}(0) & R_{n-1}\left(0; s_1^{**}\right) + \left(s_2^{**} - s_0^{**}\right) P_{n-1}'(0) \\ R_{n-2}'\left(0; s_1^{**}\right) + s_0^{**} P_{n-2}(0) & R_{n-2}\left(0; s_1^{**}\right) + \left(s_2^{**} - s_0^{**}\right) P_{n-2}'(0) \end{vmatrix}$$

is nonzero.

Proof The orthogonality of P_n^{**} is equivalent to the following condition

$$\left[P_n^{**}(t), t^k\right]_2 = 0, \quad k = 0, \ldots, n - 1,$$

which for $n \geq 3$ further reduces to

$$\left(P_n^{**}(t), t^{k-2}\right)_0 = 0, \quad k = 2, \ldots, n - 1.$$

The latter relation is obviously equivalent to the representation

$$P_n^{**}(t) = P_n(t) + B_n P_{n-1}(t) + C_n P_{n-2}(t). \tag{4.5}$$

Therefore one can see that the coefficients B_n and C_n are uniquely determined by the relations

$$\left[P_n^{**}(t), 1\right]_2 = 0,$$
$$\left[P_n^{**}(t), t\right]_2 = 0,$$

which can be rewritten as follows

$$[P_n(t), 1]_2 + B_n[P_{n-1}(t), 1]_2 + C_n[P_{n-2}(t), 1]_2 = 0,$$
$$[P_n(t), t]_2 + B_n[P_{n-1}(t), t]_2 + C_n[P_{n-2}(t), t]_2 = 0. \tag{4.6}$$

Since the monic orthogonal polynomial P_n^{**} of degree n is uniquely defined, the system (4.6) has a unique solution. Indeed, on the one hand, it is clear that the system has at least one solution because there exists a monic orthogonal polynomial of degree n. On the other hand, if it has two different solutions then these solutions would give two different monic orthogonal polynomials of degree n. The latter fact is not possible according the uniqueness of such a sequence. So, we conclude that the determinant

$$d_n^{**} = \begin{vmatrix} [P_{n-1}(t), 1]_2 & [P_{n-1}(t), t]_2 \\ [P_{n-2}(t), 1]_2 & [P_{n-2}(t), t]_2 \end{vmatrix}$$

is nonzero and the orthogonality of P_n^{**} is equivalent to the representation

$$P_n^{**}(t) = \frac{1}{d_n^{**}} \begin{vmatrix} P_n(t) & [P_n(t), 1]_2 & [P_n(t), t]_2 \\ P_{n-1}(t) & [P_{n-1}(t), 1]_2 & [P_{n-1}(t), t]_2 \\ P_{n-2}(t) & [P_{n-2}(t), 1]_2 & [P_{n-2}(t), t]_2 \end{vmatrix}.$$

Now, to get formula (4.4) it remains to re-express the entries of the corresponding determinants. To this end, for the last column one can get

$$
\begin{aligned}
[P_n(t), t]_2 &= \left[P_n(t) - P_n(0) - t P_n'(0), t\right]_2 + P_n(0)[1, t]_2 + P_n'(0)[t, t]_2 \\
&= \left(\tfrac{P_n(t) - P_n(0) - t P_n'(0)}{t^2}, t\right)_0 + s_1^{**} P_n(0) + s_2^{**} P_n'(0) \\
&= \left(\tfrac{P_n(t) - P_n(0) - t P_n'(0)}{t}, 1\right)_0 + s_1^{**} P_n(0) + s_2^{**} P_n'(0) \\
&= Q_n(0) + s_1^{**} P_n(0) + \left(s_2^{**} - s_0^{**}\right) P_n'(0) \\
&= R_n\left(0; s_1^{**}\right) + \left(s_2^{**} - s_0^{**}\right) P_n'(0).
\end{aligned}
$$

Next, we also have that

$$
\begin{aligned}
[P_n(t), 1]_2 &= \left[P_n(t) - P_n(0) - t P_n'(0), 1\right]_2 + P_n(0)[1, 1]_2 + P_n'(0)[t, 1]_2 \\
&= \left(\tfrac{P_n(t) - P_n(0) - t P_n'(0)}{t^2}, 1\right)_0 + s_0^{**} P_n(0) + s_1^{**} P_n'(0).
\end{aligned}
$$

On the other hand, notice that

$$
\begin{aligned}
Q_n'(0) &= \lim_{\epsilon \to 0} \tfrac{Q_n(\epsilon) - Q_n(0)}{\epsilon} \\
&= \lim_{\epsilon \to 0} \tfrac{1}{\epsilon} \left(\left(\tfrac{P_n(t) - P_n(\epsilon)}{t - \epsilon}, 1\right)_0 - \left(\tfrac{P_n(t) - P_n(0)}{t}, 1\right)_0 \right) \\
&= \lim_{\epsilon \to 0} \tfrac{1}{\epsilon} \int_{\mathbb{R}} \tfrac{\epsilon(P_n(t) - P_n(0)) - t(P_n(\epsilon) - P_n(0))}{t(t - \epsilon)} d\mu(t) \\
&= \int_{\mathbb{R}} \tfrac{P_n(t) - P_n(0) - t P_n'(0)}{t^2} d\mu(t) \\
&= \left(\tfrac{P_n(t) - P_n(0) - t P_n'(0)}{t^2}, 1\right)_0.
\end{aligned}
$$

Note that we can interchange the limit and integral due to Lebesgue's dominated convergence theorem. Finally, we get

$$
\begin{aligned}
[P_n(t), 1]_2 &= Q'_n(0) + s_0^{**} P_n(0) + s_1^{**} P'_n(0) \\
&= R'_n\left(0; s_1^{**}\right) + s_0^{**} P_n(0),
\end{aligned}
$$

which completes the proof. □

5 The structure of the transformed pentadiagonal matrix

In the case when the sequence of polynomials $\left\{P_n^{**}\right\}_{n=0}^{\infty}$ is orthogonal with respect to an inner product of the form (4.3), it is quite natural to consider the matrix representation of the square of the multiplication operator [12, 13]. Indeed, according to formula (4.3) the multiplication operator is not necessarily symmetric with respect to $[\cdot, \cdot]_2$ and as a consequence one cannot apply some classical tricks in this case. Nevertheless, the square of the multiplication operator is symmetric by the definition of the double Geronimus transformation. Thus, the classical machinery works for this symmetric operator. So, assuming $[\cdot, \cdot]_2$ is positive definite, let us introduce the following symmetric matrix

$$
J^{**} = \left(\left[t^2 \widehat{P}_n^{**}(t), \widehat{P}_m^{**}(t)\right]_2\right)_{n,m=0}^{\infty},
$$

where the corresponding orthonormal polynomials $\left\{\widehat{P}_n^{**}\right\}_{n=0}^{\infty}$ are given by

$$
\widehat{P}_n^{**}(t) = \frac{1}{h_n^{**}} P_n^{**}(t), \quad \left(h_n^{**}\right)^2 = \left[P_n^{**}, P_n^{**}\right]_2, \quad h_n^{**} > 0.
$$

In fact, J^{**} is pentadiagonal and the following statement holds true.

Theorem 5.1 *Let us assume that* $(\cdot, \cdot)_0$, $[\cdot, \cdot]_2$ *are positive definite and* $\{P_n\}_{n=0}^{\infty}$, $\left\{P_n^{**}\right\}_{n=0}^{\infty}$, *respectively, are the corresponding sequences of monic orthogonal polynomials. Then the matrix* J^{**} *admits the following Cholesky decomposition*

$$
J^{**} = LL^{\top}, \tag{5.1}
$$

where the lower triangular matrix L has only three nonvanishing diagonals and is of the form

$$
L = \begin{pmatrix}
\frac{h_0}{h_0^{**}} & 0 & & & \\
B_1 \frac{h_0}{h_0^{**}} & \frac{h_1}{h_1^{**}} & 0 & & \\
C_2 \frac{h_0}{h_0^{**}} & B_2 \frac{h_1}{h_1^{**}} & \frac{h_2}{h_2^{**}} & 0 & \\
0 & C_3 \frac{h_1}{h_1^{**}} & B_3 \frac{h_2}{h_2^{**}} & \frac{h_3}{h_3^{**}} & \ddots \\
& & & \ddots & \ddots
\end{pmatrix}, \tag{5.2}
$$

where the ratio $h_{n+2}/h_n^{**}{}_2$ can be expressed in terms of the coefficients B_n and C_n, $n = 1, 2, \ldots$ of the linear^{+} combination as follows

$$\frac{h_{n+2}}{h_{n+2}^{**}} = \frac{h_{n+2}}{\sqrt{C_{n+2}}h_n}, \quad n = 0, 1, \ldots. \tag{5.3}$$

Remark 5.2 It should be stressed that h_0^{**} and h_1^{**} can be parametrized by the free parameters:

$$\left(h_0^{**}\right)^2 = s_0^{**}, \quad \left(h_1^{**}\right)^2 = s_2^{**} + s_1^{**}\left(B_1 - \frac{s_1}{s_0}\right).$$

Proof Obviously, we have that

$$J^{**} = \begin{pmatrix} \frac{1}{h_0^{**}} & 0 & \\ 0 & \frac{1}{h_1^{**}} & \ddots \\ & \ddots & \ddots \end{pmatrix} \begin{pmatrix} [t^2 P_0^{**}(t), P_0^{**}(t)]_2 & [t^2 P_0^{**}(t), P_1^{**}(t)]_2 & \\ [t^2 P_1^{**}(t), P_0^{**}(t)]_2 & [t^2 P_1^{**}(t), P_1^{**}(t)]_2 & \ddots \\ & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \frac{1}{h_0^{**}} & 0 & \\ 0 & \frac{1}{h_1^{**}} & \ddots \\ & \ddots & \ddots \end{pmatrix}. \tag{5.4}$$

Since $\left[t^2 P_n^{**}(t), P_m^{**}(t)\right]_2 = \left(P_n^*(t), P_m^*(t)\right)_0$, the pentadiagonal matrix in the middle of the right hand side of (5.4) reduces to

$$\left(\left(P_n^{**}, P_m^{**}\right)_0\right)_{n,m=0}^{\infty} = \begin{pmatrix} \left(P_0^{**}, P_0^{**}\right)_0 & \left(P_0^{**}, P_1^{**}\right)_0 & \left(P_0^{**}, P_2^{**}\right)_0 & 0 \\ \left(P_1^{**}, P_0^{**}\right)_0 & \left(P_1^{**}, P_1^{**}\right)_0 & \left(P_1^{**}, P_2^{**}\right)_0 & \left(P_1^{**}, P_3^{**}\right)_0 \\ \left(P_2^{**}, P_0^{**}\right)_0 & \left(P_2^{**}, P_1^{**}\right)_0 & \left(P_2^{**}, P_2^{**}\right)_0 & \left(P_2^{**}, P_3^{**}\right)_0 & \ddots \\ 0 & \left(P_3^{**}, P_1^{**}\right)_0 & \left(P_3^{**}, P_2^{**}\right)_0 & \left(P_3^{**}, P_3^{**}\right)_0 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} h_0^2 & B_1 h_0^2 & C_2 h_0^2 & 0 \\ B_1 h_0^2 & h_1^2 + B_1^2 h_0^2 & B_2 h_1^2 + B_1 C_2 h_0^2 & C_3 h_1^2 \\ C_2 h_0^2 & B_2 h_1^2 + B_1 C_2 h_0^2 & h_2^2 + B_2^2 h_1^2 + C_2^2 h_0^2 & B_3 h_2^2 + B_2 C_3 h_1^2 & \ddots \\ 0 & C_3 h_1^2 & B_3 h_2^2 + B_2 C_3 h_1^2 & h_3^2 + B_3^2 h_2^2 + C_3^2 h_1^2 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} h_0 & 0 & & \\ B_1 h_0 & h_1 & 0 & \\ C_2 h_0 & B_2 h_1 & h_2 & 0 \\ 0 & C_3 h_1 & B_3 h_2 & h_3 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} h_0 & B_1 h_0 & C_2 h_0 & \\ 0 & h_1 & B_2 h_1 & C_3 h_1 \\ & 0 & h_2 & B_3 h_2 & \ddots \\ & & 0 & h_3 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

in view of formula (4.5). Hence it is clear that (5.1) holds with

$$
L = \begin{pmatrix} \frac{1}{h_0^{**}} & 0 & \\ 0 & \frac{1}{h_1^{**}} & \ddots \\ & \ddots & \ddots \end{pmatrix} \begin{pmatrix} h_0 & 0 & & & \\ B_1 h_0 & h_1 & 0 & & \\ C_2 h_0 & B_2 h_1 & h_2 & 0 & \\ 0 & C_3 h_1 & B_3 h_2 & h_3 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}.
$$

(5.5)

Thus we arrive at (5.2) after simple computations. Finally, it remains to see that

$$
\begin{aligned}
\left(h_{n+1}^{**}\right)^2 &= \left[P_{n+2}^{**}, P_{n+2}^{**}\right]_2 = \left[t^2 P_n^{**}(t), P_{n+2}^*(t)\right]_2 = \left(P_n^{**}, P_{n+2}^{**}\right)_0 \\
&= (P_n + B_n P_{n-1} + C_n P_{n-2}, P_{n+2} + B_{n+2} P_{n+1} + C_{n+2} P_n)_0 \\
&= C_{n+2} h_n^2,
\end{aligned}
$$

which gives (5.3). $\qquad\square$

The next step will be to establish a direct connection between the matrices J_{mon} and J_{mon}^{**} associated with the monic orthogonal polynomial sequences $\{P_n\}_{n=0}^{\infty}$, $\{P_n^{**}\}_{n=0}^{\infty}$, respectively.
Let

$$
L_{mon} = \begin{pmatrix} 1 & 0 & & & \\ B_1 & 1 & 0 & & \\ C_2 & B_2 & 1 & 0 & \\ 0 & C_3 & B_3 & 1 & \ddots \\ & & \ddots & \ddots \end{pmatrix}
$$

(5.6)

be an infinite matrix such that $P^{**} = L_{mon} P$, where $P^{**} = \left(P_0^{**}, P_1^{**}, \dots\right)^{\top}$ and $P = (P_0, P_1, \dots)^{\top}$. At the same time, from the equality

$$
\left[t^2 P_n(t), P_m^{**}(t)\right]_2 = \left[P_n(t), P_m^{**}(t)\right]_0 = 0, \quad m = 0, 1, \dots, n-1,
$$

one concludes that

$$
t^2 P_n(t) = P_{n+2}^{**}(t) + D_{n+1} P_{n+1}^{**}(t) + E_{n+1} P_n^{**}(t), \quad E_{n+1} \neq 0, \quad n = 0, 1, \dots.
$$

The above connection formula reads in a matrix form as $t^2 P = U_{mon} P^{**}$, where

$$
U_{mon}^{\top} = \begin{pmatrix} E_1 & 0 & & & & \\ D_1 & E_2 & 0 & & & \\ 1 & D_2 & E_3 & 0 & & \\ 0 & 1 & D_3 & E_4 & 0 & \\ 0 & 0 & 1 & D_4 & E_5 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}
$$

(5.7)

Thus, we can state the following.

Theorem 5.3 *We have that*

$$J_{mon}^2 = U_{mon} L_{mon}, \tag{5.8}$$

$$J_{mon}^{**} = L_{mon} U_{mon}. \tag{5.9}$$

Proof Notice that

$$t^2 P = U_{mon} P^{**} = U_{mon} L_{mon} P. \tag{5.10}$$

Thus, one sees that

$$J_{mon}^2 = U_{mon} L_{mon}. \tag{5.11}$$

On the other hand, we have

$$t^2 P^{**} = L_{mon} t^2 P = L_{mon} U_{mon} P^{**}. \tag{5.12}$$

As a consequence, one gets

$$J_{mon}^{**} = L_{mon} U_{mon}. \tag{5.13}$$

$\square$

Notice that this is the analogue for pentadiagonal matrices of the Darboux transformation with parameter considered in Section 3. Moreover, the structure of the matrix representing the multiplication operator by t^2 with respect to the orthonormal polynomial basis associated with the inner product $[\cdot, \cdot]_2$ is stated in terms of the corresponding UL factorization of the pentadiagonal matrix J^2.

References

1. Alfaro, M., Marcellán, F., Rezola, M.L., Ronveaux, A.: On orthogonal polynomials of Sobolev type: Algebraic properties and zeros. SIAM J. Math. Anal. **23**(3), 737–757 (1992)
2. Alfaro, M., Marcellán, F., Peña, A., Rezola, M.L.: When do linear combinations of orthogonal polynomials yield new sequences of orthogonal polynomials? J. Comput. Appl. Math. **233**(6), 1446–1452 (2010)
3. Alfaro, M., Peña, A., Rezola, M.L., Marcellán, F.: Orthogonal polynomials associated with an inverse quadratic spectral transform. Comput. Math. Appl. **61**(4), 888–900 (2011)
4. Beghdadi, D., Maroni, P.: On the inverse problem of the product of a semi-classical form by a polynomial. J. Comput. Appl. Math **88**(2), 377–399 (1998)
5. Branquinho, A., Marcellán, F.: Generating new classes of orthogonal polynomials. Int. J. Math. Math. Sci. **19**(4), 643–656 (1996)
6. Bultheel, A., Van Barel, M.: Formal orthogonal polynomials and rational approximation for arbitrary bilinear forms, Tech. Report TW 163. Department of Computer Science, KU Leuven (Belgium) (1991)
7. Bueno, M.I., Marcellán, F.: Darboux transformation and perturbation of linear functionals. Linear Algebra Appl. **384**, 215–242 (2004)
8. Bueno, M.I., Marcellán, F.: Polynomial perturbations of bilinear functionals and Hessenberg matrices. Linear Algebra Appl. **414**(1), 64–83 (2006)

9. Chihara, T.S.: An introduction to orthogonal polynomials. In: Mathematics and its Applications, vol. 13. Gordon and Breach Science Publishers, New York (1978)
10. Derevyagin, M.: On the relation between Darboux transformations and polynomial mappings. J. Approx. Theory **172**, 4–22 (2013)
11. Derevyagin, M., Derkach, V.: Darboux transformations of Jacobi matrices and Padé approximation. Linear Algebra Appl. **435**(12), 3056–3084 (2011)
12. Durán, A.J.: A generalization of Favard's theorem for polynomials satisfying a recurrence relation. J. Approx. Theory **74**(1), 83–109 (1993)
13. Evans, D., Littlejohn, L.L., Marcellán, F., Markett, C., Ronveaux, A.: On recurrence relations for Sobolev polynomials, SIAM. J. Math. Anal **26**(2), 446–467 (1995)
14. Geronimus, J.: On polynomials orthogonal with regard to a given sequence of numbers. Comm. Inst. Sci. Math. Mec. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] **17**, 3–18 (1940). Series 4. (Russian)
15. Hounga, C., Hounkonnou, M.N., Ronveaux, A.: New families of orthogonal polynomials. J. Comput. Appl. Math. **193**(2), 474–483 (2006)
16. Iliev, P.: Krall-Laguerre commutative algebras of ordinary differential operators. Ann. Mat. Pura Appl. **192**(2), 203–224 (2013). Series 4
17. Kwon, K.H., Lee, D.W., Marcellán, F., Park, S.B.: On kernel polynomials and self-perturbation of orthogonal polynomials. Ann. Mat. Pura Appl. **180**(2), 127–146 (2001). Series 4
18. Maroni, P.: Sur la suite de polynômes orthogonaux associée à la forme $u = \delta_c + \lambda(x - c)^{-1}$. Period. Math. Hungar. **21**(3), 223–248 (1990). (French)
19. Maroni, P., Nicolau, I.: On the inverse problem of the product of a form by a polynomial: the cubic case. Appl. Numer. Math. **45**(4), 419–451 (2003)
20. Maroni, P., Sfaxi, R.: Diagonal orthogonal polynomial sequences. Methods Appl. Anal. **7**(4), 769–791 (2000)
21. Shohat, J.: On mechanical quadratures, in particular, with positive coefficients. Trans. Amer. Math. Soc **42**(3), 461–496 (1937)
22. Spiridonov, V., Zhedanov, A.: Discrete Darboux transformations, the discrete-time Toda lattice, and the Askey-Wilson polynomials. Methods Appl. Anal. **2**(4), 369–398 (1995)
23. Spiridonov, V., Zhedanov, A.: Discrete-time Volterra chain and classical orthogonal polynomials. J. Phys. A. Math. Gen **30**, 8727–8737 (1997)
24. Zhedanov, A.: Rational spectral transformations and orthogonal polynomials. J. Comput. Appl. Math **85**(1), 67–86 (1997)