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RESEARCH MEMORANDUM

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A NOTE ON THE IDENTIFICATION OF
RESTRICTED FACTOR LOADING MATRICES

by Paul A. Bekker

1985

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Abstract

It is shown that problems of rotational equivalence of restricted factor loading matrices in orthogonal factor analysis are equivalent to problems of identification in simultaneous equations systems with covariance restrictions. A necessary and sufficient condition for local uniqueness is given and a counterexample is provided to a theorem by J. Algina concerning necessary and sufficient conditions for global uniqueness.

Key Words: identification, orthogonal rotation, confirmatory factor analysis

1. Introduction

Consider the orthogonal factor analysis model $\Sigma = \Lambda\Lambda' + \psi$, where Σ is a positive semi-definite p×p-matrix which can be consistently estimated, ψ is a p×p-diagonal matrix of unique variances, Λ is a p×m-factor loading matrix of full column rank ($p > m$). Identifiability of the matrices Λ and ψ amounts to the question whether these matrices can be solved uniquely from these equations. In what follows, it is assumed that the matrix ψ is identified. Thus identifiability of Λ stems from the properties of the equations $C = \Sigma - \psi = \Lambda\Lambda'$. In general, these equations do not yield a unique solution for Λ. Identifiability may be aided by invoking prior information represented by linear restrictions on the columns of $\Lambda = (\lambda_1,\ldots,\lambda_m)$. The identifying equations for Λ are thus given by

$$(1) \qquad C = \Lambda\Lambda',$$

$$(2) \qquad R_j\lambda_j = r_j \quad , \; j = 1,\ldots,m,$$

where R_j is a k_j×p-matrix and r_j a k_j-vector, representing k_j linear restrictions on λ_j.

In his article on confirmatory factor analysis Jöreskog (1969) attempted to give conditions for uniqueness of Λ based on the specification of $m(m-1)/2$ elements. Dunn (1973) and Jennrich (1978) gave counterexamples showing that the conditions given by Jöreskog were not sufficient. Dunn also gave sufficient conditions for identification (up to column sign changes) based on the specification of $m(m-1)/2$ zeros. Jennrich showed that when the $m(m-1)/2$ specified values are not necessarily zero, the number of solutions for Λ equals 2^m. Geweke and Singleton (1981) gave the same conditions as Dunn had given and noticed that these conditions are analogous to sufficient conditions for the identifiability of simultaneous equations systems with uncorrelated disturbances. Recently, Hausman and Taylor (1983), Hausman, Newey and Taylor (1983) and Bekker and Pollock (1984) treated the problem of identification in simultaneous equations systems with zero covariance restrictions on the disturbances. It may therefore be helpful to further develop the analogy noticed by Geweke and Singleton (1981). Another purpose of this note is to demonstrate that Algina's (1980) necessary and sufficient conditions for identification, in his theorem 3, are, in fact, not necessary and consequently, that the necessity part of his proof is wrong.

2. An Analogy

The equations in (1) can be seen as a special case of the more general equations $C = \Lambda\Phi\Lambda'$, where Φ is a positive definite m×m-matrix. Let Λ be partitioned as $\Lambda' = (\Lambda_1', \Lambda_2')$, with a corresponding partitioning of C. It is assumed, without loss of generality, that Λ_1 and C_{11} are nonsingular m×m-matrices. Furthermore, let A be an arbitrary positive definite (p-m)×(p-m)-matrix. Then

$$(3) \qquad H \equiv \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}+A \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ \Lambda_2 & I \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \Lambda_1' & \Lambda_2' \\ 0 & I \end{bmatrix} \quad ,$$

and H is positive definite. Clearly then

$$(4) \qquad \begin{bmatrix} \Lambda_1' & \Lambda_2' \\ 0 & I \end{bmatrix} H^{-1} \begin{bmatrix} \Lambda_1 & 0 \\ \Lambda_2 & I \end{bmatrix} = \begin{bmatrix} \Phi^{-1} & 0 \\ 0 & A^{-1} \end{bmatrix} .$$

The classical simultaneous equations system of econometrics is given by $y'B + x'\Gamma = \varepsilon'$, where y' is a m-vector of endogenous variables, x' is a (p-m)-vector of exogenous variables, B is a nonsingular m×m-matrix, Γ is a (p-m)×m-matrix, and the m disturbances in ε' are uncorrelated with the exogenous variables in x'. Denoting covariance matrices by Σ, with subscripts refering to the variables, it follows that

$$(5) \qquad \begin{bmatrix} \Gamma' & B' \\ 0 & I \end{bmatrix} \begin{bmatrix} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{bmatrix} \begin{bmatrix} B & 0 \\ \Gamma & I \end{bmatrix} = \begin{bmatrix} \Sigma_{\varepsilon\varepsilon} & 0 \\ 0 & \Sigma_{xx} \end{bmatrix} .$$

The analogy with (4) is obvious. The structural coefficients in B and Γ correspond to Λ_1 and Λ_2 respectively and the covariances of the disturbances in $\Sigma_{\varepsilon\varepsilon}$ correspond to Φ^{-1}.

When Φ is unrestricted as in oblique factor analysis, then also Φ^{-1} is unrestricted and necessary and sufficient conditions for global identification, with a unique solution for Λ, may be adapted from similar identification conditions for simultaneous equations systems. These conditions were given by many authors, among others by Koopmans and Reiersøl (1950), Anderson and Rubin (1956) and, more recently by Algina (1980) and Geweke and Singleton (1981).

When Φ is restricted to be block diagonal, then also Φ^{-1} is block diagonal and conditions for identification of Λ are similar to conditions for identification in simultaneous equations systems where $\Sigma_{\varepsilon\varepsilon}$ is block diagonal. Conditions for identification in these cases were given by F.M. Fisher (1966). For example, Dunn's condition can be seen as a special case of Fisher's conditions. Conditions for the identification of single rows of (B',Γ'), or Λ', were given by Hausman and Taylor (1983) and Bekker and Pollock (1984). A necessary and sufficient condition for local identification, based on the Jacobian matrix of the equations in (5) is given by Bekker and Pollock (1984) and Hausman, Newey and Taylor (1983). An analogous condition for orthogonal factor analysis is the following. Let K be a permutation matrix, i.e. K is a partitioned matrix of order $m^2 \times m^2$ whose (j,i)-th block is an $m\times m$-matrix which has a unit in the (i,j)-th position and zeros elsewhere: $K = \sum_i \sum_j (e_j e_i' \otimes e_i e_j')$. Let the restrictions (2) be written as $R\ \mathrm{vec}(\Lambda) = r$, where R is a $\sum_j k_j \times mp$-matrix and r is a $\sum_j k_j$ vector. It is assumed that Λ is a regular point of the matrix:

$$(6)\qquad F(\Lambda) \equiv \begin{bmatrix} I_{m^2}+K \\ \\ R(I_m \otimes \Lambda) \end{bmatrix},$$

i.e. the rank of $F(\Lambda)$ is constant in an open neighbourhood of Λ. The condition is the following.

<u>Proposition</u>: A necessary and sufficient conditions for a solution Λ of the equations (1) and (2) to be locally unique is that rank $\{F(\Lambda)\} = m^2$.

The proof is omitted.

Algina (1980) has given a much simpler necessary and sufficient condition for global identification of Λ. However, by using an example, similar to an example given by Bekker and Pollock (1984), I will show that Algina's condition is not necessary.

3. A counterexample

Algina's theorem 3 is as follows: Let E_j denote the first $(j-1)$ columns of the m×m identity matrix, then, under the restrictions given by (2), where $r_j \neq 0$, a necessary and sufficient condition for identification of Λ is that $\mathrm{rank}(R_1\Lambda) = m$ and $\mathrm{rank}(\Lambda'R_j', E_j) = m$ for $j = 2,\ldots,m$.

In order to give a counterexample to this theorem, let Λ be partitioned as above, and let Λ_1 be a 3×3 nonsingular matrix which is restricted as

$$(7) \qquad \Lambda_1 = \begin{bmatrix} \lambda_{11} & \lambda_{12} & 0 \\ 0 & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & 0 & \lambda_{33} \end{bmatrix} .$$

Given these restrictions, one of the equations in $C_{11} = \Lambda_1\Lambda_1'$ is

$$(8) \qquad (C_{11})_{31} = \lambda_{31}\lambda_{11}.$$

Considering the equations $\Lambda_1'C_{11}^{-1}\Lambda_1 = I$, it follows that

$$(9) \qquad (C_{11}^{-1})_{11}\lambda_{11}^2 + 2(C_{11}^{-1})_{13}\lambda_{11}\lambda_{31} + (C_{11}^{-1})_{33}\lambda_{31}^2 = 1.$$

Any two solutions $\Lambda_1^{(1)}$ and $\Lambda_1^{(2)}$, say, must thus satisfy

$$(10) \qquad \lambda_{31}^{(1)}\lambda_{11}^{(1)} = \lambda_{31}^{(2)}\lambda_{11}^{(2)} = (C_{11})_{31},$$

and

$$(11) \qquad (C_{11}^{-1})_{11}\lambda_{11}^{(1)^2} + (C_{11}^{-1})_{33}\lambda_{31}^{(1)^2} = (C_{11}^{-1})_{11}\lambda_{11}^{(2)^2} + (C_{11}^{-1})_{33}\lambda_{31}^{(2)^2} .$$

Multiplying (11) by $\lambda_{11}^{(1)^2}\lambda_{11}^{(2)^2}$ and using (10) yields

$$(12) \qquad (C_{11}^{-1})_{11}\lambda_{11}^{(1)^2}\lambda_{11}^{(2)^2}(\lambda_{11}^{(1)^2}-\lambda_{11}^{(2)^2}) = (C_{11}^{-1})_{33}(C_{11})_{31}^2(\lambda_{11}^{(1)^2}-\lambda_{11}^{(2)^2}).$$

Therefore, if $\lambda_{11}^{(1)^2} \neq \lambda_{11}^{(2)^2}$, it must hold true that

$$\lambda_{11}^{(1)^2}\lambda_{11}^{(2)^2} = \frac{(C_{11}^{-1})_{33}(C_{11})_{31}^2}{(C_{11}^{-1})_{11}} . \tag{13}$$

This equation shows that if $\lambda_{11}^{(1)}$, or $-\lambda_{11}^{(1)}$, is a solution for λ_{11}, then there can be at most two other solutions $\lambda_{11}^{(2)}$ and $-\lambda_{11}^{(2)}$ which differ only by sign. Analogous expressions hold for the other parameters. Furthermore, equation (10) shows that, within a single column of Λ_1, the value of one parameter, λ_{11} say, fixes the value of the other free parameter λ_{31}. Therefore, up to column sign changes, there are no more than two distinct solutions for Λ_1. As $\Lambda_2 = C_{21}\Lambda_1'^{-1}$, each solution for Λ_1 corresponds to a solution for Λ_2. Consequently, if Λ_1 is restricted according to (7), there are, up to column sign changes, no more than two distinct solutions for Λ, i.e. Λ is locally identified.

A nummerical example will show that the two distinct solutions may indeed exist. Let C_{11} be the following matrix:

$$C_{11} = \frac{1}{100}\begin{bmatrix} 81 & 54 & 45 \\ 54 & 86 & 5 \\ 45 & 5 & 50 \end{bmatrix} , \tag{14}$$

then two solutions for Λ_1 are given by:

$$\Lambda_1^{(1)} = \frac{1}{10}\begin{bmatrix} 3.(5)^{\frac{1}{2}} & 6 & 0 \\ 0 & 9 & (5)^{\frac{1}{2}} \\ 3.(5)^{\frac{1}{2}} & 0 & (5)^{\frac{1}{2}} \end{bmatrix} , \tag{15}$$

and

$$\Lambda_1^{(2)} = \frac{1}{10}\begin{bmatrix} 9.(\frac{10}{19})^{\frac{1}{2}} & \frac{27}{(19)^{\frac{1}{2}}} & 0 \\ 0 & 2.(19)^{\frac{1}{2}} & (10)^{\frac{1}{2}} \\ (\frac{95}{2})^{\frac{1}{2}} & 0 & \frac{1}{2}(10)^{\frac{1}{2}} \end{bmatrix} .$$

Of course, within each column signs may be changed.

Dunn's specification of zeros leads to local identification of Λ where Λ is unique up to column sign changes. The example shows that specification of zeros may also lead to local identification of Λ where Λ is not unique up to column sign changes. Further specification of a value, e.g. $\lambda_{11} = 3.(5)^{\frac{1}{2}}$ leads to uniqueness of Λ up to sign changes of the second and third columns. If also the values $\lambda_{22} = 9$ and $\lambda_{33} = (5)^{\frac{1}{2}}$ are specified, then Λ is globally unique. This last specification may serve as a counterexample to Algina's theorem 3. The restrictions on the columns, $R_j\lambda_j = r_j$, are such that $r_j \neq 0$, so that, according to the theorem, global identification of Λ implies the existence of at least one column, λ_k say, such that $\mathrm{rank}(R_k\Lambda) = 3$. That is to say, at least one column must have three specified values. In the example, specification of only two values in each column leads to global identification. Therefore Algina's conditions are not necessary for identification and consequently, the necessity part of his proof is wrong.

4. Discussion

The sufficient conditions of Dunn (1973), Jennrich (1978), and Algina (1980) for identification of Λ are all based on the same principle. The columns of Λ are identified sequentially. In the first step a column is identified by means of linear restrictions $R_1\lambda_1 = r_1$ in (2) and $(0,I)H^{-1}\lambda_1 = 0$ in (4) and a normalization $\lambda_1'H^{-1}\lambda_1 = 1$ in (4). In the second step another column is identified using similar restrictions and an additional bilinear restriction $\lambda_1'H^{-1}\lambda_2 = 0$ in (4), which becomes linear if λ_1 is previously identified. If λ_1 and λ_2 are previously identified, then the third column has two additional linear restrictions $(\lambda_1,\lambda_2)'H^{-1}\lambda_3 = 0$, etc..

In this procedure the bilinear restrictions $\Lambda'H^{-1}\Lambda = I$ in (4) behave as if they were linear restrictions on separate columns of Λ. As all restrictions are linear, apart from the normalizations, the consequent conditions for uniqueness of Λ are relatively simple. However, these conditions, including Algina's condition, are not necessary. The example presented here shows that Λ may be locally identified while no column of Λ is previously identified by means of linear restrictions and a normalization. In such cases the restrictions in $\Lambda'H^{-1}\Lambda = I$ truly behave as bilinear restrictions which cannot be reduced to linear restrictions, and the columns of Λ are identified simultaneously instead of sequentially. A necessary and sufficient condition for identification must therefore be a system-wide condition instead of a sequence of relatively simple conditions for separate columns.

The necessary and sufficient condition for local identification of Λ that is presented here is system-wide. However, the evaluation of the rank of the matrix $F(\Lambda)$ may be very complicated. Therefore, in my opinion, the problem of identification of Λ, or the equivalent problem of identifiability of simultaneous equations systems with covariance restrictions, cannot be considered as completely solved. Moreover, an additional problem is encountered in confirmatory factor analysis. Usually it is assumed, just as in this note, that the matrix of unique variances ψ is identified. Although it is not hard to give necessary and sufficient conditions for the identification of ψ in case there are no prior restrictions on elements of Λ (see for instance Shapiro (1983)), restrictions on Λ may affect the identification of ψ. Therefore the overall identification problem in confirmatory factor analysis is far from being solved.

References

Algina, J., 1980, A note on identification in the oblique and orthogonal factor analysis models, Psychometrika 45, 393-396.

Anderson, T.W. and H. Rubin, 1956, Statistical inference in factor analysis, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability 5, 111-150, (Jerzy Neyman, ed.), University of California Press.

Bekker, P.A. and D.S.G. Pollock, 1984, Identification of linear stochastic models with covariance restrictions, Research Memorandum 159, Tilburg University.

Dunn, J.E., 1973, A note on a sufficiency condition for uniqueness of a restricted factor matrix, Psychometrika 38, 141-143.

Fisher, F.M., 1966, The identification problem in econometrics, McGraw-Hill, New York.

Geweke, J.F. and K.J. Singleton, 1981, Maximum likelihood confirmatory factor analysis of economic time series, International Economic Review 22, 37-54.

Hausman, J.A. and W.E. Taylor, 1983, Identification in linear simultaneous equations models with covariance restrictions: an instrumental variables interpretation, Econometrica 51, 1527-1549.

Hausman, J.A., W. Newey and W. Taylor, 1983, Efficient estimation and identification of simultaneous equation models with covariance restrictions, MIT Working Paper 331.

Jennrich, R.I., 1978, Rotational equivalence of factor loading matrices with specified values, Psychometrika 43, 421-426.

Jöreskog, K.G., 1969, A general approach to confirmatory factor analysis, Psychometrika 34, 183-202.

Koopmans, T.C. and O. Reiersøl, 1950, The identification of structural characteristics, Annals of Math. Stat. 21, 165-181.

Reiersøl, O., 1950, On the identifiability of parameters in Thurstone's multiple factor analysis, Psychometrika 15, 121-149.

Shapiro, A., 1983, On local identifiability in structural models, Research Report, University of South Africa, Dept. of Math. and Appl. Math..

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