soit:

$$\sigma_n = \frac{1}{n}\sum_{j=1}^{\lambda_n} \varepsilon_j(A)\exp 2i\pi x\varphi(j).$$

Comme $\varepsilon(A)$ est par hypothèse normale, on a $\lambda_n \sim 2n$. Par ailleurs, la proposition établie au paragraphe précédent conduit à l'estimation

$$\frac{1}{\lambda_n}\sum_{j=1}^{\lambda_n}\big(2\varepsilon_j(A)-1\big)\exp 2i\pi x\varphi(j) = o(1).$$

Par suite

$$\sigma_n = \frac{1}{\lambda_n}\sum_{j=1}^{\lambda_n}\exp 2i\pi x\varphi(j)+o(1).$$

φ étant un polynôme, on sait que lorsque l'entier p tend vers l'infini, la moyenne

$$\frac{1}{p}\sum_{j=1}^{p}\exp 2i\pi x\varphi(j)$$

tend vers une limite. Donc:

$$\sigma_n = \lim_{p\to\infty}\frac{1}{p}\sum_{j=1}^{p}\exp 2i\pi x\varphi(j)+o(1).$$

Cette égalité prouve bien la double implication

$$x\in B\big(\varphi(N)\big) \Leftrightarrow x\in B\big(\varphi(A)\big), \qquad \text{C.Q.F.D.}$$

En particulier, si φ est un polynôme non constant défini de $\boldsymbol{Z}$ dans $\boldsymbol{Z}$, on sait que $B\big(\varphi(N)\big) = \boldsymbol{R}-\boldsymbol{Q}$. Cette remarque, associée aux lemmes 3 et 6, prouve le théorème 2.

Travaux cités

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A note on the least prime in an arithmetic progression with a prime difference

by

Yoichi Motohashi (Tokyo)

Let $P(k,l)$ be the least prime in the arithmetic progression $n\equiv l \pmod k$, where $(k,l)=1$. The estimate of $P(k,l)$ is one of the most important problems in the theory of numbers, and so many results have been obtained. But it seems that the following fact has not been observed before.

THEOREM. *For any fixed l there exist infinitely many primes q such that*

$$P(q,l) < c(\varepsilon)q^{\vartheta+\varepsilon},$$

where $\vartheta = 2e^{1/4}(2e^{1/4}-1)^{-1} = 1.63773\ldots$

We will prove this in detail only in the case $l=1$, but the general case is merely an easy extension.

Let us consider the product

$$\prod_{p\leqslant N}(p-1),$$

where p runs over all primes not exceeding large N. Defining $\pi(N,k)$ to be the number of primes not exceeding N and $\equiv 1 \pmod k$, we have

$$(1)\qquad \sum_{q^a<N}\pi(N,q^a)\log q = \sum_{p\leqslant N}\log p + O\Big(\sum_{p\leqslant N}p^{-1}\Big) = N+O\big(N\exp(-c(\log N)^{1/2})\big),$$

where the sum of the left side is taken over prime q and integer $a\geqslant 1$.

Now

$$(2)\qquad \sum_{q^a<N}\pi(N,q^a)\log q = \Big\{\sum_{\substack{q\leqslant N^{1/2}(\log N)^{-B}\\ a=1}} + \sum_{\substack{N^{1/2}(\log N)^{-B}<q\leqslant N^{\zeta}\\ a=1}} + \sum_{\substack{N^{\zeta}<q<N\\ a=1}} + \sum_{\substack{q^a<N\\ a\geqslant 2}}\Big\}\pi(N,q^a)\log q = \Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4, \text{ say,}$$

where B is a sufficiently large number and $\frac{1}{2} < \zeta \leqslant \frac{3}{4}$ is a number which is to be defined explicitly later.

Now to the sum Σ_1 we may apply the mean-value theorem of Bombieri [1] (with $B = 44$ for example). We have

$$\Sigma_1 = \frac{N}{\log N} \sum_{q \leqslant N^{1/2}(\log N)^{-B}} \frac{\log q}{q-1} + O\big(N(\log N)^{-1}\big) \tag{3}$$
$$= \tfrac{1}{2}N + O\big(N\log\log N(\log N)^{-1}\big).$$

To the sum Σ_2 we apply the following result of Klimov [2]:

$$\pi(N, q) \leqslant 2\,\frac{N}{(q-1)\log(N/q)}\left\{1+4\,\frac{\log\log N}{\log(N/q)} + \frac{e^{\gamma}+\varepsilon}{2}\cdot\frac{1}{\log N}\right\},$$

where γ is the Euler constant. Hence we have

$$\Sigma_2 \leqslant 2N \sum_{N^{1/2}(\log N)^{-B} < q \leqslant N^{\zeta}} \frac{\log q}{(q-1)\log(N/q)} + O\left(\frac{\log\log N}{(\log N)^2}\sum_{q\leqslant N}\frac{\log q}{q}\right) \tag{4}$$
$$= 2N \sum_{N^{1/2}(\log N)^{-B} < q \leqslant N^{\zeta}} \frac{\log q}{q\log(N/q)} + O\big(N\log\log N(\log N)^{-1}\big).$$

By the partial summation, putting $y = N^{1/2}(\log N)^{-B}$ and $z = N^{\zeta}$, we have

$$\sum_{y<q\leqslant z} \frac{\log q}{q\log(N/q)} \tag{5}$$
$$= \frac{1}{z\log(N/z)} \sum_{q\leqslant z}\log q - \frac{1}{y\log(z/y)}\sum_{q\leqslant y}\log q - \int_y^z f'(\lambda)\sum_{q\leqslant\lambda}\log q\, d\lambda$$

where $f(\lambda) = \left(\lambda\log\frac{N}{\lambda}\right)^{-1}$.

Now

$$\int_y^z f'(\lambda)\sum_{q\leqslant\lambda}\log q\, d\lambda = \int_y^z \lambda f'(\lambda)\,d\lambda + O\left\{\exp\big(-c(\log N)^{1/2}\big)\int_y^z \lambda\,|f'(\lambda)|\,d\lambda\right\} \tag{6}$$
$$= -\int_y^z f(\lambda)\,d\lambda + O\big((\log N)^{-1}\big)$$
$$= \log\log\frac{N}{z} - \log\log\frac{N}{y} + O\big((\log N)^{-1}\big)$$
$$= \log 2(1-\zeta) + O\big(\log\log N(\log N)^{-1}\big).$$

Hence we have

$$\sum_{N^{1/2}(\log N)^{-B} < q \leqslant N^{\zeta}} \frac{\log q}{q\log(N/q)} = -\log 2(1-\zeta) + O\big(\log\log N(\log N)^{-1}\big).$$

And this gives

$$\Sigma_2 \leqslant -2\log 2(1-\zeta)N + O\big(N\log\log N(\log N)^{-1}\big). \tag{7}$$

In order to estimate the sum Σ_4 we divide this into two parts, namely

$$\Sigma_4 = \Big\{\sum_{\substack{q^{\alpha}\leqslant N^{2/3}\\ \alpha\geqslant 2}} + \sum_{\substack{N^{2/3}<q^{\alpha}<N\\ \alpha\geqslant 2}}\Big\}\pi(N, q^{\alpha})\log q$$
$$= \Sigma_{41} + \Sigma_{42}, \text{ say}.$$

By the theorem of Brun–Titchmarsh

$$\Sigma_{41} = O\Big\{\sum_{q\leqslant\sqrt{N}}\log q\cdot\frac{N}{\log N}\sum_{\alpha\geqslant 2}\frac{1}{\varphi(q^{\alpha})}\Big\} \tag{8}$$
$$= O\Big\{\frac{N}{\log N}\sum_{q\leqslant\sqrt{N}}\frac{\log q}{q^2}\Big\} = O\big(N(\log N)^{-1}\big).$$

And also

$$\Sigma_{42} = O\Big\{\sum_{q\leqslant\sqrt{N}}\log q\sum_{N^{2/3}<q^{\alpha}<N}\frac{N}{q^{\alpha}}\Big\} = O\Big\{N^{1/3}\sum_{q\leqslant\sqrt{N}}\log q\cdot\frac{\log N}{\log q}\Big\} = O(N^{5/6}). \tag{9}$$

From (8) and (9) we have

$$\Sigma_4 = O\big(N(\log N)^{-1}\big). \tag{10}$$

Consequently from (3), (7) and (10) we obtain

$$\sum_{N^{\zeta}<q<N}\pi(N, q)\log q \geqslant 2N\log\big(2e^{1/4}(1-\zeta)\big) + O\big(N\log\log N(\log N)^{-1}\big).$$

Hence if $\zeta < 1-\frac{1}{2}e^{-1/4}$, we have

$$\sum_{N^{\zeta}<q<N}\pi(N, q)\log q > 0.$$

From this inequality the theorem follows at once.

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