A NOTE ON THE LIE ALGEBRAS OF ALGEBRAIC GROUPS

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0. In his book [1] C. Chevalley defined the replicas for any elements of Lie algebras of algebraic groups of matrices which are defined over fields of characteristic 0, and he characterized algebraic subalgebras as those subalgebras of the general linear algebras which are closed with respect to "replica operation" i.e. those which contain all replicas of any elements of themselves. In this paper we shall define the replica in the case of any algebraic groups defined over fields of characterestic 0 and show that the same characterization of algebraic subalgebras is true in this case too.

1. Let G be a connected algebraic group[1]; let $\Omega\ (G)$ be the field of rational functions on G where Ω is the universal domain; let $k(G)$ be the subfield of $\Omega(G)$ consisting of all rational functions defined over k where k is a field of definition for G. Then $\Omega(G)$ is the union of $k(G)$ for all fields k of definition for G, and the mapping $f\rightarrow f(p)$ is a k-isomorphism of $k(G)$ onto $k(p)$ where p is a generic point over k on G. Suppose that G is given by $[V_\alpha, \mathfrak{F}_\alpha, T_{\beta\alpha}]$, and let $\xi_1, \ldots, \xi_N$ be the coordinate functions relative to V_α, i.e. $\xi_i(p) = x_i$, where (x) is the representative of p in V_α. Then we have $\Omega(G) = \Omega(\xi)$ and $k(G) = k(\xi)$.

For any $p \in G$ we denote by $\mathfrak{o}_p$ the local ring of p on G. Let $\mathfrak{m}_p$ be the maximal ideal of $\mathfrak{o}_p$. By a tangent vector to G at p we mean an Ω-linear mapping X_p of $\mathfrak{o}_p$ into Ω such that for $f_1, f_2 \in \mathfrak{o}_p$ we have

$$X_p(f_1f_2) = (X_pf_1)\, f_2(p) + f_1(p)(X_pf_2).$$

If k is a field of definition for G such that p is rational over k, then X_p is said to be rational over k if X_p maps $\mathfrak{o}_p \cap k(G)$ into k. An Ω-derivation D is said to be finite at p if D maps $\mathfrak{o}_p$ into itself. In this case D induces a tangent vector D_p to G at p such that $D_p f = (Df)(p)$ for $f \in \mathfrak{o}_p$, which is called the local component of D at p. If further D is defined over k and maps $\mathfrak{m}_p \cap k(G)$ into itself, we have a k-derivation X of $k(p)$ such that $Xf(p) = (Df')(p)$ for $f(p) \in k(p)$, where f' is an element of $\mathfrak{o}_p \cap k(G)$ such that $f'(p) = f(p)$.

Let ρ be a rational mapping of G into another connected algebraic group G'. If ρ is generically surjective, we get an Ω-isomorphism ρ^* of $\Omega(G')$ into $\Omega(G)$ such that $\rho^*f(x) = f(\rho(x))$ for $f \in \Omega(G')$, where x is a generic point on G over some field of definition for G, G', ρ, and f. For $p \in G$ let R_p be the right translation, let L_p be the left translation, and let $\iota(p)$ be the inner

1) As for the terminology and preliminary results, cf. Nakano [2] and Rosenlicht [3], [4].

automorphism $x \to pxp^{-1}$ of G. Any Ω-derivation D of $\Omega(G)$ is called (right) invariant if $R_p^* Df = D R_p^* f$ for any $p \in G$ and $f \in \Omega(G)$. Any invariant Ω-derivation is everywhere finite and determined by the local component at one point on G. The set of all invariant Ω-derivations of $\Omega(G)$ is called the Lie algebra of G which is a Lie algebra over Ω with the bracket multiplication $[D, D] = DD' - D'D$. In the case of any algebraic group, the Lie algebra of its component containing the unit element e is called the Lie algebra of this algebraic group. In this paper we shall denote algebraic groups by $G, G', H, \ldots.,$ and their Lie algebras by $\mathfrak{g}, \mathfrak{g}', \mathfrak{h}, \ldots.$. It is known that if K is a field of definition for G, the Lie algebra $\mathfrak{g}$ of G has a base consisting of n invariant Ω-derivations defined over K, where n is the dimension of G. The set $\mathfrak{g}(\ ,K)$ of all those elements of $\mathfrak{g}$ which are defined over K is a Lie algebra over K, and $\mathfrak{g}$ is the scalar extension of $\mathfrak{g}(\ , K)$ from K to Ω.

Let ρ be a rational homomorphism of G into G', then we have a homomorphism $d\rho$ of $\mathfrak{g}$ into $\mathfrak{g}'$ such that $\rho^*(d\rho(D)f)(p) = (D\rho^* f)(p)$ for $D \in \mathfrak{g}$, $p \in G$ and $f \in \mathfrak{o}_{\rho p}$. Let H be a connected algebraic subgroup of G, and let σ be the natural injection of H int) G, then $d\sigma(\mathfrak{h})$ is a subalgebra of $\mathfrak{g}$ which is Ω-isomorphic to $\mathfrak{h}$. In this paper we identify $d\sigma(\mathfrak{h})$ with $\mathfrak{h}$. Then an element D of $\mathfrak{g}$ is in $\mathfrak{h}$ if and only if D maps $\mathfrak{m} \cap k(G)$ into itself, where k is a field of definition for G, H and D and $\mathfrak{m}$ is the maximal ideal of the local ring of a generic point over k on H. A subalgebra of $\mathfrak{g}$ is called algebraic if it is the Lie algebra of some connected algebraic subgroup of G.

Let k be a field of definition for G; let x and y be independent generic points over k on G; let φ be the rational mapping of $V_\alpha \times V_\alpha$ into V_α which is induced by the group operation $G \times G \ni x \times y \to xy \in G$; let $\varphi^i(x,y)$ be the i-th coordinate of the representative of xy in V_α; let $\varphi^i(X, Y)$ be a suitable rational expression in indeterminates $(X;\ Y)$ with coefficients in k (e.g. if the unit element e has a representative in V_α we take such $\varphi^i(X, Y) = P^i(X, Y)/Q^i(X, Y)$ that $Q^i(e, e) \neq 0$, where P^i, $Q^i \in k[X, Y]$). For any Ω-derivation D of $\Omega(G)$, put $D\xi_i = \chi_i(\xi)$. Then D is determined by $(\chi_1(\xi), \ldots., \chi_N(\xi))$. If a point z of G has a representative in V_α and D is finite at z, the local component of D at z is determined by $(\chi_1(z), \ldots., \chi_N(z))$. If D is defined over k, $\chi_i(\xi)$ is in $k(\xi)$. And D is invariant if and only if

$$(1) \qquad \chi_i(R_y^*\xi) = \sum_{j=1}^{N} (\partial\varphi^i(X,y)/\partial X_j)_{X=\xi}\, \chi_j(\xi).$$

If the unit element e has a representative in V_α, we have

$$\chi_i(y) = \sum_{j=1}^{N} (\partial\varphi^i(X,y)/\partial X_j)_{X=e}\, \chi_j(e),$$

and therefore

$$(2) \qquad \chi_i(\xi) = \sum_{j=1}^{N} (\partial\varphi^i(X,\xi)/\partial X_j)_{X=e}\, \chi_j(e).$$

Conversely if this relation holds for an Ω-derivation D defined over k such that $D\xi_i = \chi_i(\xi)$, then D is invariant.

In the following we often denote by the same letter x the point of G and its representative in some affine variety V.

Suppose that G is a connected algebraic subgroup of $GL(n, \Omega)$. Let u_{ij} and ξ_{ij} be the coordinate functions of $GL(n, \Omega)$ and G, respectively; let D be an element of $\mathfrak{g}$ defined over k; let $\mathfrak{p}$ be the prine ideal of $k[u]$ associated with G; put $D\xi_{ij} = \chi_{ij}(\xi)$, then (2) implies that

$$D\xi_{ij} = \sum_{l=1}^{\nu} \chi_{il}(I)\,\xi_{lj},$$

where I is the unit matrix. Put

$$\Phi(D) = -(\chi_{ij}(I)), \tag{3}$$

then the k-derivation $\delta(\Phi(D))$ of $k[u]$ maps $\mathfrak{p}$ into itself[2]. A simple calculation shows that $D \to \Phi(D)$ is a k-isomorphism of $\mathfrak{g}(\ ,k)$ into $\mathfrak{gl}(n, k)$ and that image of $\mathfrak{g}(\ ,k)$ by Φ is the Lie algebra of G defined by Chevalley [1] p. 128. Thus we may imbed the Lie algebra of algebraic subgroup of $GL(n, \Omega)$ in $\mathfrak{gl}(n, \Omega)$.

The next lemma is usefull in the section 3.

LEMMA 1. *Let s be a point on G; let k be a field of definition for G. Then for a k-derivation X of $k(s)$ there exists uniquely an element D of $\mathfrak{g}$, defined over $k(s)$, such that $(D\xi_i)(s) = X s_i$, where ξ_i are coordinate functions relative to V in which s has a representative.*

PROOF. If we set $X_s f = (Xf(s))$ for $f \in \mathfrak{o}_s \cap k(G)$, weobtain a k-linear mapping X_s of $\mathfrak{o}_s \cap k(G)$ into $k(s)$ such that for $f_1, f_2 \in \mathfrak{o}_s \cap k(G)$

$$X_s(f_1 f_2) = (X_s f_1) f_2(s) + f_1(s)(X_s f_2) \tag{4}$$

Let K be any overfield of $k(s)$. Then for any $f \in K[\xi]$ we may express $f = \sum_{i=1}^{n} \alpha f_i$, where $\alpha_i \in K$ and $f_i \in k[\xi]$. If we set $X_s \bar{f} = \sum_{i=1}^{m} \alpha_i X_s f_i$, we obtain a K-linear mapping X_s of $K[\xi]$ into K with the analogous property (4) for $f_1, f_2 \in K[\xi]$. In fact; suppose that $\sum_{i=1}^{n} \alpha_i f_i = 0$. Then we may suppose that for some intger $l \leqq m$ $\alpha_1, \ldots, \alpha_l$ are linearly independent over k and $\alpha_i = \sum_{j=1}^{l} \gamma_{ij} \alpha_j$ for some $\gamma_{ij} \in k$. The equation $\sum_{i=1}^{l} \alpha\,(f_i + \sum_{j=l+1}^{m} \gamma_j f_j) = 0$ implies $f_i + \sum_{j=l+1}^{m} \gamma_{ji} f_j = 0$, since K and $k(\xi)$ are linearly disjoint over k. Thus we have $X_s f_i + \sum_{j=l+1}^{m} \gamma_{ji} X_s f_j = 0$ and $\sum_{i=1}^{n} \alpha_i X_s f_i = \sum_{i=1}^{l} \alpha_i (X_s f_i + \sum_{j=l+1}^{m} \gamma_{ji} X_s f_j) = 0$, and the mapping of $K[\xi]$ into K is defined. The linearity is clear and the equation (4) holds for such forms $\alpha_1 f_1$. $\alpha_2 f_2$ that $\alpha_1, \alpha_2 \in K$ and $f_1, f_2 \in k[\xi]$. Clearly X_s induces a tangent vector to G at s which we shall denote by the same X_s. Taking $K = k(s)$, we see that X_s is rational over $k(s)$.

Let f be an element of $\Omega(G)$; let K be a field of definition for G and f over which s and X_s are rational let x be a generic point on G over K. Then

2) As for the definition of δ cf. [1] p. 126

$R^*_{s^{-1}x} f$ is in $\mathfrak{o}_s$ and rational over $K(x)$, so $X_s R^*_{s^{-1}x} f$ is a welldefined element of $K(x)$. Let f' be the unique element of $k(G)$ such that $f'(x) = X_s R^*_{s^{-1}x} f$. It is clear that f' depends only on G, f and X_s. If we set $Df = f'$, we obtain the element D of $\mathfrak{g}$ described above. In fact, $D\,\Omega = 0$ and the linearity holds. For $f_1, f_2 \in K(G)(D, f f_1)(x) = X_s R^*_{s^{-1}x} (f_1 f_2) = X_s (R^*_{s^{-1}x} f_1 \quad R^*_{s^{-1}} f_1) = (Df_1)(x) f_2(x) + f_1(x)(Df_2)(x)$. Taking $K = k(s)$, we see that D is an Ω-derivation of $\Omega(G)$ defined over $k(s)$. If $f \in \Omega(G)$, $a \in G$, K is a field of definition for G and f over which a, s and X_s are rational, and x is generic for G over K, we have $(R^*_a Df)(x) = (Df)(xa) = X_s R^*_{s^{-1}xa} f = X_s R^*_{s^{-1}x} R^*_a f = (DR^*_a f)(x)$. Thus D is invariant. If K is a field of definition for G over which s and X_s are rational, and x is generic for G over K, we have $D_x f = X_s R^*_{s^{-1}x} f$ for $f \in K(G)$ and therefore $D_x = X_s R^*_{s^{-1}x}$. By the invariance of D we have $D_s = D_x R^*_{x^{-1}s} = X_s R^*_{s^{-1}x} R^*_{x^{-1}s} = X_s$, and in paticular $(D\xi_i)(s) = X s_i$.

Since an invariant Ω-derivation of $\Omega(G)$ is determined by its local component at one point of G, the uniqueness is clear. q.e.d.

In the following we shall denote by D_X the element of $\mathfrak{g}$ which is determined by X as described in this lemma.

Let $D \in \mathfrak{g}$, then $d_l(x)D$ is in $\mathfrak{g}$ for any $x \in G$. Let V be an affine variety in which x has a representative, then

$$d_l(x)D\xi_i = L^*_{x^{-1}} DL^*_x \xi_i = L^*_{x^{-1}}\, D\,\varphi^i(x, \xi) = L^*_{x^{-1}} \sum_{j=1}^{N} (\partial\varphi^i(x, Y)/\partial Y_j)_{Y=\xi}\, \chi_j(\xi).$$

And we have that $d_l(x)D$ is defined over $k(x)$ if D is defined over k. Further we have

LEMMN 2. *Let H be a connected algebraic subgroup of G with the Lie algebra $\mathfrak{h}$. Then for any $h \in H$, $d_l(h)$ maps $\mathfrak{y}$ iuto itself.*

PROOF. Let $D \in \mathfrak{h}$; let k be a field of definition for G, H and D over which h is rational; let $\mathfrak{m}$ be the maximal ideal of the local ring of a generic point over k on H. Then for $f \in \mathfrak{m} \cap k(G)$, $L^*_h f$ is in $\mathfrak{m} \cap k(G)$ and therefore $DL^*_h f$ is in $\mathfrak{m} \cap k(G)$, since $D \in \mathfrak{h}$. So we have $L^*_{h^{-1}} DL^*_h f \in \mathfrak{m} \cap k(G)$, i.e. $d_l(x)D$ is in $\mathfrak{h}$. q.e.d.

Let $D_1, \ldots, D_n$ be a base of $\mathfrak{g}(\ , k)$, k being a field of definition for G, then for any generic point x over k on G we may express $d_l(x)D_i = \sum_{j=1}^{n} \gamma_{ji} D_j$, where γ_{ji} is in Ω. Put $D_i \xi_j = \chi_{ij}(\xi)$ and $d_l(x)D_i \xi_j = \chi_j(\xi)$, then $\chi_{ji}(\xi)$ is in $k(\xi)$ and $\chi_j(\xi)$ is in $k(x)(\xi)$. Since $D_1, \ldots, D_n$ is a base of $\mathfrak{g}$, a family of vectors $(\chi_{11}(\xi), \ldots, \chi_{1N}(\xi)), \ldots, (\chi_{n1}(\xi), \ldots, \chi_{nN}(\xi))$ are linearly independent over Ω. Therefore $\chi_j(\xi) = \sum_{l=1}^{n} \gamma_{li} \chi_{lj}(\xi)$ implies that γ_{ij} is an element $\gamma_{ij}(x)$ of $k(x)$. If we denote by $Ad(x)$ the matrix $(\gamma_{ij}(x))$, we have that $x \to Ad(x)$ is a rational mapping Ad of G into $\mathfrak{gl}(n, \Omega)$ defined over k. For independent generic points x and y over k on G, we have $d\iota(x)d\iota(y) = d\iota(xy)$, so $Ad(x)Ad(y) = Ad(xy)$. And $d\iota(x^{-1})$ being the inverse of $d\iota(x)$, we have that $Ad(x)$ is in $GL(n, \Omega)$. Thus Ad is everywhere defined on G and the locus of $Ad(x)$ over k on $GL(n, \Omega)$ is a connected algebraic subgroup of $GL(n, \Omega)$ which is denoted by $Ad(G)$ (cf. proposition 2 of [1] p. 82). So $x \to Ad(x)$ is a rational homomorphism of G onto $Ad(G)$ defined over k.

Now for this rational homomorphism $x \to Ad(x)$ of G onto $Ad(G)$, we have the natural homomorphism ad of $\mathfrak{g}$ onto the Lie algebra of $Ad(G)$. Then we may suppose that $ad(\mathfrak{g})$ is contained in $\mathfrak{gl}(n, \Omega)$ and that $\mathrm{ad}(\mathfrak{g})$ is a subalgebra of the Lie algebra of all endomorphisms of the vector space $\mathfrak{g}$ over Ω.

Let k' be another field of definition for G and let $D'_1, \ldots\ldots, D'_n$ be a base of $\mathfrak{g}(\ ,k')$. Let Ad' and ad' be the representation of G and its differential which are defined as described above with respect to k' and $D'_1, \ldots\ldots, D'_n$. Then if K is the compositum of k and k', there exists a matrix $S = (s_{ij})$ in $GL(n, K)$ such that $D_i = \sum_{j=1}^{n} s_{ji} D'_j$. We have $Ad'(x) = S\ Ad(x)\ S^{-1}$ for any $x \in G$. Let D be an element of $\mathfrak{g}$ defined over K. Then from the definition and (3) we have that $\Phi(ad(D)) = -(\Lambda_{ij})$ and $\Phi(ad'(D)) = -(\Lambda'_{ij})$, where $\Lambda_{ij} = (D\gamma_{ij}(\xi))(e)$ and $\Lambda'_{ij} = (D\gamma'_{ij}(\xi))(e)$. It is easily seen that $\Phi(ad'(D)) = S\ \Phi(ad(D))\ S^{-1}$.

Then if we identify $Ad(x)$ with $d\iota(x)$ and $ad(x)$ with the endomorphism of the vector space $\mathfrak{g}$ over Ω such that $D_i \to \sum_{j=1}^{n} \Lambda_{ji} D_j$, we have a linear representation of G and its differential which are independent of the choice of a field k of definition for G and a base $D_1, \ldots, D_n$ of $\mathfrak{g}(\ ,k)$. We shall call Ad and ad the adjoint representation of G and $\mathfrak{g}$, respectively. Then we have

PROPOSITION 1. *Let G be a connected algebraic group; let $x \to Ad(x)$ be the adjoint representation of G. Then for any D, $D' \in \mathfrak{g}$ we have $ad(D)D' = [D, D']$.*

PROOF. Let k be a field of definition for G and D; let $D_1, \ldots, D_n$ be a base of $\mathfrak{g}(\ ,k)$; let x be a generic point over k on G. Then we have

$$Ad(x)\, D_i = \sum_{j=1}^{n} \gamma_{ji}(x)\, D_j, \tag{5}$$

where $\gamma_{ji}(x)$ is in $k(x)$. Let $\xi_1, \ldots\ldots, \xi_N$ be coordinate functions of G relative to an affine variety in which the unit element e has a representative, and put $D\xi_j = \chi_j(\xi)$, $D_i\xi_j = \chi_{ij}(\xi)$ and

$$\Lambda_{ij} = (D\gamma_{ij}(\xi))(e). \tag{6}$$

Then from (3) and the above remark it follows that

$$ad(D)D_i = -\sum_{l=1}^{n} \Lambda_{li} D_l,$$

and therefore

$$(7)\qquad ad(D)D_i\,\xi_j = -\sum_{t=1}^{n} \Lambda_{ti}\,\chi_{tj}(\xi).$$

On the other hand

$$(8)\qquad [D, D_i]\,\xi_j = \sum_{p=1}^{\nu} \{\partial\chi_{ij}(\xi)/\partial\xi_p\,\chi_p(\xi) - \partial\chi_j(\xi)/\partial\xi_p\,\chi_{ip}(\xi)\}.$$

So it is sufficient to show that these two functions (7) and (8) have the same value at e. From (5) we have

$$\sum_{t=1}^{n} \gamma_{ti}(x)\,\chi_{tj}(\xi) = L^*_{x^{-1}}\,D_i\,L^*_x\,\xi_j.$$

Applying L^*_x, we have

$$\sum_{t=1}^{n} \gamma_{ti}(x)\,L^*_x\,\chi_{tj}(\xi) = \sum_{p=1}^{\nu} (\partial\rho^j(x,\xi)/\partial\xi_p)\,\chi_{ip}(\xi),$$

so at e, we have

$$\sum_{t=1}^{n} \gamma_{ti}(x)\,\chi_{tj}(x) = \sum_{p=1}^{\nu} (\partial\varphi^j(x, Y)/\partial Y_p)_{Y=e}\,\chi_{ip}(e).$$

Applying the k-derivation of $k(x)$ induced by D, we have

$$\sum_{t=1\ p=1}^{n,\ N} \gamma_{ti}(x)/\partial x_p\,\chi_p(x)\,\chi_{tj}(x) + \sum_{t=1\ p=1}^{n,\ \nu} \gamma_{ti}(x)\partial\,\chi_{tj}(x)/\partial x_p\,\chi_p(x)$$
$$= \sum_{p,q=1}^{\nu} (\partial\varphi^j(X, Y)/\partial X_q\partial Y_p)_{X=x,\ Y=e}\,\chi_q(x)\,\chi_{ip}(e).$$

Since $\gamma_{ij}(x)$ is in the specialization ring of e in $k(x)$ and $\gamma_{ij}(e) = \delta_{ij}$, putting $x = e$, we see that the functions (7) and (8) have the same value at e (cf. (2) and (6)). q.e.d.

2. In this section we assume that the characterestic of the universal domain is 0. We first prove the proposition which affords the definition of replica.

PROPOSITION 2. *Let H_1 aud H_2 be connected algebraic subgroups of G whose Lie algebras are $\mathfrak{h}_1$ and $\mathfrak{h}_2$; let H_0 be the component of $H_1 \cap H_2$ containing the unit element e. Then we have $\mathfrak{h}_1 \cap \mathfrak{h}_2 = \mathfrak{h}_0$ where $\mathfrak{h}_0$ is the Lie algebra of H_0.*

PROOF. H_0 being an algebraic subgroup of H_1 and H_2, clearly $\mathfrak{h}_0 \subset \mathfrak{h}_1 \cap \mathfrak{h}_2$.

Let D be any element of $\mathfrak{h}_1 \cap \mathfrak{h}_2$. We shall show that D is in $\mathfrak{h}_0$. Let k be a field of definition for G, H_0, H_1, H_2 and D; let x be a generic point over k on G; let V be an affine variety in which e has a representative, then H_i also has a representative in V. Let $\mathfrak{P}$ be the ideal in $k[X]$ determined by V; let $\mathfrak{P}_i$ be those for H_i, then the set of those points of G whose represen'atives in V are zeros of any polynomials in $\mathfrak{A} = \mathfrak{P}_1 + \mathfrak{P}_2$ is the set of those points of $H_1 \cap H_2$ which have representatives in V. Let $\mathfrak{Q}_0$ be the set of all those $P(X)$ in $k[X]$, for which there exists a polynomial $L(X)$ in $k[X]$ such that $L(h_0) \neq 0$ and $L(X)P(X) \in \mathfrak{A}$, where h_0 is a generic point over k on H_0. Then by the lemma 5 of [5] F-III$_3$, $\mathfrak{Q}_0$ is a $\mathfrak{P}_0$-primary ideal in $k[X]$. Let h_i be a generic point over k on H_i; let Q_i be the specialization ring of h_i in $k(x)$; let $\mathfrak{m}_i$ be the maximal ideal in Q_i. Then we have $\mathfrak{m}_i = \mathfrak{P}_{i|x} Q_i$, where $\mathfrak{P}_{i|x}$ is the set of those elements $P(x)$ of $k[x]$ for some $P(X) \in \mathfrak{P}_i$. We shall denote by the same D the k-derivation of $k(x)$ induced by D. Then, since D is everywhere finite and in particular $D\, Q_0 \subset Q_0$, for D to be contained in $\mathfrak{h}_0$, it is sufficent that $D\mathfrak{P}_{0|x} \subset \mathfrak{m}_0$. Take any element $P(x)$ of $\mathfrak{P}_{0|x}$, then $DP(x) \in Q_0$, so we may express $DP(x) = F_1(x)/F_2(x)$ for some $F_1(X)$, $F_2(X) \in k[X]$ such that $F_2(h_0) \neq 0$. Let $P(X)$ be an element of $k[X]$ which has a specialization $P(x)$ over $(X) \to (x)$. Let r be the minimal positive integer such that $P(X)^r \in \mathfrak{Q}_0$, then there exists $L(X) \in k[X]$ such that $L(h_0) \neq 0$ and $P(X)^r L(X) \in \mathfrak{A}$, and we may write

$$P(X)^r L(X) = P_1(X) + P_2(X),$$

where some $P_i(X) \in \mathfrak{P}_i$, therefore we have

$$P(x)^r L(x) = P_1(x) + P_2(x).$$

Applying D on this relation, we have

$$r\, P(x)^{r-1} DP(x) L(x) + P(x)^r DL(x) = DP_1(x) + DP_2(x), \text{ i.e.}$$

$$r P(x)^{r-1} L(x) F_1(x)/F_2(x) = DP_1(x) + DP_2(x) - P(x)^r DL(x).$$

Since D is in $\mathfrak{h}_1 \cap \mathfrak{h}_2$, $DP_i(x)$ is in $\mathfrak{m}_i$ $(i = 1, 2)$, and therefore we may write $DP_i(x) = P_i'(x)/P_i''(x)$ for some $P_i'(X) \in \mathfrak{P}_i$, $P_i''(X) \in k[X]$ such that $P_i''(h_0) \neq 0$, since $DP_i(x)$ is in Q_0. So, $DL(x)$ being in Q_0, in the relation

$$r\, P(x)^{r-1} L^2(x) F_1(x)/F_2(x) = L(x)\{DP_1(x) + DP_2(x)\} - L(x) P(x)^r DL(x),$$

the right hand side may be expressed as $A(x)/B(x)$ for some $A(X) \in \mathfrak{A}$ and $B(X) \in k[X]$ such $B(h_0) \neq 0$. Then we have

$$r\, P(x)^{r-1} L^2(x) B(x) F_1(x) = A(x) F_2(x),$$

and, x being generic for G over k, we have

$$r\, P(X)^{r-1} L^2(X) B(X) F_1(X) = A(X) F_2(X) + P_0(X),$$

where $P_0(X)$ is some element of $\mathfrak{P}$. Since $\mathfrak{P}$ is contained in $\mathfrak{P}_i$, the right hand side is contained in $\mathfrak{A}$, and therefore $r\, P(X)^{r-1} F_1(X)$ is in $\mathfrak{Q}_0$. But $r\, P(X)^{r-1} \notin \mathfrak{Q}_0$, so we have that $F_1(X)$ is in $\mathfrak{P}_0$. Thus we have shown that D is in $\mathfrak{h}_0$.

q. e. d.

Now we have two corollaries; and the first as follows;

COROLLARY 1. *Let be a connected algebraic gropu For any element D of the Lie algebra of G, there exists the smallest connected algebraic subgroup of G whose Lie algebra contains D.*

PROOF. Let $\mathfrak{M}$ be the family of connected algebraic subgroups of G whose Lie algebra contains D; then, as $\mathfrak{M}$ is not empty, there exists an element H of $\mathfrak{M}$ whose dimension is > 0 and the minimal in $\mathfrak{M}$. Take any $H' \in \mathfrak{M}$, and let $(H \cap H')_0$ be the component of $H \cap H'$ containing e, then $(H \cap H')_0 \subseteqq H_0$ By the proposition 2, the Lie algebra of $(H \cap H')_0$ contains D, so $(H \cap H')_0$ $\in \mathfrak{M}$ and dim $(H \cap H')_0 \geqq$ dim H. Thus we have that $(H \cap H') = H$ and H is contained in H'.

q. e. d.

Let us denote by G_D the smallest connected algebraic subgroup H of G in the corollary and by $\mathfrak{g}_D$ the Lie algebra of G_D. We call any element of $\mathfrak{g}_D$ replica of D. This is an extension of the definition 2 of [1] p. 180.

Let H be a connected algebraic subgroup of G with Lie algebra $\mathfrak{h}$; let H′ be the connected algebraic subgroup of G generated by G_D for all $D \in \mathfrak{h}$: let $\mathfrak{h}'$ be the Lie algebra of H'. Then H' is contained in H, since each G_D is contained in H. But, as $\mathfrak{h}$ is generated by $\mathfrak{g}_D$ for all $D \in \mathfrak{h}$, $\mathfrak{h}'$ contains $\mathfrak{h}$, therefore $\dim H = \dim \mathfrak{h} \leqq \dim \mathfrak{h}' = \dim H'$, and $H = H'$. Thus we have proved that H is the connected algebraic subgroup of G generated by G_D for all $D \in \mathfrak{h}$, and we have another corollary of the proposition 2:

COROLLARY 2. *Let H_1 and H_2 be connected algebraic subgroups of G, let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be the Lie algebras of H_1 and H_2, respectively. Then if $\mathfrak{h}_1$ contains $\mathfrak{h}_2$, H_1 contains H_2.*

The next proposition shows the relation between G_D and rational homomorphism:

PROPOSITION 3. *Let G and $\bar{G}$ be connected algebraic groups; let ρ be a rational homomorphism of G onto $\bar{G}$. Then for any D of $\mathfrak{g}$, we have $\rho(G_D) = G_{d\rho D}$.*

PROOF. As $d\rho D$ is in the Lie algebra $d\rho\, \mathfrak{g}_D$ of the algebraic subgroup $\rho(G_D)$ of $\bar{G}$, $G_{d\rho D}$ is contained in $\rho(G_D)$.

We shall show that $G_{d\rho D}$ contains $\rho(G_D)$. Let H be the algebraic subgroup of G consisting of those z such that $\rho(z)$ is in $G_{d\rho D}$ and H_0 be the component of H containing the unit element e of G. Let k be a field of definition for ρ, D, and all these algebraic groups concerned; let x and h be generic points over k on G and H_0, then $y = \rho(x)$ and $\rho(h)$ are those on $\bar{G}$ and $G_{d\rho D}$. Let V and $\bar{V}$ be affine varieties in which the unit elements of G and $\bar{G}$ have their representatives. We shall denote by the same ρ the rational mapping of V into $\bar{V}$ induced by ρ. Let $\mathfrak{P}$ and $\mathfrak{P}_0$ be the ideals in $k[X]$ determined by x and h; let $\bar{\mathfrak{P}}_0$ be the ideal in $k[Y]$ determined by $\rho(h)$. Let $\mathfrak{S}$ be the subset of $k[X]$ consisting of those $P(X)$ for which there exists $P_0(X) \in k[X]$ such that $P_0(x) \neq 0$ and $P(x)/P_0(x) \in \bar{\mathfrak{P}}_{0|y}$, where $\bar{\mathfrak{P}}_{0|y}$ is the ideal in $k[y]$ consisting of those $F(y)$ for some $F(Y) \in \bar{\mathfrak{P}}_0$; let $\mathfrak{A}$ be the ideal in $k[X]$ generated by $\mathfrak{P}$ and $\mathfrak{S}$.

Then for a point z of G which has a representative in V, z is in H if and only if z is a zero of $\mathfrak{A}$. In fact; suppose that z is in H, then $P(z) = 0$ for $P \in \mathfrak{P}$. If $P \in \mathfrak{S}$, we have $R(x) = P(x)/P_0(x) \in k(x)$ such that $R(x) \in \bar{\mathfrak{P}}_{0|y}$. Since ρ is everywhere defined on G and therefore $R(x)$ is in the specialization ring of z in $k(x)$, we have another expression $R(x) = P'(x)/P'_0(x)$ where $P'_0(z) \neq 0$, if necessary. Since $\rho(z)$ is a specialization of $\rho(h)$ over k, we have $P'(z) = 0$ and therefore $P(z) = 0$. Conversely, suppose that z is a zero of $\mathfrak{A}$. For $F \in \bar{\mathfrak{P}}_0$ there exist $P(X)$ and $P_0(X)$ in $k[X]$ such that $F(y) = P(x)/P_0(x)$ and $P_0(z) \neq 0$, since $F(y)$ is in the specialization ring of z in $k(x)$. Then z being a specialization of x over k, we have $P(X) \in \mathfrak{C}$, and $P(z) = 0$. Thus $F(\rho(z)) = 0$ and $\rho(z)$ is a specialization of $\rho(h)$ over k.

Let $\mathfrak{D}$ be the ideal in $k[X]$ consisting of those $P(X)$ for which there exists $L(X) \in k[X]$ such that $L(h) \neq 0$ and $L(X)P(X) \in \mathfrak{A}$, then $\mathfrak{D}$ is $\mathfrak{P}_0$-primary. Let $\bar{\mathfrak{m}}$ be the maximal ideal of the specialization ring of $\rho(h)$ in $k(y)$, We shall denote by the same D and $d\rho D$ the k-derivations of $k(x)$ and $k(y)$ induced by D and $d\rho D$, respectively. Then, from the definition of $d\rho$, we have $D\mathfrak{m} = d\rho D\bar{\mathfrak{m}} \subset \bar{\mathfrak{m}}$, since $d\rho D$ is in the Lie algebra of $G_{d\rho D}$. Now if $P(X) \in \mathfrak{P}$, we have $D\cdot P(x) = 0$ and if $P(X) \in \mathfrak{C}$, we have $F(Y) \in \bar{\mathfrak{P}}_0$ and $P_0(X) \in k[X]$ such that $P_0(x) \neq 0$ and $F(y) = P(x)/P_0(x)$. So we have

$$D\, P(x) = (DP_0(x))\, F(y) + P_0(x)\, DF(y).$$

Since D is finite at h, $D\, P_0(x)$ is in the specialization ring of h in $k(x)$. On the other hand we have $D\,F(y) = D\,F(\rho(x)) = d\rho D\ F(y) = F_1(y)/F_2(y)$ for some $F_1(Y) \in \bar{\mathfrak{P}}_0$ and $F_2(Y) \in k[Y]$ such that $F_2(\rho(h)) \neq 0$. And ρ being defined at h, $F(y)$ and $F_1(y)$ is in the specialization ring of h in $k(x)$. Thus we may express $DP(x) = A(x)/B(x)$ for some $A(X) \in \mathfrak{A}$ and $B(X) \in k[X]$ such that $B(h) \neq 0$. D being finite at h, we see that this is true for any $P(x) \in \mathfrak{A}$. So the argument which has run in the proof of the proposition 2 shows that D is in the Lie algebra of H_0. Thus G_D is in H, and $\rho(G_D)$ is contained in $G_{d\rho D}$.

q. e. d.

We have an application of this proposition as follows;

PROPOSITION 4. *Let ρ be a rational homomorphism of a connected algebraic group G onto another $\bar{G}$; let $\bar{H}$ be an algebraic subgroup of $\bar{G}$ with the Lie algebra $\bar{\mathfrak{h}}$. Then the set of those elements y of G such that $\rho(y)$ is in $\bar{H}$ is an algebraic subgroup H of G with the Lie algebra consisting of those elements D of $\mathfrak{g}$ such that $d\rho D$ is in $\bar{\mathfrak{h}}$.*

PROOF. It is known that such a set H is algebraic. We may suppose that H and $\bar{H}$ are connected. Let $\mathfrak{h}'$ be the set of those D of $\mathfrak{g}$ such that $d\rho D \in \bar{\mathfrak{h}}$; let $\mathfrak{h}$ be the Lie algebra of H. Since $\rho(H) = \bar{H}$, we have $d\rho\ \mathfrak{h} = \bar{\mathfrak{h}}$ and therefore $\mathfrak{h}$ is contained in $\mathfrak{h}'$. Conversely for any D of $\mathfrak{h}'$ we have $\rho(G_D) = G_{d\rho D}$ by the proposition 3. As $\bar{H}$ is algebraic, $\bar{H}$ contains $G_{d\rho D}$. Thus we have that G_D is contained in H and D is in $\mathfrak{h}$.

q. e d.

From this proposition follows the next corollary;

COROLLARY. *Let G be a connected algebraic group with the Lie algebra $\mathfrak{g}$; let $\mathfrak{s}$ be a subspace of $\mathfrak{g}$. Then the set H of those elements y of G such that $Ad(y)$ maps $\mathfrak{s}$ into itself is an algebraic subgroup of G with the Lie algebra consitsing of those elements D of $\mathfrak{g}$ such that $[D, D']$ is in $\mathfrak{s}$ for any $D' \in \mathfrak{s}$.*

PROOF. Let H be the set of $Ad(y)$ such that $Ad(y)\mathfrak{s}\subset\mathfrak{s}$. Then it is known that $\overline{H}$ is algebraic, and therefore that H is algebraic. We may suppose that $\overline{H}$ is contained in $GL(n,\Omega)$ where n is the dimension of G. By the example of §10 of [1], the Lie algebra $\overline{\mathfrak{h}}$ of $\overline{H}$ is the set of those X of $\mathfrak{gl}(n,\Omega)$ such that $X\mathfrak{s}\subset\mathfrak{s}$. By the proposition 4 the Lie algebra $\mathfrak{h}$ of H is the set of those D of $\mathfrak{g}$ such that $ad(D)\in\overline{\mathfrak{h}}$. So, for D of $\mathfrak{g}$, D is in $\mathfrak{h}$ if and only if $ad\,(D)\mathfrak{s}\subset\mathfrak{s}$ i.e. $[D,\mathfrak{s}]\subset\mathfrak{s}$ (cf. Proposition 1).

q.e.d.

3. In this section we assume that the characterestic of the universal domain is 0. Then we have

PROPOSITION 5. *Let U be a subvariety of a connected algebraic group G which contains the unit element e: let k be a field of definition for G and U. Then the Lie algebra of the connected algebraic subgroup H of G generated by U is the minimal subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that* (i) *for any $y\in U$, $Ad(y)$ maps $\mathfrak{h}$ into itself,* (ii) *for any overfield k' of k, generic point u over k' on U and k'-derivation X of $k'(u)$, D_X* is in $\mathfrak{h}$.

PROOF. Evidently the intersection of those subspaces of $\mathfrak{g}$ with these properties also has these properties too, so there exists the unique minimal subspace $\mathfrak{h}$ of $\mathfrak{g}$. By the lemma 1 and 2 the Lie algebra of H satisfies these two conditions, and therefore $\mathfrak{h}$ is contained in the Lie algebra of H.

We shall show the converse. Let D be an element of the Lie algebra of H, defined over k; let $u_1,\ldots,u_r$ be independent generic points over k on U such that product $h=u_1\ldots u_r$ is a generic point over k on H; let $\xi_1,\ldots,\xi_N$ be coordinate functions of G relative to an affine variety V in which e has a representative; let X be the k-derivation of $k(h)$ induced by D; put $K=k(u_1,\ldots,u_r)$ and $K_i=k(u_1,\ldots,\hat{u}_i,\ldots,u_r)$, where $\wedge$ means that the letter under $\wedge$ is to be omitted. Then there exists a k-derivation of K which is an extension of X, and which we shall denote by the same X. Let X_i be the K_i-derivation of K such that $X_i\,u_{ij}=X\,u_{ij}$, where u_{ij} is the j-th coordinate of the representative of u_i in $V\,(1\leqq i\leqq r, 1\leqq j\leqq N)$. In fact there exists such X_i, since K_i and $k(u_i)$ are linearly disjoint over k. Then we have $X=\sum_{i=1}^r X_i$. Put $s_1=e$, $s_i=\Pi_{p<i}\,u_p$, $t_i=\Pi_{i<q}\,u_q$ and $t_r=e$ $(1\leqq i\leqq r)$, then we have

$$D=\sum_{i=1}^r Ad(s_i)\,D_{X_i}. \tag{9}$$

In fact; it is sufficent to prove that the eqaulity of these two invariant derivations holds at h, i. e. putting $D\xi_i=\chi_i(\xi)$ and $Ad(s_i)D_{X_i}\cdot\xi_j=\chi_{ij}(\xi)$, we have to show that $\chi_j(h)=\sum_{i=1}^r\chi_{ij}(h)$. We have

$$\chi_j(h)=Xh_j=\sum_{i=1}^r X_i(s_iu_it_i)_j$$

$$= \sum_{i=1}^{r,\ v}{}_{p,q=1} (\partial\varphi^j(s_i, Y)/\partial Y_p)_{Y=u_i t_i} (\partial\varphi^p(Z, t_i)/\partial Z_q)_{Z=u_i} Xu_{iq}.$$

On the other hand we have

$$\chi_{ij}(\xi) = L^*_{s_i^{-1}} D_{x_i} L^*_{s_i} \xi_j = L^*_{s_i^{-1}} \left(\sum_{p=1}^{v} (\partial\varphi^j(s_i, Y)/\partial Y_p)_{Y=\xi} \chi_{ip}(\xi)\right),$$

where $\bar{\chi}_{ip}(\xi) = D_{x_i}\xi_p$. So, by the invariance of D_{x_i}

$$\chi_{ij}(h) = \sum_{p=1}^{v} (\partial\varphi^j(s_i, Y)/\partial Y_p)_{Y=u_i t_i} \chi_{ip}(u_i t_i)$$

$$= \sum_{p,q=1}^{v} (\partial\varphi^j(s_i, Y)/\partial Y_p)_{Y=u_i t_i} (\partial\varphi^p(Z, t_i)/\partial Z_q)_{Z=u_i} \bar{\chi}_{iq}(u_i).$$

By the lemma 1, $\bar{\chi}_{iq}(u_i) = X_i u_{iq} = Xu_{iq}$. Thus we have proved (9).

For $y_1, y_2 \in U$, we have $Ad(y_1 y_2) = Ad(y_1)Ad(y_2)$ and therefore $Ad(y_1 y_2)\mathfrak{h} \subset \mathfrak{h}$. Thus, from (i) and (ii) it follows that D is in $\mathfrak{h}$.

q. e. d.

Now we prove the main theorem:

THEOREM. *Let G be a connected algebraic group with the Lie algebra $\mathfrak{g}$; let H_i be connected algebraic subgroups of G with the Lie algebras $\mathfrak{h}_i$, where i runs through a set I of indecies; let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ generated by all $\mathfrak{h}_i$' s; let H be the connected algebraic subgroup of G generated by all H_i' s. Then the Lie algebra of H is $\mathfrak{h}$.*

PROOF. Suppose that I is a finite set, say $I = \{1, \ldots, r\}$. Let k be a field of definition for $H_1, \ldots, H_r$ and G; let $h_1, \ldots, h_r$ be independent generic points over k on $H_1, \ldots, H_r$, respectively; let U be the locus of the product $u = h_1 \ldots h_r$ over k on G, then H is the connected algebraic subgroup of G generated by U. Let $\mathfrak{h}'$ be the Lie algebra of H. Since H contains H_i, so $\mathfrak{h}'$ contains $\mathfrak{h}_i$ and therefore $\mathfrak{h}'$ contains $\mathfrak{h}$.

To prove the converse we have to show that $\mathfrak{h}$ satisfies the two conditions of the proposition 5.

As for (i); Let $\bar{H}$ be the set of those elements x of G such that $Ad(x)$ maps $\mathfrak{h}$ into itself, then $\bar{H}$ is an algebraic group (cf. the corollary of the proposition 4). Let $\bar{H}_0$ be the component of $\bar{H}$ containing e, then, by the same corollary the Lie algebra $\bar{\mathfrak{h}}_0$ of $\bar{H}_0$ is the subalgebra of $\mathfrak{g}$ consisting of those D of $\mathfrak{g}$ such that $[D, \mathfrak{h}] \subset \mathfrak{h}$, so all $\mathfrak{h}_i$'s are in $\bar{\mathfrak{h}}_0$, and by the corollary 2 of the proposition 2, all H_i's are in $\bar{H}_0$ and therefore H is contained in $\bar{H}_0$. Thus we proved that the condition (i) is satisfied.

As for (ii); let k' be an overfield of k; let u be a generic point over k' on U; let X be any k'-derivation of $k'(u)$. Let $h_1', \ldots, h_r'$ be independent generic points over k' on $H_1, \ldots, H_r$, respectively, then $u' = h_1' \ldots h_r'$ is a generic point over k' on U. So we have a k'-isomorphism of $k'(u')$ which transforms u into u'. Let X' be the k'-derivation of $k'(u')$ induced by the X. As in the proof of the proposition 5 we extend X' into K and define the K_1-

derivation X_i' of K, where $K = k'(h_1', \dots, h_r')$ and $K_i = k'(h_1', \dots, \hat{h}_i', \dots h_r')$, and we have $D_{X'} = \sum_{i=1}^{r} Ad(s_i)\ D_{X'_i}$, where s_i is a product of some finite points of U. From the definition $D_{X'_i}$ is in $\mathfrak{h}_i$, and now that the condition (i) has been proved, we have $D_{X'} \in \mathfrak{h}$. But from the definition the local components of D_X and $D_{X'}$ at u' are same, and therefore we have $D_X = D_{X'}$. Thus the condition (ii) has been proved.

In the infinite case, for any finite subset E of I, let $\mathfrak{h}_E$ be the subalgebra of $\mathfrak{g}$ generated by all $\mathfrak{h}_i$ for $i \in E$; let H_E be the connected algebraic subgroup of G with the Lie algebra $\mathfrak{h}_E$; let $\mathfrak{h}_{E_0}$ be such one of those $\mathfrak{h}_E$ that dim $\mathfrak{h}_{E_0}$ is the maximal. Then if E_0 is a subset of E, have $\mathfrak{h}_{E_0} = \mathfrak{h}_E$. So we have $\mathfrak{h} = \mathfrak{h}_{E_0}$.

q. e. d.

The next corollary gives a characterization of algebraic subalgebra of $\mathfrak{g}$, which is a generalization of the proposition 2 of [1] p. 181.

COROLLARY. *Let $\mathfrak{g}$ be the Lie algebra of a connected algebraic group; let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is algebraic if and only if for $D \in \mathfrak{h}$ any replica of D is in $\mathfrak{h}$.*

PROOF. The necessity is trivial from the definition of replica. Conversely, $\mathfrak{h}$ being generated by $\mathfrak{h}_D's$ for $D \in \mathfrak{h}$, $\mathfrak{h}$ is algebraic (cf. the theorem).

q. e. d.

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