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A Note on the Limit of the Core of an Exchange Economy

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1. INTRODUCTION

In his Mathematical Psychics Edgeworth [2] considered an exchange economy with 2r consumers and two commodities. The consumers were divided into types, everyone of the same type having identical preferences and initial resources. Edgeworth showed that as r increases the core or the contract curve shrinks and has the set of competitive allocations as its limit.

Using the Edgeworth procedure for enlarging the market, Debreu and Scarf [1] studied an exchange economy consisting of m rather than two types of consumers, each repeated r times, and an arbitrary number of commodities. Assuming insatiability, strong-convexity and continuity of preferences and strict positivity of initial resources, they proved that

(1) An allocation in the core gives every consumer of the same type the same commodity bundle.

(2) As r passes to infinity the core has the set of competitive allocations as its limit.

The purpose of this paper is to simplify the second proof given

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* I am indebted to Herbert Scarf and Lloyd Shapley for reading preliminary drafts of this paper and making valuable comments.

by Debreu and Scarf by assuming that the preference orderings may be represented by differentiable, strictly concave utility functions. For convenience all points on the boundary of the commodity orthant are assumed to have zero utility. This assumption may be weakened, though this would complicate the proofs slightly.

It is proved that in the limit an allocation of the total supply that cannot be blocked by the restricted class of coalitions consisting first of all consumers of type i and all but one of the other types, secondly, the coalitions consisting of one consumer of each type and finally the consumers by themselves is a competitive allocation. Only a small fraction of the total number of coalitions therefore has to be considered as r passes to the limit. On the other hand it is easy to construct examples with nondifferentiable preferences in which blocking by this restricted class of coalitions is not sufficient to produce a competitive equilibrium.

2. THE CORE IN A PURE EXCHANGE ECONOMY

We shall consider an exchange economy with m consumers each with specific preferences for commodity bundles consisting of nonnegative quantities of a finite number of commodities, v. Such a commodity bundle is represented by a vector in the nonnegative orthant of the commodity space and the preferences of i^{th} consumer by a complete preordering, $\underset{i}{\gtrsim}$.

We shall assume that the preordering, $\underset{i}{\gtrsim}$, may be represented by a utility function $u_i(x)$ that is defined over the nonnegative orthant of the commodity space. For convenience all points on the boundary of the nonnegative orthant are assumed to have zero utility. For interior points of the nonnegative orthant we shall assume that

2.1 $u_i(x)$ is strictly concave

2.2 $u_i(x)$ is differentiable

2.3 $$0 < \frac{\partial u_i}{\partial x_{ij}} < \infty$$

$$i = 1, .., m$$
$$j = 1, .., v$$

where the subscript j refers to commodity j .

Each consumer owns a commodity bundle which he is interested in exchanging for preferred commodity bundles. The vector w_i will represent the resources of the i^{th} consumer.

2.4 We shall assume that every consumer owns a strictly positive quantity of every commodity.

The core can now be defined. Let $(x_1,\ldots,x_m)$ with

2.5 $$\sum_{i=1}^{m} (x_i - w_i) = 0$$

be an assignment of the total supply to the various consumers and let S

be an arbitrary set of consumers. We say that the allocation is blocked by S if it is possible to find commodity bundles x_i' for all i in S such that

2.6 $$\sum_{i \in S} (x_i' - w_i) = 0$$

$u_i(x') \geq u_i(x)$ for all i in S with strict inequality for at least one member of S.

The core of the economy is defined as the collection of all allocations of the total supply which cannot be blocked by any set S. One immediate consequence of this definition is that an allocation in the core is Pareto optimal. Further our assumptions on initial resources and on the utility functions imply that an allocation in the core gives all consumers a strictly positive quantity of every commodity, otherwise the allocation $(x_1, \ldots, x_m)$ would be blocked by the consumers themselves.

It is known that given our assumptions on the utility functions and initial holdings there is a competitive allocation, which must necessarily be in the core.

3. THE CORE AS THE NUMBER OF CONSUMERS BECOMES INFINITE

We shall now follow the procedure first used by Edgeworth for enlarging the market. We imagine the economy to be composed of m types of consumers with r consumers of each type. For two consumers to be of the same type we require them to have precisely the same preferences and precisely the same vector of initial resources. The economy therefore consists of m r consumers.

The argument given by Debreu and Scarf, considering the coalition of all consumers and coalitions of one consumer of each type, proves

Theorem 1. An allocation in the core assigns the same consumption vector to all consumers of the same type.

Theorem 1 implies that an allocation in the core for the repeated economies considered here may be described by a collection of m commodity bundles $(x_1, \ldots, x_m)$ such that

3.1 $$\sum_{1}^{m} (x_i - w_i) = 0$$

As observed by Debreu and Scarf an allocation in the core for $(r + 1)$ is contained in the core for r.

If a Pareto optimal allocation of the total supply in the economy consisting of one participant of each type is repeated when we enlarge the economy to r participants of each type, the resulting allocation is Pareto optimal in the larger economy.

In addition if a competitive allocation in the economy consisting of one consumer of each type is repeated when the economy is enlarged to r consumers of each type, the resulting allocation is competitive in the larger economy and consequently is in the core.

We shall now prove Theorem 2.

Theorem 2. If $(x_1, \ldots, x_n)$ is a Pareto optimal allocation of the total supply, giving every consumer a strictly positive quantity of every commodity and $(x_1, \ldots, x_m)$ cannot be blocked when r passes to the limit by the $m \cdot r$ coalitions consisting of all consumers of type i and all but one of the other types then $(x_1, \ldots, x_m)$ is a competitive allocation.

From the definition of Pareto optimality we have that the allocation $(x_1, \ldots, x_m)$ is associated with a set of numbers $\Pi_2, \ldots, \Pi_v$ such that

$$3.2 \qquad \frac{\frac{\partial u_i}{\partial x_{ij}}}{\frac{\partial u_i}{\partial x_{ij}}} = \Pi_j \qquad \begin{array}{l} i = 1, \ldots, m \\ j = 2, \ldots, v \end{array}$$

where the partial derivatives $\frac{\partial u_i}{\partial x_{ij}}$ are evaluated at x_i. From 2.3 follows that $\Pi_j > 0$ for all j.

Consider now the coalition of all consumers of the k^{th} type, and all but one of the other types. Let all consumers of type i $(i=1,..,m \; i \neq k)$ get x_i and let what is left be divided among the r consumers of the k^{th} type. We then have

$$3.3 \qquad x_k' = x_k - \frac{1}{r}(x_k - w_k)$$

We must have

3.4 $$u_k(x') \leq u_k(x)$$

otherwise the allocation $(x_1, \ldots, x_{k-1}, x_k', x_{k+1} \ldots x_n)$ would block.

When r passes to the limit we must have

3.5 $$\lim_{r \to \infty} \frac{u_k(x) - u_k(x')}{\frac{1}{r}} \geq 0$$

But

3.6 $$\lim_{r \to \infty} \frac{u_k(x) - u_k(x')}{\frac{1}{r}} = \sum_{j=1}^{v} \frac{\partial u_k}{\partial x_{kj}} (x_{kj} - w_{kj}) \geq 0$$

Dividing through by $\frac{\partial u_k}{\partial x_{k1}}$ and substituting in from 3.2 we get

3.7 $$\sum_{j=1}^{v} \Pi_j (x_{kj} - w_{kj}) \geq 0 \qquad \Pi_1 = 1$$

Since 3.7 is true for all k and $\sum_{i=1}^{m} (x_i - w_i) = 0$ we must have

3.8 $$\sum_{j=1}^{v} \Pi_j (x_{kj} - w_{kj}) = 0$$

$$k = 1, \ldots m$$

Given our assumption on the utility functions and initial resources a competitive allocation may be interpreted as a Pareto optimal allocation of the total supply associated with a price vector Π $(\Pi_1 = 1)$ such that

3.9 $$\sum_{j=1}^{v} \Pi_j(x_{ij} - w_{ij}) = 0 \qquad i = 1, \ldots, m$$

3.10 $$\frac{\dfrac{\partial u_i}{\partial x_{ij}}}{\dfrac{\partial u_i}{\partial x_{i1}}} = \Pi_j \qquad \begin{array}{l} j = 2, \ldots, v \\ i = 1, \ldots, m \end{array}$$

Since our allocation satisfies the conditions, Theorem 2 is demonstrated.

4. NONDIFFERENTIABLE PREFERENCES

The purpose of this section is to give an example, due to Scarf and Shapley, with nondifferentiable utility functions in which blocking by the restricted class of coalitions, mentioned above, is not sufficient to produce a competitive equilibrium.

Consider an economy with 3 types of consumers and 2 commodities. Let the utility functions and initial resources be

4.1 $$\begin{array}{ll} u_1(x) = \min(x_1, x_2) & w_1 = (1,7) \\ u_2(x) = \min(x_1, 1/2\, x_2) & w_2 = (5,1) \\ u_3(x) = \min(1/2\, x_1, x_2) & w_3 = (3,2) \end{array}$$

The competitive allocation of this economy is calculated to be

$$x_1 = (3.17,\ 3.17)$$
$$x_2 = (2.61,\ 5.22) \qquad (4.2)$$
$$x_3 = (3.22,\ 1.61)$$

The allocation

$$x_1' = (2,\ 2)$$
$$x_2' = (3,\ 6) \qquad (4.3)$$
$$x_3' = (4,\ 2)$$

is a Pareto optimal allocation of the total supply. It is easily verified that this allocation cannot be blocked by the restricted class of coalitions above.

[1] Debreu, Gerard and Scarf, Herbert, "A Limit Theorem on the Core of an Economy," International Economic Review, September 1963.

[2] Edgeworth, F. Y., Mathematical Psychics (London: Kegan Paul, 1881).