A NOTE ON THE METHOD OF MULTIPLE SCALES*

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Abstract. A modification of the method of multiple scales makes use of the expansion of parameters of the system in order to remove undesirable characteristics of the solution obtained by the usual multiple-scale method. When applied to the damped harmonic oscillator, the modification leads to the exact solution. For the Duffing equation it leads to an approximation which can be reduced to the solution reported by Nayfeh [1]. However, the solution derived here appears to be more accurate and the frequency takes on a form without nonuniformities.

The method of multiple scales produces a uniformly valid expansion for systems in which a troublesome term is multiplied by a small parameter and for which an ordinary perturbation expansion leads to a nonuniformly valid series solution. For example, the weakly damped harmonic oscillator

$$\ddot{x} + 2\varepsilon\dot{x} + \omega^2 x = 0, \qquad \varepsilon \ll 1, \tag{1}$$

has the general solution

$$x = [a_0 e^{i\sqrt{\omega^2-\varepsilon^2}t} + a_0^* e^{-i\sqrt{\omega^2-\varepsilon^2}t}]e^{-\varepsilon t}, \tag{2}$$

where a_0^* is the complex conjugate of a_0. An ordinary perturbation in ε yields results which correspond to expanding $e^{-\varepsilon t}$ and $\exp[\pm i(\omega^2 - \varepsilon^2)^{1/2}t$ in power series which are nonuniformly valid since t can always be large enough to offset the smallness of ε or ε^2.

The derivative expansion method of multiple scales [1, pp. 236–240] makes use of the time expansion

$$t_n = \varepsilon^{n t}, \qquad x = x(t_0, t_1, t_2 \cdots), \qquad dx/dt = D_0 x + \varepsilon D_1 x + \varepsilon^2 D_2 x + \cdots, \tag{3}$$

where $D_n \equiv \partial/\partial t_n$. The time t_0 is the unstretched time coordinate and the remaining t_n correspond to longer time scales. To $O(\varepsilon^2)$ the solution

$$\begin{aligned} x &= Ae^{-t_1} \exp\left[i\omega\left(t_0 - \frac{1}{2\omega^2} t_2\right)\right] \\ &= Ae^{-\varepsilon t} \exp[i\omega(t - \varepsilon^2 t/2\omega^2)] + CC, \end{aligned} \tag{4}$$

where CC stands for complex conjugate, is obtained by Nayfeh [1] with the derivative-expansion method, the two-variable expansion method and the generalized method of multiple scales.

It is clear that (4) is a distinct improvement over the ordinary perturbation solution since the important exponentially decaying part of the solution emerges and the amplitude A does not involve the time. However, it is also clear that the solution still contains a secular type of behavior since the frequency term is not uniformly valid. The fact that

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the essential physics is already contained in (4) is gratifying, but that is not a feature that can generally be obtained for more complicated problems. Hence, it will be helpful to eliminate the inadequacy in (4) if that is possible.

In this example it is obvious that, since the lowest-order equation is

$$D_0^2 x + \omega^2 x = 0, \tag{5}$$

the lowest-order solution is proportional to $\exp[\pm i\omega t_0]$ and subsequent orders can only modify this result. Since the inadequacy is caused by the "wrong" lowest-order frequency, it is plausible to try to correct the lowest-order frequency. We do so by expanding the coefficient of the last term in (1), viz. the parameter ω^2, in powers of ε by writing

$$\omega^2 = \omega_0^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \tag{6}$$

where the first term is written as a square for convenience. At the outset none of the ω_n are known. We shall evaluate them by imposing the condition that the frequency be nonsecular, thereby eliminating the inadequacy in (4).

Making use of (3) and (6) in (1), we obtain

$$[D_0^2 + 2\varepsilon D_1 D_0 + \varepsilon^2(D_1^2 + 2D_0 D_2) + \cdots]x + \omega_0^2 x$$
$$= -2\varepsilon(D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots)x - \varepsilon\omega_1 x - \varepsilon^2\omega_2 x - \cdots. \tag{7}$$

Then equating to zero the coefficients of like powers of ε yields to $O(\varepsilon^2)$

$$D_0^2 x + \omega_0^2 x = 0, \tag{8}$$

$$2D_0 D_1 x = -2D_0 x - \omega_1 x, \tag{9}$$

$$(D_1^2 + 2D_0 D_2)x = -2D_1 x - \omega_2 x. \tag{10}$$

The general solution of (8) is

$$x = A(t_1, t_2)\exp[i\omega_0 t_0] + CC. \tag{11}$$

Substituting (11) into (9) yields

$$(2i\omega_0 D_1 A + 2i\omega_0 A + \omega_1 A)\exp[i\omega_0 t_0] + CC = 0 \tag{12}$$

and, since the coefficients of $\exp[i\omega_0 t_0]$ and $\exp[-i\omega_0 t_0]$ must vanish separately,

$$D_1 A + \left(1 - \frac{i\omega_1}{2\omega_0}\right)A = 0 \tag{13}$$

and

$$A = a(t_2)\exp\left[-t_1\left(1 - \frac{i\omega_1}{2\omega_0}\right)\right]. \tag{14}$$

As pointed out earlier, the coefficient ω_1 is to be evaluated so that the frequency is nonsecular. Hence,

$$\omega_1 = 0, \qquad A = a(t_2)e^{-t_1}. \tag{15}$$

Substituting (11) and (14) into (10) yields

$$[2i\omega_0 D_2 a + (\omega_2 - 1)a]\exp[-t_1 + i\omega_0 t_0] + CC = 0. \tag{16}$$

Hence,

$$D_2 a - \frac{i(\omega_2 - 1)}{2\omega_0} a = 0 \tag{17}$$

and

$$a = a_0 \exp[i(\omega_2 - 1)t_2/2\omega_0]. \tag{18}$$

In order to suppress secular terms in the frequency we must have

$$\omega_2 = 1, \qquad a = a_0. \tag{19}$$

Hence, to $O(\varepsilon^2)$ the solution is

$$x = a_0 \exp[i\omega_0 t_0 - t_1] + CC \tag{20}$$

with

$$\omega^2 = \omega_0^2 + \varepsilon^2 \qquad \text{or} \qquad \omega_0^2 = \omega^2 - \varepsilon^2. \tag{21}$$

Therefore, the solution (20) can be written

$$x = a_0 \exp[i\sqrt{\omega^2 - \varepsilon^2}\, t - \varepsilon t] + CC \tag{22}$$

which is identical to (2).

The fact that the exact solution emerges here is fortuitous. However, the reasoning is applicable to more complicated problems for which the exact solution cannot be obtained, and it yields results that are at least an improvement over the usual methods of multiple scales. In each case, the procedure is equivalent: expand the available parameters in the equation(s) and use the unknown coefficients of the expansion to remove undesirable traits of the approximate solution. To a certain extent the choice of undesirable traits that one removes may be subjective. In the foregoing example there was only one inadequacy, the secular frequency, so the choice was obvious.

A second example is the Duffing equation

$$\ddot{u} + \omega^2 u + \varepsilon u^3 = 0 \tag{23}$$

with a nonlinear spring described by the term εu^3. To $O(\varepsilon^2)$ the solution obtained by Nayfeh [1] by the three multiple-scale methods is

$$u = a \cos(\bar{\omega} t + \chi) + \frac{\varepsilon a^3}{32\omega^2}\left(1 - \frac{21a^2\varepsilon}{32\omega^2}\right)\cos 3(\bar{\omega} t + \chi) + \frac{a^5\varepsilon^2}{1024\omega^4} \cos 5(\bar{\omega} t + \chi), \tag{24}$$

where

$$\bar{\omega} = \omega + \frac{3a^2\varepsilon}{8\omega} - \frac{150^4\varepsilon^2}{256\omega^2}. \tag{25}$$

Here again, the frequency contains secular terms which limit the validity of the solution, so we shall use (6) to suppress this secularity since the lowest-order frequency arises from the coefficient of u in (23).

Because of the nonlinearity in this problem it is necessary to expand u in a series of ε as well,

$$u = \sum_{n=0}^{N} \varepsilon^n u_n. \tag{26}$$

Then with expansions equivalent to (3) we obtain the equations to $O(\varepsilon^2)$

$$D_0^2 u_0 + \omega_0^2 u_0 = 0, \tag{27}$$

$$D_0^2 u_1 + \omega_0^2 u_1 = -2D_0 D_1 u_0 - u_0^3 - \omega_1 u_0, \tag{28}$$

$$D_0^2 u_2 + \omega_0^2 u_2 = -2D_0 D_1 u_1 - 2D_0 D_2 u_0 - D_1^2 u_0 - 3u_0^2 u_1 - \omega_1 u_1 - \omega_2 u_0. \tag{29}$$

Eq. (27) has the solution

$$u_0 = A(t_1, t_2)\exp[i\omega_0 t_0] + CC \tag{30}$$

and (28) becomes

$$D_0^2 u_1 + \omega_0^2 u_1 = (-2i\omega_0 D_1 A - \omega_1 A - 3A^2 A^*)\exp[i\omega_0 t_0] - A^3 \exp[3i\omega_0 t_0] + CC. \tag{31}$$

To suppress secular terms of the form $t_0 \exp[i\omega_0 t_0]$ we set the coefficient of $\exp[i\omega_0 t_0]$ on the right-hand side equal to zero:

$$2i\omega_0 D_1 A + \omega_1 A + 3A^2 A^* = 0. \tag{32}$$

Now write

$$A = \frac{a}{2} e^{i\phi} \tag{33}$$

where $a(t_1, t_2)$ and $\phi(t_1, t_2)$ are real. Then equating real and imaginary parts of (32), we obtain

$$D_1 a = 0, \qquad D_1 \phi = \frac{\omega_1}{2\omega_0} + \frac{3a^2}{8\omega_0} \tag{34}$$

with solutions

$$a = a(t_2), \qquad \phi = \left(\frac{\omega_1}{2\omega_0} + \frac{3a^2}{8\omega_0}\right)t_1 + \phi_0(t_2), \tag{35}$$

so that

$$A = \frac{a(t_2)}{2} \exp\left[i\left(\frac{\omega_1}{2\omega_0} + \frac{3a^2}{8\omega_0}\right)t_1 + i\phi_0(t_2)\right]. \tag{36}$$

To suppress secular terms in the frequency, we require

$$\omega_1 = -3a^2/4 \tag{37}$$

so that $D_1 A = 0$ and

$$A = \frac{a(t_2)}{2} \exp[i\phi_0(t_2)]. \tag{38}$$

The remaining terms in (31) lead to the solution

$$u_1 = B(t_1, t_2)\exp[i\omega_0 t_0] + \frac{A^3}{8\omega_0^2} \exp[3i\omega_0 t_0] + CC. \tag{39}$$

With (30), (37), (38) and (39) the right-hand side of (29) becomes

$$\left\{-2i\omega_0(D_1 B + D_2 A) - 3\left(A^2B^* + 2AA^*B + \frac{A^{*2}A^3}{8\omega_0^2}\right) - \omega_2 A - \omega_1 B\right\}\exp[i\omega_0 t_0]$$

$$-\left\{\left(3A^2B + \frac{3A^4A^*}{4\omega_0^2} + \frac{\omega_1 A^3}{8\omega_0^2}\right)\exp[3i\omega_0 t_0] + \frac{3A^5}{8\omega_0^2}\exp[5i\omega_0 t_0]\right\} + CC. \quad (40)$$

The coefficient of $\exp[i\omega_0 t_0]$ must vanish if secular terms are to be eliminated. This is accomplished by writing $B \equiv 0$ and with A given by (38), we obtain

$$D_2 a = 0, \qquad \omega_0 D_2 \phi_0 = \frac{3a^5}{256\omega_0^2} + \frac{\omega_2 a}{2}. \quad (41)$$

Thus, a is a constant and in order to suppress secular behavior in frequency we must have $D_2(\phi_0) = 0$ or

$$\omega_2 = -3a^4/128\omega_0^2. \quad (42)$$

The terms multiplying $\exp[3i\omega_0 t_0]$ and $\exp[5i\omega_0 t_0]$ in (40) yield

$$u_2 = \frac{a^5}{2048\omega_0^4}\exp[5i(\omega_0 t_0 + \phi_0)] + \frac{3a^5}{2048\omega_0^4}\exp[3i(\omega_0 t_0 + \phi_0)]. \quad (43)$$

Hence, the solution of (23) to $O(\varepsilon^2)$ is

$$u = a\left[\cos(\omega_0 t + \phi_0) + \frac{\varepsilon a^2}{32\omega_0^2}\left(1 + \frac{3\varepsilon a^2}{32\omega_0^2}\right)\cos 3(\omega_0 t_0 + \phi_0) + \left(\frac{\varepsilon a^2}{32\omega_0^2}\right)\cos 5(\omega_0 t_0 + \phi_0)\right], \quad (44)$$

where

$$\omega^2 = \omega_0^2 - \frac{3a^2}{4}\varepsilon\left(1 + \frac{a^2\varepsilon}{32\omega_0^2}\right). \quad (45)$$

Eq. (45) can be solved for ω_0. To this order it suffices to set $\omega_0^2 = \omega^2$ in the last term and obtain

$$\omega_0 = \left(\omega^2 + \frac{3a^2\varepsilon}{4}\left(1 + \frac{a^2\varepsilon}{32\omega^2}\right)\right)^{1/2}. \quad (46)$$

This solution reduces to (24) if ω_0 is expanded in powers of ε and the expansion is substituted into (44) wherever ω_0 appears. However, the present form is more accurate in the sense that it contains (naturally) selected higher-order terms in the expression for ω_0. Furthermore, the frequency ω_0 is expressed in a uniformly valid form rather than as a power series of terms multiplying t. The solution is still valid only to $O(\varepsilon^2)$ but the present form may be more useful, especially if one uses it to evaluate u for values of ε that are not so small.

In the two examples discussed above the unknown coefficients in the expansion (6) are used to suppress secularity in the frequency. In other problems the coefficients may be used for removing other undesirable traits of the approximate solution. The suggested procedure is to carry out the usual multiple-scales analysis, find where the inadequacies (if any) of the approximate solution occur and then use expansions analogous to (6) to remove those inadequacies.

Reference

[1] A. H. Nayfeh, *Perturbation methods*, Wiley-Interscience, John Wiley and Sons, New York, 1973