A NOTE ON THE NON-EQUIVALENCE OF THE NEYMAN-PEARSON AND GENERALIZED LIKELIHOOD RATIO TESTS FOR TESTING A SIMPLE NULL VERSUS A SIMPLE ALTERNATIVE HYPOTHESIS

by

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1. Introduction

Some introductory textbooks in mathematical statistics pose a problem equivalent to the following [1]: "Show that the likelihood ratio principle leads to the same test, when testing a simple hypothesis H_0 against an alternative simple hypothesis H_1, as that given by the Neyman-Pearson theorem." It is the object of this note to observe that a more careful wording of the problem would assume the existence of a (generalized) likelihood ratio test of a given size and to note that this existence is a non-trivial matter.

Suppose that $f(x;\theta_0)$ and $f(x;\theta_1)$ represent specified (joint) probability densities associated with the (perhaps vector valued) datum x and corresponding to the two states of nature θ_0 and θ_1. For observed x, we wish to test the simple null hypothesis that $f(\cdot;\theta_0)$ produced x, against the simple alternative that $f(\cdot;\theta_1)$ is the true underlying density. We write $H_0 : \theta = \theta_0$, $H_1 : \theta = \theta_1$ and define

$$\lambda(x) = \frac{f(x;\theta_0)}{f(x;\theta_1)}, \quad \Lambda(x) = \frac{f(x;\theta_0)}{\max\{f(x;\theta_0), f(x;\theta_1)\}} .$$

Note that $0 \le \Lambda(x) \le 1$ and that $\Lambda(x) = \min\{\lambda(x), 1\}$. Finally, for specified $0 \le \alpha \le 1$, define the Neyman-Pearson (NP) and Generalized Likelihood Ratio (LR) tests as those with critical regions respectively

$$R_{NP} = \{x | \lambda(x) < A_\alpha\} , \quad R_{LR} = \{x | \Lambda(x) < B_\alpha\} ,$$

where A_α and B_α are chosen (<u>if they exist</u>) to make

$$P_{\theta_0}\left(\lambda(X) < A_\alpha\right) = P_{\theta_0}\left(\Lambda(X) < B_\alpha\right) = \alpha \quad .$$

2. Example

We will restrict attention to a continuous random variable to emphasize that the possible non-existence of B_α is not due to its discreteness. Rather, it is that $P_{\theta_1}\left(\Lambda(X) = 1\right) > 0$. Suppose, for example, that we seek size $\alpha = \frac{1}{2}$ tests of $H_0 : \theta = 0$ versus $H_1 : \theta = 1$ for one observation from $X \sim N(\theta,1)$. Then $\lambda(x) = e^{\frac{1}{2}-x}$ and $P_0\left[\lambda(X) < e^{\frac{1}{2}}\right] = P_0\left[X > 0\right] = \frac{1}{2}$, i.e. $A_{\frac{1}{2}} = e^{\frac{1}{2}}$ and $R_{NP} = \left\{x \mid x > 0\right\}$. But $P_0\left[\Lambda(X) = 1\right] = P_0\left[f(X;0) \geq f(X;1)\right] = P_0\left[X \leq \frac{1}{2}\right] = 0.691$. Thus there does not exist a real number B_α for which $P_0\left[\Lambda(X) < B_\alpha\right] = \frac{1}{2}$. In fact there exist likelihood ratio tests only of size $\alpha \leq 1 - .691 = .309$ and $\alpha = 1$.

3. The Result

We shall now show that if both NP and LR tests exist, then they are equivalent and establish conditions for their existence. First note that $\Lambda(x) = \min\left\{\lambda(x),1\right\} \leq \lambda(x)$ so that if $\lambda(x) < c$, then $\Lambda(x) < c$. Thus

$$P_{\theta_0}\left[\Lambda(X) < B_\alpha\right] = \alpha = P_{\theta_0}\left[\lambda(X) < A_\alpha\right] \leq P_{\theta_0}\left[\Lambda(X) < A_\alpha\right], \text{ and so } B_\alpha \leq A_\alpha \quad .$$

Next observe that B_α is a non-decreasing function of α and that for $B_\alpha > 1$, $P_{\theta_0}\left[\Lambda(X) < B_\alpha\right] = 1$, for $B_\alpha = 1$, $P_{\theta_0}\left[\Lambda(X) < B_\alpha\right] = 1 - P_{\theta_0}\left[\Lambda(X) = 1\right] = \alpha_0$ say, and $B_\alpha \leq 1$ if and only if $P_{\theta_0}\left[\Lambda(X) < B_\alpha\right] \leq \alpha_0$. Thus, there do not exist LR tests of size $\alpha > \alpha_0$, except the test of size $\alpha = 1$.

Now, except for non-existence due to the possible discreteness of X, there are LR tests of size $\alpha \le \alpha_0$, and in this case $B_\alpha \le 1$. So suppose $\alpha \le \alpha_0$ and thus

$$\begin{aligned} \alpha = P_{\theta_0}\left[\Lambda(X) < B_\alpha\right] &= P_{\theta_0}\left[\min\{\lambda(X),1\} < B_\alpha(\le 1)\right] \\ &= P_{\theta_0}\left[\lambda(X) < B_\alpha\right] \\ &\le P_{\theta_0}\left[\lambda(X) < A_\alpha\right] \qquad \text{since } B_\alpha \le A_\alpha \\ &= \alpha \ . \end{aligned}$$

Therefore $P_{\theta_0}\left[\lambda(X) < B_\alpha\right] = P_{\theta_0}\left[\lambda(X) < A_\alpha\right]$ and we may take $A_\alpha = B_\alpha \le 1$. In this case, $\Lambda(x) < B_\alpha$ if and only if $\min\{\lambda(x),1\} < B_\alpha$ if and only if $\lambda(x) < B_\alpha = A_\alpha$, i.e. $x \epsilon R_{LR}$ if and only if $x \epsilon R_{NP}$, so the tests are equivalent.

4. Summary

In summary, we have shown that there exist generalized likelihood ratio tests only of size $\alpha = 1$ and $\alpha \le \alpha_0$ where $\alpha_0 = 1 - P_{\theta_0}\left[\Lambda(X) = 1\right] = 1 - P_{\theta_0}\left[f(X;\theta_0) \ge f(X;\theta_1)\right]$, and that if such a test exists, it is equivalent to the Neyman-Pearson (most powerful) test of the same size.

REFERENCE

Craig, A. T. and R. V. Hogg. Introduction to Mathematical Statistics, 3rd ed., p. 307, Macmillan Company [1970].