# 35. A Note on the Normal Generation of Ample Line Bundles on Abelian Varieties

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Let $k$ be an algebraically closed field, let $A$ be an abelian variety defined over $k$ and let $L$ be an ample line bundle on $A$. It is well known that $L^{\otimes n}$ is normally generated if $n \geqq 3$ (see Koizumi [2] or Sekiguchi [5], [6]). But $L^{\otimes 2}$ is not normally generated in general because $L^{\otimes 2}$ is not very ample in general. For the very ampleness of $L^{\otimes 2}$, the following result is obtained (see Ohbuchi [3]).

**Theorem A.** *$L^{\otimes 2}$ is not very ample if and only if $(A, L)$ is isomorphic to $(A_1 \times A_2, \mathcal{O}(\Theta_1 \times A_2 + A_1 \times D_2))$ where $A_1$ and $A_2$ are abelian varieties with $\dim(A_1) > 0$ and $\Theta_1$ is a theta divisor.*

Our purpose is to give a condition for the normal generation of $L^{\otimes 2}$. The result is as follows:

**Theorem.** *If $\operatorname{char}(k) \neq 2$ and $L$ is a symmetric ample line bundle, then $L^{\otimes 2}$ is normally generated if and only if the origine $0$ of $A$ is not contained in $\operatorname{Bs}|L \otimes P_\alpha|$ for any $\alpha \in \hat{A}_2 = \{\alpha \in \hat{A}\,;\, 2\alpha = 0\}$ where $\hat{A}$ is the dual abelian variety of $A$, $P$ is the Poincaré bundle on $A \times \hat{A}$, $P_\alpha = P_{|A \times \{\alpha\}}$ for $\alpha \in \hat{A}$ and $\operatorname{Bs}|L \otimes P_\alpha|$ is the set of all base points of $L \otimes P_\alpha$.*

To prove this theorem, we need three lemmas.

**Lemma 1.** *If $\operatorname{char}(k) \neq 2$ and $L$ is a symmetric ample line bundle, then $\xi^*(p_1^* L \otimes p_2^* L) \simeq p_1^*(L^{\otimes 2}) \otimes p_2^*(L^{\otimes 2})$ where $p_i : A \times A \to A$ is the i-th projection $(i=1,2)$ and $\xi : A \times A \to A \times A$ is defined by $\xi(x, y) = (x+y, x-y)$ for all S-valued points $x, y$ where $S$ is a k-scheme.*

*Proof.* As $\xi^*(p_1^* L \otimes p_2^* L)_{|A \times \{y\}} \simeq T_y^* L \otimes T_{-y}^* L \simeq L^{\otimes 2}$ for any closed point $y \in A$, therefore $\xi^*(p_1^* L \otimes p_2^* L) \otimes (p_1^*(L^{\otimes 2}))^{-1} \simeq p_2^* M$ for some line bundle $M$ on $A$ by See-Saw theorem. Moreover $\xi^*(p_1^* L \otimes p_2^* L)_{|\{0\} \times A} \simeq L \otimes (-1_A)^* L \simeq L^{\otimes 2}$, hence $M \simeq L^{\otimes 2}$.

**Lemma 2.** *If $\operatorname{char}(k) \neq 2$ and $L$ is an ample line bundle, then*

$$\sum_{\alpha \in \hat{A}_2} \Gamma(A, L \otimes P_\alpha) \xrightarrow{2_A^*} \Gamma(A, 2_A^* L)$$

*is an isomorphism.*

*Proof.* This is a well known fact (see Mumford [1]).

**Lemma 3.** *If $L$ is an ample line bundle, then*

$$\Gamma(A, L^{\otimes n}) \otimes \Gamma(A, L^{\otimes m}) \longrightarrow \Gamma(A, L^{\otimes (n+m)})$$

*is surjective if $n \geqq 2, m \geqq 3$.*

*Proof.* See Koizumi [2] or Sekiguchi [5], [6].

*Proof of Theorem.* If the canonical map $\Gamma(A,L^{\otimes 2})\otimes\Gamma(A,L^{\otimes 2})\to\Gamma(A,L^{\otimes 4})$ is surjective, then $L^{\otimes 2}$ is normally generated by Lemma 3. Hence we prove that the canonical map $\Gamma(A, L^{\otimes 2})\otimes\Gamma(A, L^{\otimes 2})\to\Gamma(A, L^{\otimes 4})$ is surjective if and only if the origine 0 of $A$ is not contained in $\mathrm{Bs}\,|L\otimes P_\alpha|$ for any $\alpha\in\hat{A}_2$. Since $L$ is symmetric, there exists an isomorphism $2_A^*L\simeq L^{\otimes 4}$ (see Mumford [1]). As $(L\otimes P_\alpha)^{\otimes 2}\simeq L^{\otimes 2}$ for any $\alpha\in\hat{A}_2$, therefore we obtain the following commutative diagram:

$$\begin{array}{ccc}\Gamma(p_1^*(L\otimes P_\alpha)\otimes p_2^*(L\otimes P_\alpha)) & \xrightarrow{\xi^*} & \Gamma(p_1^*(L^{\otimes 2})\otimes p_2^*(L^{\otimes 2})) \\ {\scriptstyle 2_{A\times A}^*}\downarrow & & \downarrow{\scriptstyle \xi^*} \\ \Gamma(p_1^*(2_A^*L)\otimes p_2^*(2_A^*L)) & \xrightarrow{\simeq} & \Gamma(p_1^*(L^{\otimes 4})\otimes p_2^*(L^{\otimes 4})).\end{array}$$

By Künneth's formula, we obtain the following commutative diagram:

$$\begin{array}{ccc}\Gamma(L\otimes P_\alpha)\otimes\Gamma(L\otimes P_\alpha) & \xrightarrow{\xi^*} & \Gamma(L^{\otimes 2})\otimes\Gamma(L^{\otimes 2}) \\ {\scriptstyle 2_A^*\otimes 2_A^*}\downarrow & & \downarrow{\scriptstyle \xi^*} \\ \Gamma(2_A^*L)\otimes\Gamma(2_A^*L) & \xrightarrow{\simeq} & \Gamma(L^{\otimes 4})\otimes\Gamma(L^{\otimes 4}).\end{array}$$

Let $V_\alpha$ be a vector subspace of $\Gamma(2_A^*L)$ generated by $e^*(s)2_A^*(s')$ where $s$, $s'\in\Gamma(A, L\otimes P_\alpha)$ and $e^*:\Gamma(A, L\otimes P_\alpha)\to k$ is the evaluation map defined by the origine 0 of $A$ for any $\alpha\in\hat{A}_2$. As the canonical map $\Gamma(A, L^{\otimes 2})\otimes\Gamma(A, L^{\otimes 2})\to\Gamma(A, L^{\otimes 4})$ is obtained by

$$\Gamma(L^{\otimes 2})\otimes\Gamma(L^{\otimes 2})\xrightarrow{\xi^*}\Gamma(L^{\otimes 4})\otimes\Gamma(L^{\otimes 4})\xrightarrow{e^*\otimes id}\Gamma(L^{\otimes 4})$$

where $e^*:\Gamma(A, L^{\otimes 4})\to k$ is the evaluation map defined by the origine 0 of $A$, the image of $\Gamma(A,L^{\otimes 2})\otimes\Gamma(A,L^{\otimes 2})\to\Gamma(A,L^{\otimes 4})\simeq\Gamma(A,2_A^*L)$ is $\sum_{\alpha\in\hat{A}_2}V_\alpha$ by Lemma 2 and the above diagram because $e^*$ satisfies that $e^*(2_A^*s)=e^*(s)$. Hence the canonical map $\Gamma(A,L^{\otimes 2})\otimes\Gamma(A,L^{\otimes 2})\to\Gamma(A,L^{\otimes 4})$ is surjective if and only if $V_\alpha=2_A^*\Gamma(A, L\otimes P_\alpha)$ for any $\alpha\in\hat{A}_2$ because $V_\alpha$ is contained in $2_A^*\Gamma(A, L\otimes P_\alpha)$. If there exists an $s\in\Gamma(A, L\otimes P_\alpha)$ such that $e^*(s)\neq 0$, then it is clear that $V_\alpha=2_A^*\Gamma(A, L\otimes P_\alpha)$. Moreover if any $s\in\Gamma(A, L\otimes P_\alpha)$ satisfies that $e^*(s)=0$, then $V_\alpha=\{0\}\neq 2_A^*\Gamma(A, L\otimes P_\alpha)$. Hence $V_\alpha=2_A^*\Gamma(A, L\otimes P_\alpha)$ if and only if the origine 0 of $A$ is not contained in $\mathrm{Bs}\,|L\otimes P_\alpha|$. Therefore $L^{\otimes 2}$ is normally generated if and only if the origine 0 of $A$ is not contained in $\mathrm{Bs}\,|L\otimes P_\alpha|$ for any $\alpha\in\hat{A}_2$.

**Corollary.** *If $L$ is an ample line bundle on $A$ and $L$ is base point free, then $L^{\otimes 2}$ is normally generated.*

## References


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