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Convex Domains

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In this article a proof for the Poincaré inequality with explicit constant for convex domains is given. This proof is a modification of the original proof [5], which contains a mistake.

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1 Introduction

The classical proof for the Poincaré inequality

$$\|u\|_{L^2(\Omega)} \leq c_\Omega \|\nabla u\|_{L^2(\Omega)},$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $u \in H^1(\Omega)$ with vanishing mean value over Ω , is based on the compact embedding of $H^1(\Omega)$ in $L^2(\Omega)$ which is valid under quite general assumptions on Ω (cf. [6]). However, the constant c_Ω depends on the domain Ω , and the proof based on compactness does not provide insight into this dependency.

For practical purposes it is important to know an explicit expression for this constant (see for example [2], [7]). Therefore, the special case of convex domains is interesting, since in [5] this constant is proved to be d/π , where d is the diameter of Ω . Though this proof is elegant, it contains a mistake. The same mistake can also be found in [1], in which the L^1 estimate is considered.

The goal of this article is to fix the error made in [5]. Luckily, the constant d/π in the Poincaré inequality remains valid.

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2 The one-dimensional case

We first prove the Poincaré inequality for the one-dimensional case. In fact we will prove a generalization which the multidimensional case can be reduced to.

Lemma 2.1 *Let $m \in \mathbb{N}$ and ρ be a non-negative concave function on the interval $[0, L]$. Then for all $u \in H^1(0, L)$ satisfying*

$$\int_0^L \rho^m(x) u(x) \, dx = 0 \quad (2.1)$$

it holds that

$$\int_0^L \rho^m(x) |u(x)|^2 \, dx \leq \frac{L^2}{\pi^2} \int_0^L \rho^m(x) |u'(x)|^2 \, dx. \quad (2.2)$$

Furthermore, the constant L^2/π^2 is optimal.

Proof. (a) Let us first assume that ρ is strictly positive and twice differentiable. Then each non-zero function v minimizing the quotient

$$\frac{\int_0^L \rho^m(x) |u'(x)|^2 \, dx}{\int_0^L \rho^m(x) |u(x)|^2 \, dx} \quad (2.3)$$

and satisfying (2.1) must satisfy the Sturm-Liouville system (cf. [3])

$$[\rho^m v']' + \lambda \rho^m v = 0 \quad \text{with } v'(0) = v'(L) = 0, \quad (2.4)$$

where λ is the minimum of the quotient (2.3). After dividing (2.4) by ρ^m and differentiating, we introduce the new variable $w = \rho^{m/2} v'$ and obtain

$$w'' + \frac{m}{2} \left[\frac{\rho''}{\rho} - \left(1 + \frac{m}{2}\right) \frac{(\rho')^2}{\rho^2} \right] w + \lambda w = 0 \quad \text{with } w(0) = w(L) = 0.$$

Since ρ is concave, $\rho'' \leq 0$. Hence, $w'' + \lambda w \geq 0$ and integration by parts leads to

$$\lambda \geq \frac{\int_0^L |w'(x)|^2 \, dx}{\int_0^L |w(x)|^2 \, dx}.$$

The last quotient is bounded by the first eigenvalue of the vibrating string with fixed ends, which gives $\lambda \geq \pi^2/L^2$.

(b) If ρ is a non-negative concave function, we may represent it as the L^∞ -limit of strictly positive concave C^2 -functions ρ_k , cf. [4]. From Part (a) one has

$$\int_0^L \rho_k^m(x) |\hat{u}(x)|^2 \, dx \leq \frac{L^2}{\pi^2} \int_0^L \rho_k^m(x) |u'(x)|^2 \, dx,$$

where $\hat{u}(x) := u(x) - \bar{u}$ and

$$\bar{u} := \frac{\int_0^L \rho_k^m(x) u(x) dx}{\int_0^L \rho_k^m(x) dx}.$$

Hence,

$$\int_0^L \rho_k^m(x) |u(x)|^2 dx \leq \frac{L^2}{\pi^2} \int_0^L \rho_k^m(x) |u'(x)|^2 dx + \bar{u} \int_0^L \rho_k^m(x) u(x) dx.$$

In the limit $k \rightarrow \infty$ we obtain (2.2).

(c) To see that the constant L^2/π^2 is optimal, choose $\rho^m \equiv 1$, $L = 1$ and $u(x) = \cos(\pi x)$. Then $\int_0^1 \rho^m(x) u(x) dx = 0$ and

$$\frac{\int_0^1 \rho^m(x) |u(x)|^2 dx}{\int_0^1 \rho^m(x) |u'(x)|^2 dx} = \frac{1}{\pi^2} \frac{\int_0^1 \cos^2(\pi x) dx}{\int_0^1 \sin^2(\pi x) dx} = \frac{1}{\pi^2}.$$

■

3 The n -dimensional case

In the rest of this article we will consider the case $n \geq 2$. By the following lemma we are able to reduce the n -dimensional problem to the one-dimensional case.

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^n$ be a convex domain with diameter d . Assume that $u \in L^1(\Omega)$ satisfies $\int_{\Omega} u(x) dx = 0$. Then for any $\delta > 0$ there are disjoint convex domains Ω_i , $i = 1, \dots, k$, such that*

$$\bar{\Omega} = \bigcup_{i=1}^k \bar{\Omega}_i, \quad \int_{\Omega_i} u(x) dx = 0, \quad i = 1, \dots, k,$$

and for each Ω_i there is rectangular coordinate system such that

$$\Omega_i \subset \{(x, y) \in \mathbb{R}^n : 0 \leq x \leq d \text{ and } |y_j| \leq \delta, j = 1, \dots, n-1\}.$$

Proof. For each $\alpha \in [0, 2\pi]$ there is a unique hyperplane $H_{\alpha} \subset \mathbb{R}^n$ with normal $(0, \dots, 0, \cos(\alpha), \sin(\alpha))$ that divides Ω into two convex sets Ω'_{α} and Ω''_{α} of equal volume. Since $I(\alpha) = -I(\alpha + \pi)$, where $I(\alpha) = \int_{\Omega'_{\alpha}} u(x) dx$, by continuity there is α_0 such that $I(\alpha_0) = 0$. Applying this procedure recursively to each of the parts Ω'_{α_0} and Ω''_{α_0} , we are able to subdivide Ω into convex sets Ω_i such that each of the sets is contained between two parallel hyperplanes with normal of the form $(0, \dots, 0, \cos(\beta), \sin(\beta))$ at distance at most δ , and the average of u vanishes on each of them.

Consider one of these sets. By rotating the coordinate system we can assume that the normal of the enclosing hyperplanes is $(0, \dots, 0, 1)$. In these coordinates we apply the above arguments using hyperplanes with normals of the form $(0, \dots, 0, \cos(\alpha), \sin(\alpha), 0)$. Continuing this procedure we end up with the desired decomposition of Ω . \blacksquare

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^n$ be a convex domain with diameter d . Then*

$$\|u\|_{L^2(\Omega)} \leq \frac{d}{\pi} \|\nabla u\|_{L^2(\Omega)}$$

for all $u \in H^1(\Omega)$ satisfying $\int_{\Omega} u(x) \, dx = 0$.

Proof. Let us first assume that u is twice continuously differentiable. According to the previous Lemma 3.1 we are able to decompose Ω into convex subsets Ω_i such that for each Ω_i there is a rectangular coordinate system in which Ω_i is contained in

$$\{(x, y) \in \mathbb{R}^n : 0 \leq x \leq d_i, |y_j| \leq \delta \text{ for } j = 1, \dots, n-1\}.$$

We may assume that the interval $[0, d_i]$ on the x -axis is contained in Ω_i . Let $R(t)$ be the $(n-1)$ -volume of the intersection of Ω_i with the hyperplane $x = t$. In polar coordinates $R(t)$ can be written in the form

$$R(t) = \int_{\mathbb{S}^{n-2}} \int_0^{\rho(t, \omega)} r^{n-2} \, dr \, d\omega = \frac{1}{n-1} \int_{\mathbb{S}^{n-2}} \rho^{n-1}(t, \omega) \, d\omega,$$

where $\rho(t, \omega)$ is the distance of the boundary point of Ω_i at (t, ω) to the x -axis. Since Ω_i is convex, ρ is a concave function with respect to t .¹

From the smoothness of u it can be seen that there are constants c_1 , c_2 and c_3 such that

$$\left| \int_{\Omega_i} u(x, y) \, dx \, dy - \int_0^{d_i} u(x, 0) R(x) \, dx \right| \leq c_1 |\Omega_i| \delta \quad (3.1)$$

$$\left| \int_{\Omega_i} \left| \frac{\partial u}{\partial x}(x, y) \right|^2 \, dx \, dy - \int_0^{d_i} \left| \frac{\partial u}{\partial x}(x, 0) \right|^2 R(x) \, dx \right| \leq c_2 |\Omega_i| \delta \quad (3.2)$$

$$\left| \int_{\Omega_i} |u(x, y)|^2 \, dx \, dy - \int_0^{d_i} |u(x, 0)|^2 R(x) \, dx \right| \leq c_3 |\Omega_i| \delta \quad (3.3)$$

Let $\omega \in \mathbb{S}^{n-2}$. Since $u(\cdot, 0) \in H^1(0, d_i)$, we can apply Lemma 2.1 to $\hat{u}_\omega(x) := u(x, 0) - \bar{u}_\omega$, where

$$\bar{u}_\omega := \frac{\int_0^{d_i} u(x, 0) \rho^{n-1}(x, \omega) \, dx}{\int_0^{d_i} \rho^{n-1}(x, \omega) \, dx}.$$

¹In [5] it is claimed that $R(t)$ is a concave function, which is not true for $n \geq 3$.

Hence,

$$\int_0^{d_i} |\hat{u}_\omega(x)|^2 \rho^{n-1}(x, \omega) dx \leq \frac{d_i^2}{\pi^2} \int_0^{d_i} \left| \frac{\partial u}{\partial x}(x, 0) \right|^2 \rho^{n-1}(x, \omega) dx.$$

Applying Fubini's theorem we obtain

$$\begin{aligned} \frac{d_i^2}{\pi^2} \int_0^{d_i} \left| \frac{\partial u}{\partial x}(x, 0) \right|^2 R(x) dx &= \frac{1}{n-1} \int_{\mathbb{S}^{n-2}} \frac{d_i^2}{\pi^2} \int_0^{d_i} \left| \frac{\partial u}{\partial x}(x, 0) \right|^2 \rho^{n-1}(x, \omega) dx d\omega \\ &\geq \frac{1}{n-1} \int_{\mathbb{S}^{n-2}} \int_0^{d_i} |\hat{u}_\omega(x)|^2 \rho^{n-1}(x, \omega) dx d\omega \\ &= \frac{1}{n-1} \int_{\mathbb{S}^{n-2}} \int_0^{d_i} \hat{u}_\omega(x) u(x, 0) \rho^{n-1}(x, \omega) dx d\omega \\ &\geq \int_0^{d_i} |u(x, 0)|^2 R(x) dx - M \left| \int_0^{d_i} u(x, 0) R(x) dx \right|, \end{aligned}$$

where $M := \max_{\omega \in \mathbb{S}^{n-2}} |\bar{u}_\omega|$. By $\int_{\Omega_i} u(x, y) dx dy = 0$, (3.1) and (3.2) we are lead to

$$\begin{aligned} \int_0^{d_i} |u(x, 0)|^2 R(x) dx &\leq \frac{d_i^2}{\pi^2} \int_0^{d_i} \left| \frac{\partial u}{\partial x}(x, 0) \right|^2 R(x) dx + c_1 |\Omega_i| M \delta \\ &\leq \frac{d_i^2}{\pi^2} \int_{\Omega_i} |\nabla u(x, y)|^2 dx dy + \left(c_1 M + c_2 \frac{d_i^2}{\pi^2} \right) |\Omega_i| \delta. \end{aligned}$$

From (3.3) and the summation over i we obtain

$$\int_{\Omega} |u(x, y)|^2 dx dy \leq \frac{d^2}{\pi^2} \int_{\Omega} |\nabla u(x, y)|^2 dx dy + (c_1 M + c_2 \frac{d^2}{\pi^2} + c_3) |\Omega| \delta$$

and, since $\delta > 0$ is arbitrary, the desired estimate is proven. The assertion follows from the density of $C^\infty(\bar{\Omega})$ in $H^1(\Omega)$. \blacksquare

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