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A NOTE ON THE q-ANALOGUE OF KIM'S p-ADIC log GAMMA TYPE FUNCTIONS ASSOCIATED WITH q-EXTENSION OF GENOCCHI AND EULER NUMBERS WITH WEIGHT α

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Abstract. In this paper, we introduce the q-analogue of p-adic log gamma functions with weight alpha. Moreover, we give a relationship between weighted p-adic q-log gamma functions and q-extension of Genocchi and Euler numbers with weight alpha.

1. Introduction

Assume that p is a fixed odd prime number. Throughout this paper $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Also we denote $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ and $\exp(x) = e^x$. Let $v_p : \mathbb{C}_p \to \mathbb{Q} \cup \{\infty\}$ ($\mathbb{Q}$ is the field of rational numbers) denote the p-adic valuation of $\mathbb{C}_p$ normalized so that $v_p(p) = 1$. The absolute value on $\mathbb{C}_p$ will be denoted as $|\cdot|$, and $|x|_p = p^{-v_p(x)}$ for $x \in \mathbb{C}_p$. When one talks of q-extensions, q is considered in many ways, e.g. as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume $|1-q|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We use the following notation

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q}, \tag{1.1}$$

where $\lim_{q\to 1} [x]_q = x$; cf. [1-21].

For a fixed positive integer d, we set

$$X = X_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N\mathbb{Z}, \quad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p$$

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and

$$a+dp^N\mathbb{Z}_p=\left\{x\in X\mid x\equiv a\left(\operatorname{mod} dp^N\right)\right\},$$

where $a\in\mathbb{Z}$ satisfies the condition $0\leq a<dp^N$ (see [6, Section 2]).

It is known that

$$\mu_q\left(x+p^N\mathbb{Z}_p\right)=\frac{q^x}{\left[p^N\right]_q}$$

is a distribution on X for $q\in\mathbb{C}_p$ with $\left|1-q\right|_p\leq 1$.

Let $UD\left(\mathbb{Z}_p\right)$ be the set of uniformly differentiable function on $\mathbb{Z}_p$. We say that f is a uniformly differentiable function at a point $a\in\mathbb{Z}_p$, if the difference quotient

$$F_f\left(x,y\right)=\frac{f\left(x\right)-f\left(y\right)}{x-y}$$

has a limit $f^{'}\left(a\right)$ as $\left(x,y\right)\to\left(a,a\right)$ and denote this by $f\in UD\left(\mathbb{Z}_p\right)$. The p-adic q-integral of the function $f\in UD\left(\mathbb{Z}_p\right)$ is defined by

$$I_q\left(f\right)=\int_{\mathbb{Z}_p}f\left(x\right)d\mu_q\left(x\right)=\lim_{N\to\infty}\frac{1}{\left[p^N\right]_q}\sum_{x=0}^{p^N-1}f\left(x\right)q^x. \tag{1.2}$$

The bosonic integral is considered by Kim as the bosonic limit $q\to 1$, $I_1\left(f\right)=\lim_{q\to 1}I_q\left(f\right)$. Similarly, the p-adic fermionic integration on $\mathbb{Z}_p$ was defined by Kim as follows:

$$I_{-q}\left(f\right)=\lim_{q\to -q}I_q\left(f\right)=\int_{\mathbb{Z}_p}f\left(x\right)d\mu_{-q}\left(x\right).$$

Let $q\to 1$. Then we have p-adic fermionic integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}\left(f\right)=\lim_{q\to -1}I_q\left(f\right)=\lim_{N\to\infty}\sum_{x=0}^{p^N-1}f\left(x\right)\left(-1\right)^x.$$

Stirling asymptotic series are defined by

$$\log\left(\frac{\Gamma\left(x+1\right)}{\sqrt{2\pi}}\right)=\left(x-\frac{1}{2}\right)\log x+\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}}{n\left(n+1\right)}\frac{B_{n+1}}{x^n}-x, \tag{1.3}$$

where B_n are familiar n-th Bernoulli numbers (cf. [5, 6, 21]).

Recently, Araci, Acikgoz and Seo defined q-Genocchi polynomials with weight α in [1, 2] by the means of generating function:

$$\sum_{n=0}^{\infty}\widetilde{G}_{n,q}^{(\alpha)}\left(x\right)\frac{t^n}{n!}=t\int_{\mathbb{Z}_p}e^{[x+\xi]_{q^\alpha}t}d\mu_{-q}\left(\xi\right). \tag{1.4}$$

So from above, we easily get Witt's formula of q-Genocchi polynomials with weight α as follows:

$$\frac{\widetilde{G}_{n,q}^{(\alpha)}\left(x\right)}{n+1}=\int_{\mathbb{Z}_p}\left[x+\xi\right]_{q^\alpha}^n d\mu_{-q}\left(\xi\right), \tag{1.5}$$

where $\widetilde{G}_{n,q}^{(\alpha)}(0) := \widetilde{G}_{n,q}^{(\alpha)}$ are called the q-extension of Genocchi numbers with weight α (cf. [1, 2]).

For any non-negative integer n, Ryoo [17] defined the q-Euler numbers with weight α as follows:

$$\widetilde{E}_{n,q}^{(\alpha)} = \int_{\mathbb{Z}_p} [\xi]_{q^\alpha}\, d\mu_{-q}(\xi). \tag{1.6}$$

By (1.5) and (1.6), we get the following proposition:

Proposition 1. The following identity holds:

$$\widetilde{E}_{n,q}^{(\alpha)} = \frac{\widetilde{G}_{n+1,q}^{(\alpha)}}{n+1}. \tag{1.7}$$

In recent years, T. Kim studied the new formula of the p-adic q-analogue of $\log\left(\frac{\Gamma(x+1)}{\sqrt{2\pi}}\right)$, in which he derivatived interesting properties of q-Euler and q-Bernoulli numbers. By the same motivation, we introduce the q-analogue of p-adic log gamma functions with weight alpha. Furthermore, we get interesting properties of q-extension of Genocchi numbers with weight alpha.

On p-adic $\log\Gamma$ function with weight α

In this section, from (1.2), we start by the following expression:

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} q^l (-1)^{n-1-l} f(l), \tag{1.8}$$

where $f_n(x) = f(x+n)$ and $n \in \mathbb{N}$ (see [3, 5, 7, 15]).

In particular for $n = 1$ into (1.8), we easily see that

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0). \tag{1.9}$$

By the easy application, it is simple to indicate as follows:

$$((1+x)\log(1+x))' = 1 + \log(1+x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^n, \tag{1.10}$$

where $((1+x)\log(1+x))' = \frac{d}{dx}((1+x)\log(1+x))$.

By the expression of (1.10), we can derive

$$(1+x)\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^{n+1} + x + c, \text{ where } c \text{ is a constant.} \tag{1.11}$$

If we substitute $x = 0$, we have $c = 0$. By (1.10) and (1.11), we easily see that

$$(1+x)\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} x^{n+1} + x. \tag{1.12}$$

It is considered by T. Kim for q-analogue of p adic locally analytic function on $\mathbb{C}_p \backslash \mathbb{Z}_p$ as follows:

(1.13)
$$G_{p,q}(x) = \int_{\mathbb{Z}_p} [x+\xi]_q \left(\log [x+\xi]_q - 1\right) d\mu_{-q}(\xi) \text{ (for details, see [5, 6]).}$$

By the same motivation of (1.13), q-analogue of p-adic locally analytic function on $\mathbb{C}_p \backslash \mathbb{Z}_p$ with weight α as

$$G_{p,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} [x+\xi]_{q^\alpha} \left(\log [x+\xi]_{q^\alpha} - 1\right) d\mu_{-q}(\xi). \tag{1.14}$$

In particular $\alpha = 1$ into (1.14), we easily see that, $G_{p,q}^{(1)}(x) = G_{p,q}(x)$.

It is easy to show that,

$$\begin{aligned}
&[x+\xi]_{q^\alpha} \\
&= 1 + q^\alpha + q^{2\alpha} + \cdots + q^{\alpha(x+\xi-1)} \\
&= 1 + q^\alpha + q^{2\alpha} + \cdots + q^{\alpha(x-1)} + q^{\alpha x}\left(1 + q^\alpha + q^{2\alpha} + \cdots + q^{\alpha(\xi-1)}\right) \\
&= [x]_{q^\alpha} + q^{\alpha x}[\xi]_{q^\alpha}.
\end{aligned} \tag{1.15}$$

We set $x \to \frac{q^{\alpha x}[\xi]_{q^\alpha}}{[x]_{q^\alpha}}$ into (1.12) and by using (1.15), we get an interesting formula:

$$\begin{aligned}
&[x+\xi]_{q^\alpha}\left(\log [x+\xi]_{q^\alpha} - 1\right) \\
&= \left([x]_{q^\alpha} + q^{\alpha x}[\xi]_{q^\alpha}\right)\log [x]_{q^\alpha} + \sum_{n=1}^{\infty} \frac{(-q^{\alpha x})^{n+1}}{n(n+1)} \frac{[\xi]_{q^\alpha}^{n+1}}{[x]_{q^\alpha}^n} - [x]_{q^\alpha}.
\end{aligned} \tag{1.16}$$

If we substitute $\alpha = 1$ into (1.16), we get Kim's q-analogue of p-adic log gamma function (for details, see [5]).

From expressions of (1.2) and (1.16), we obtain worthwhile and interesting theorems as follows:

Theorem 1. *For $x \in \mathbb{C}_p \backslash \mathbb{Z}_p$ the following*

(1.17)
$$G_{p,q}^{(\alpha)}(x) = \left([x]_{q^\alpha} + q^{\alpha x}\frac{\widetilde{G}_{2,q}^{(\alpha)}}{2}\right)\log [x]_{q^\alpha} + \sum_{n=1}^{\infty} \frac{(-q^{\alpha x})^{n+1}}{n(n+1)(n+2)} \frac{\widetilde{G}_{n+2,q}^{(\alpha)}}{[x]_{q^\alpha}^n} - [x]_{q^\alpha}$$

is true.

Theorem 2. *For $x \in \mathbb{C}_p \backslash \mathbb{Z}_p$ the following*

$$G_{p,q}^{(\alpha)}(x) = \left([x]_{q^\alpha} + q^{\alpha x}\widetilde{E}_{1,q}^{(\alpha)}\right)\log [x]_{q^\alpha} + \sum_{n=1}^{\infty} \frac{(-q^{\alpha x})^{n+1}}{n(n+1)} \frac{\widetilde{E}_{n+1,q}^{(\alpha)}}{[x]_{q^\alpha}^n} - [x]_{q^\alpha} \tag{1.18}$$

is true.

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