performed both acoustic-only and visual-only speech recognition systems for both speaker independent and dependent cases. It is important to note that a set of as few as three labial geometric features is sufficient to improve the recognition rate by as much as 20% (from 62%, with acoustic-only information, to 82%, with audio-visual information at $\mathrm{SNR} = 0$ dB).

Finally, we note that, although we used markers to extract the geometric visual features, geometric visual features can be extracted in real time accurately without using any markers [26], [27].

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A Note on the Robust Stability of Uncertain Stochastic Fuzzy Systems With Time-Delays

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***Abstract*—Takagi–Sugeno (T-S) fuzzy models are now often used to describe complex nonlinear systems in terms of fuzzy sets and fuzzy reasoning applied to a set of linear submodels. In this note, the T-S fuzzy model approach is exploited to establish stability criteria for a class of nonlinear stochastic systems with time delay. Sufficient conditions are derived in the format of linear matrix inequalities (LMIs), such that for all admissible parameter uncertainties, the overall fuzzy system is stochastically exponentially stable in the mean square, independent of the time delay. Therefore, with the numerically attractive Matlab LMI toolbox, the robust stability of the uncertain stochastic fuzzy systems with time delays can be easily checked.**

***Index Terms*—Fuzzy systems, linear matrix inequality (LMI), nonlinear systems, robust stability, stochastic systems, time-delay systems.**

I. INTRODUCTION

Stability analysis of stochastic systems has been well investigated in the past three decades, since stochastic modeling has come to play

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an important role in many real-world systems, including nuclear, thermal, chemical processes, biology, socio-economics, immunology, etc. (see [16] and [32] for more details). In particular, the systems with stochastic parameter systems, also called bilinear stochastic systems (with bilinearity in terms of the state and the noise disturbance input), state-dependent noise systems, or multiplicative noise systems, have been dealt with by many authors. Among them, we quote Boukas and Liu [3], Bernstein and Haddad [2], DeKoning [10], Yasuda *et al.* [34], Skelton *et al.* [22], Yaz [35], and Wang and Burnham [30]. Note that bilinear stochastic models are widely used to model nonlinear processes in signal and image processing, as well as communication systems analysis, such as channel equalization, echo cancellation, nonlinear tracking, electroencephalogram (EEG) signal classification, multiplicative disturbance tracking, etc. (see [31] and references therein).

As is well known now, time delay and system uncertainty are commonly encountered and are often the sources of instability [15]. In the past few decades, the robust stability analysis problems for linear uncertain and/or time-delay systems have received considerable attention. For example, in [1], the engineering insight for robust stability was well explained with some practical examples. The stability radii problem for linear perturbed systems was discussed in detail in [21]. Recently, robust stability criteria for uncertain time-delay systems were studied in [19], [20], and [28]. In the stochastic setting, the robust stochastic stability problems were studied in [12] and [13], respectively, for linear and semilinear systems. Moreover, Mao [18] investigated the robust stability for uncertain stochastic differential delay systems, and proposed several test criteria for various types of system uncertainties.

On the other hand, the stability-analysis problem of nonlinear time-delay systems has gained much research attention in recent years. One typical approach for analysis of nonlinear systems with time delay is the so-called linearization approach [9]. In the past two decades, the fuzzy logic theory [36] has been proven to be a practical approach to dealing with the analysis and synthesis problems for nonlinear systems. Recently, the popular Takagi-Sugeno (T-S) fuzzy model [23] has been employed in most model-based fuzzy analysis approaches. The innovative idea proposed in [23], which is shown to be conceptually simple and effective, is that a linear system is adopted as the consequent part of a fuzzy rule. Typically, a nonlinear plant is first approximated by a T-S fuzzy linear model, and then the analysis will be based on the T-S linear model (see, e.g., [24], [25], [27]). Recently, the robust stability analysis and synthesis problem was investigated in [5] and [14], for a class of uncertain continuous- and discrete-time fuzzy systems, respectively. Note that the uncertainties may exist in the real systems, or come from the fuzzy modeling procedure. Furthermore, in [6], the T-S fuzzy model with time delay was presented, and sufficient stabilization conditions were established for fuzzy feedback-control systems.

However, the stability-analysis problem for *nonlinear stochastic* systems is difficult. In the existing works, some strong assumptions are required on the nonlinearities, such as the smoothness and Lipschitz continuity, to ensure the desired stochastic stability (usually in the mean square), and this limits the application scope of the conventional nonlinear stochastic stability theory [29]. Moreover, when the system parameter uncertainties, as well as time delay, also appear in the nonlinear stochastic model, the stability-analysis problem becomes even more complex and challenging. Motivated by the T-S fuzzy model-based analysis technique, we shall introduce a fuzzy stochastic system by extending the ordinary T-S fuzzy model, and deal with the stability analysis problem of nonlinear uncertain stochastic time-delay systems via fuzzy logic approach. To the authors' knowledge, so far, the stability analysis issue for a *fuzzy uncertain stochastic time-delay* systems has not been addressed in the existing literature.

In this note, we first define a fuzzy stochastic system by extending the ordinary T-S fuzzy model. The given uncertain nonlinear stochastic time-delay systems are represented by the linear stochastic T-S model with time delay and uncertainties. The system dynamics is captured by a set of fuzzy implications that characterize local relations in the state space, and the local dynamics of each fuzzy rule is expressed by a linear uncertain stochastic system with time delay. The overall fuzzy model can be achieved by fuzzy "summing" of the linear models. We aim at deriving sufficient conditions such that, for all admissible parameter uncertainties, the overall fuzzy system is stochastically exponentially stable in the mean square, independent of the time delay. The conditions are given in terms of the solutions to a set of linear matrix inequalities (LMIs). Note that LMIs can now be solved efficiently by Matlab LMI toolbox (see [4] and [11]) and, therefore, our proposed design procedure is practically useful.

Notation: The notations in this note are quite standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (respectively, positive definite). I is the identity matrix with compatible dimension. We let $h > 0$ and $C([-h,0];\mathbb{R}^n)$ denote the family of continuous functions φ from $[-h,0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-h\leq\theta\leq 0}|\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. If A is a matrix, denote by $\|A\|$ its operator norm, i.e., $\|A\| = \sup\{|Ax| : |x| = 1\} = \sqrt{\lambda_{\max}(A^TA)}$ where $\lambda_{\max}(\cdot)$ [respectively, $\lambda_{\min}(\cdot)$] means the largest (respectively, smallest) eigenvalue of A. $l_2[0,\infty]$ is the space of square integrable vector. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). Denoted by $L^p_{\mathcal{F}_0}([-h,0];\mathbb{R}^n)$, the family of all $\mathcal{F}_0$-measurable $C([-h,0];\mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -h \leq \theta \leq 0\}$ such that $\sup_{-h\leq\theta\leq 0}\mathbb{E}|\xi(\theta)|^p < \infty$, where $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

II. Problem Formulation and Assumptions

A fuzzy dynamic model has been proposed by Takagi and Sugeno [23] to represent local linear input/output relations of nonlinear systems. This fuzzy linear model is described by fuzzy IF-THEN rules and has been employed to deal with the control-design problems of nonlinear systems with exogenous disturbance [7]–[27]. The ordinary T-S fuzzy model has further been extended to other kinds of systems, such as parameter uncertain systems [5], [14], time-delay systems [6], descriptor systems [26], etc. Motivated by this, we shall generalize the T-S fuzzy model to represent a fuzzy uncertain stochastic system whose consequent parts are linear stochastic uncertain time-delay systems.

As in [5], [6], and [14], we consider a T-S fuzzy stochastic model with time delay and parameter uncertainty, in which the ith rule is formulated in the following form:

Plant Rule i:
IF $\theta_1(t)$ is η_{i1} and $\ldots\theta_p(t)$ is η_{ip}
THEN

$$\dot{x}(t) = (A_i + \Delta A_i)x(t) + A_{di}x(t-h) + J_ix(t)w(t) \tag{1}$$

$$x(t) = \varphi(t), \quad t \in [-h,0], \quad i = 1,\ldots,r \tag{2}$$

where η_{ij} is the fuzzy set, $x(t) \in \mathbb{R}^n$ is the state, h denotes the *unknown* state delay, $\varphi(t)$ is a continuous vector valued initial function. $w(t)$ is a scalar zero mean Gaussian white noise process with unit covariance, A_i, A_{di}, and J_i are known constant matrices with appropriate dimensions, r is the number of IF-THEN rules, and $\theta(t) = [\theta_1(t)\ \theta_2(t)\ \ldots\ \theta_p(t)]$ are the premise variables. It is assumed that the premise variables do not depend on the noise-input variables $w(t)$ explicitly. ΔA_i is a real-valued matrix function that represents norm-bounded parameter uncertainty and satisfies [5], [14]

$$\Delta A_i = M_i H_i(t) N_i \tag{3}$$

where $M_i \in \mathbb{R}^{n\times y}$, and $N_i \in \mathbb{R}^{z\times n}$ are known real constant matrices which characterize how the deterministic uncertain parameter in $H_i(t)$ enters the nominal matrix $A(r(t))$; and $F(t, r(t)) \in \mathbb{R}^{y\times z}$ is an unknown time-varying matrix function meeting

$$H_i^T(t) H_i(t) \leq I. \tag{4}$$

Remark 1: We point out that the theory proposed later works for more general system

$$\dot{x}(t) = (A_i + \Delta A_i)x(t) + A_{di}x(t-h) + \sum_{k=1}^{q} J_{ki}x(t)w_k(t) \tag{5}$$

$$x(t) = \varphi(t), \quad t \in [-h, 0], \quad i = 1, \ldots, r \tag{6}$$

where $(w_1(t)\ w_2(t) \cdots w_q(t))$ is a q-dimensional white noise. Within the same framework to be developed, we can also consider the case when the uncertainties exist in the time-delay term [14], the case when the time delay is time-varying [5], and the case for designing the feedback controllers in order to stabilize the uncertain stochastic fuzzy time-delay systems. The reason why we discuss the system in (1) and (2) is to make our theory more understandable and to avoid unnecessarily complicated notations.

The defuzzified output of the T-S fuzzy system (1) is represented as shown in (7) at the bottom of the page, where

$$v_i\Big(\theta(t)\Big) = \prod_{j=1}^{p} \eta_{ij}\Big(\theta_j(t)\Big)$$

$$h_i\Big(\theta(t)\Big) = \frac{v_i\Big(\theta(t)\Big)}{\sum_{j=1}^{r} v_j\Big(\theta(t)\Big)}$$

in which $\eta_{ij}\Big(\theta_j(t)\Big)$ is the grade of membership of $\theta_j(t)$ in η_{ij}.

A basic property of $v_i(\theta(t))$ is that

$$v_i\Big(\theta(t)\Big) \geq 0, \qquad i = 1, 2, \ldots, r, \qquad \sum_{j=1}^{r} v_j\Big(\theta(t)\Big) > 0 \tag{8}$$

and, therefore

$$h_i\Big(\theta(t)\Big) \geq 0, \qquad i = 1, 2, \ldots, r, \qquad \sum_{j=1}^{r} h_j\Big(\theta(t)\Big) = 1, \qquad \forall t. \tag{9}$$

In this note, we assume that all membership functions are continuous and piecewise continuously differential, and the defuzzified model is also continuous. Observe the defuzzified T-S model (7) and let $x(t;\xi)$ denote the state trajectory from the initial data $x(\zeta) = \xi(\zeta)$ on $-h \leq \zeta \leq 0$ in $L^2_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$. Clearly, the system (7) admits a trivial solution $x(t; 0) \equiv 0$, corresponding to the initial data $\xi = 0$.

We now introduce the following stability concepts.

Definition 1: For the uncertain time-delay fuzzy stochastic system (7) and every $\xi \in L^2_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$, the trivial solution is asymptotically stable in the mean square if

$$\lim_{t\to\infty} \mathbb{E}|x(t;\xi)|^2 = 0 \tag{10}$$

and is exponentially stable in the mean square if there exist scalars $\alpha > 0$ and $\beta > 0$, such that

$$\mathbb{E}|x(t;\xi)|^2 \leq \alpha e^{-\beta t} \sup_{-h\leq\zeta\leq 0} \mathbb{E}|\xi(\zeta)|^2. \tag{11}$$

The purpose of this note is to derive easy-to-check conditions such that, for all admissible parameter uncertainties, the overall fuzzy system (7) is exponentially stable in the mean square, independent of the unknown time delay.

III. Main Results and Proofs

Before proceeding, we recall some lemmas that will be frequently used throughout the proofs of our main results.

Lemma 1 (Schur Complement): Given constant matrices $\Omega_1, \Omega_2, \Omega_3$, where $\Omega_1 = \Omega_1^T$ and $0 < \Omega_2 = \Omega_2^T$, then $\Omega_1 + \Omega_3^T \Omega_2^{-1} \Omega_3 < 0$ if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0$$

or equivalently

$$\begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^T & \Omega_1 \end{bmatrix} < 0.$$

Lemma 2 (See, e.g., [33])*:* Let M, N, and F be real matrices of appropriate dimensions with $F^T F \leq I$, where F may be time-varying. Then, for any scalar $\mu \neq 0$, we have

$$MFN + N^T F^T M^T \leq \mu^2 M M^T + \mu^{-2} N^T N.$$

For stochastic systems, Itô's differential formula (see, e.g., [16]) will be used in our derivation. To facilitate the reader, we introduce the Itô's differential operator here. For a general stochastic system $\dot{x}(t) = f(x(t), t) + g(x(t))w(t)$ on $t \geq 0$ with initial value $x(0) = x_0 \in \mathbb{R}^n$, where $w(t)$ is an m-dimensional white noise with unit intensity, $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ satisfy the local Lipschitz condition and the linear growth condition. Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}; \mathbb{R}_+)$ denote the family of all nonnegative functions $Y(x, t, i)$ on $\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}$, which are continuously twice differentiable in x and once differentiable in t. Define an Itô' differential operator $\mathcal{L}$ acting on $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}; \mathbb{R}_+)$ function by

$$\begin{aligned} \mathcal{L}V(x,t) = {} & V_t(x,t) + V_x(x,t)f(x,t) \\ & + \frac{1}{2}[g^T(x,t)V_{xx}(x,t)g(x,t)] + V_x(x,t)g(x,t)w(t) \end{aligned} \tag{12}$$

$$\begin{aligned} \dot{x}(t) &= \frac{\sum_{i=1}^{r} v_i\Big(\theta(t)\Big)[(A_i + \Delta A_i)x(t) + A_{di}x(t-h) + J_i x(t)w(t)]}{\sum_{i=1}^{r} v_i\Big(\theta(t)\Big)} \\ &= \sum_{i=1}^{r} h_i\Big(\theta(t)\Big)[(A_i + \Delta A_i)x(t) + A_{di}x(t-h) + J_i x(t)w(t)] \end{aligned} \tag{7}$$

where $V_x = (V_{x_1}, \ldots, V_{x_n})$, and $V_{xx} = (V_{x_i x_j})_{n\times n}$. Recall that, compared to the usual differential rule, there is an extra term, i.e.,

$$\frac{1}{2} g^T(x,t) V_{xx}(x,t) g(x,t)$$

appearing in the Itô' stochastic differential.

We now deal with the stability-analysis problem, that is, derive the stability conditions for the system (7) with parameter uncertainty and time delay

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(\theta(t))[(A_i + \Delta A_i)x(t) + A_{di}x(t-h) + J_i x(t) w(t)]. \quad (13)$$

Theorem 1: Consider the fuzzy uncertain time-delay stochastic system (13). If there exist a positive scalar $\mu > 0$ and positive definite matrices $P > 0, Q > 0$ such that the following matrix inequalities

$$A_i^T P + PA_i + J_i^T P J_i + P\left(\mu^2 M_i M_i^T + A_{di} Q^{-1} A_{di}^T\right) P + \mu^{-2} N_i^T N_i + Q < 0 \quad (14)$$

hold for $i = 1, 2, \ldots, r$, then the stochastic fuzzy system (13) is exponentially stable in the mean square.

Proof: Fix $\xi \in L^2_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$ arbitrarily and write $x(t;\xi) = x(t)$. Define a Lyapunov function candidate $Y(x,t,i) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathcal{S}; \mathbb{R}_+)$ by

$$Y\Big(x(t), t\Big) = x^T(t) P x(t) + \int_{t-h}^{t} x^T(s) Q x(s)\, ds. \quad (15)$$

where $P > 0$ and $Q > 0$ are the solutions to (14).

By Itô's formula (see, e.g., [16]), the stochastic derivative of Y along a given trajectory of (13) is obtained from (12) as follows:

$$\begin{aligned}\frac{d}{dt} Y\Big(x(t), t\Big) &= \mathcal{L}Y\Big(x(t), t, i\Big) \\ &= \sum_{i=1}^{r} h_i(\theta(t)) \Big\{ x^T(t) \Big[(A_i + \Delta A_i)^T P \\ &\quad + P(A_i + \Delta A_i) + J_i^T P J_i \Big] x(t) \\ &\quad + x^T(t-h) A_{di}^T P x(t) + x^T(t) P A_{di} x(t-h) \Big\} \\ &\quad + x^T(t) Q x(t) - x^T(t-h) Q(t-h) \\ &\quad + 2 \sum_{i=1}^{r} h_i\Big(\theta(t)\Big) x^T(t) P J_i x(t) w(t). \end{aligned} \quad (16)$$

Define

$$\begin{aligned}\Upsilon\Big(x(t), t\Big) := &\sum_{i=1}^{r} h_i\Big(\theta(t)\Big) \Big\{ x^T(t) \Big[(A_i + \Delta A_i)^T P \\ &+ P(A_i + \Delta A_i) + J_i^T P J_i \Big] x(t) \\ &+ x^T(t-h) A_{di}^T P x(t) + x^T(t) P A_{di} x(t-h) \Big\} \\ &+ x^T(t) Q x(t) - x^T(t-h) Q(t-h). \end{aligned} \quad (17)$$

Noting that $\sum_{i=1}^{r} h_i(\theta(t)) = 1$, we have

$$\begin{aligned}\Upsilon\Big(x(t), t\Big) = &\sum_{i=1}^{r} h_i\Big(\theta(t)\Big) \Big\{ x^T(t) \Big[(A_i + \Delta A_i)^T P \\ &+ P(A_i + \Delta A_i) + J_i^T P J_i \Big] x(t) \\ &+ x^T(t-h) A_{di}^T P x(t) + x^T(t) P A_{di} x(t-h) \\ &+ x^T(t) Q x(t) - x^T(t-h) Q(t-h) \Big\}. \end{aligned} \quad (18)$$

Since $\Delta A_i = M_i H_i(t) N_i$ and $H_i^T H_i \le I$, it then follows from Lemma 2 that, for any scalar $\mu \neq 0$

$$\begin{aligned} P(\Delta A_i) + (\Delta A_i)^T P_i &= P M_i F_i N_i + N_i^T F_i^T P M_i^T \\ &\le \mu^2 P M_i M_i^T P + \mu^{-2} N_i^T N_i. \end{aligned} \quad (19)$$

Denote

$$\begin{aligned}\Theta_i :=\ &A_i^T P + P A_i + J_i^T P J_i \\ &+ \mu^2 P M_i M_i^T P + \mu^{-2} N_i^T N_i + Q \end{aligned} \quad (20)$$

$$S_i := \begin{bmatrix} \Theta_i & P A_{di} \\ A_{di}^T P & -Q \end{bmatrix}. \quad (21)$$

Then, substituting (19) into (18) results in

$$\begin{aligned}\Upsilon\Big(x(t), t\Big) \le &\sum_{i=1}^{r} h_i\Big(\theta(t)\Big) \Big\{ x^T(t) \Theta x(t) \\ &+ x^T(t-h) A_{di}^T P_i x(t) + x^T(t) P_i A_{di} x(t-h) \\ &- x^T(t-h) x(t-h) \Big\} \\ = &\sum_{i=1}^{r} h_i\Big(\theta(t)\Big) \left\{ [x^T(t) \quad x^T(t-h)] S_i \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \right\}. \end{aligned} \quad (22)$$

From the Schur Complement Lemma (Lemma 1), we know that $S_i < 0$ if and only if

$$\Theta_i + P A_{di} Q^{-1} A_{di}^T P < 0 \quad (23)$$

which is the same as the inequality (14). Therefore, we arrive at the conclusion that $\Upsilon(x(t), t) < 0$ and

$$\mathbb{E}\left[\frac{d}{dt} Y(x(t), t)\right] < 0.$$

This means that the fuzzy stochastic system (13) is asymptotically stable in the mean square for all admissible uncertainties ΔA_i, independent of the unknown time delay, provided that the inequalities (14) are met for $i = 1, 2, \ldots, r$.

Having obtained the inequality (22), we are able to prove the exponential stability (in the mean square) of the system (13) by using the techniques developed in [17].

Define

$$\lambda_P = \lambda_{\max} P, \qquad \lambda_S = \min_{i=1}^{r}(-\lambda_{\max} S_i), \qquad \lambda_p = \lambda_{\min} P$$

where P is the solution to (14) and S_i is defined in (21). Let δ be the unique root to the equation

$$\delta(\lambda_P + h e^{\delta h}) = \lambda_S + \min(1, \lambda_S e^{\delta h}).$$

To prove the mean square exponential stability, we modify the Lyapunov function candidate (15) as

$$Y_1(x(t), t, i) = e^{\delta t}\left(x^T(t) P_i x(t) + \int_{t-h}^{t} \mathbb{E}(x^T(s) Q x(s))\, ds \right). \quad (24)$$

Following the similar line for the proof of Theorem 3.1 in [17], and noticing the fact that

$$\sum_{i=1}^{r} h_i(\theta(t)) = 1$$

we can show that

$$e^{\delta t} \lambda_p \mathbb{E}|x(t)|^2 \le (\lambda_P + h(1 + e^{\delta h}))\mathbb{E}\|\xi\|^2$$

or

$$\lim_{t\to\infty} \sup \frac{1}{t} \log(\mathbb{E}|x(t,\xi)|^2) \le -\delta.$$

This implies that the trivial solution of the system (13) is exponentially stable in the mean square. The proof of this theorem is complete. ∎

Remark 2: Theorem 1 provides the analysis results for the exponential stability of the system (13) via fuzzy model approach. We can see from (14) that we need to check whether there exist a common scalar $\mu \neq 0$ and two common positive definite matrices $P > 0$ and $Q > 0$, such that r decoupled matrix inequalities hold. This is certainly not a trivial task. Fortunately, we could convert the r nonlinear (on P and μ) inequalities into the associated LMIs [4], and then we are able to determine exponential stability of the system (13) readily by checking the solvability of the LMI's [11].

Theorem 2: For the fuzzy uncertain time-delay stochastic system (13), if there exist a positive scalar $\varepsilon > 0$ and positive definite matrices $P > 0, Q > 0$ satisfying the following LMIs

$$\begin{bmatrix} \Lambda_{1i} & PA_{di} & \varepsilon N_i^T & PM_i \\ A_{di}^T P & -Q & 0 & 0 \\ \varepsilon N_i & 0 & -\varepsilon I & 0 \\ M_i^T P & 0 & 0 & -\varepsilon I \end{bmatrix} < 0 \tag{25}$$

where Λ_i is defined by

$$\Lambda_{1i} := A_i^T P + PA_i + J_i^T PJ_i + Q \tag{26}$$

for $i = 1, 2, \dots, r$, then the stochastic fuzzy system (13) is exponentially stable in the mean square.

Proof: First, we rearrange (14) as

$$\Lambda_{1i} + [PA_{di} \quad \mu^{-1}N_i^T \quad \mu PM_i] \begin{bmatrix} Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \times \begin{bmatrix} A_{di}^T P \\ \mu^{-1} N_i \\ \mu M_i^T P \end{bmatrix} < 0. \tag{27}$$

It follows from the Schur Complement Lemma (Lemma 1) that the above inequality holds if and only if

$$\begin{bmatrix} \Lambda_{1i} & PA_{di} & \mu^{-1} N_i^T & \mu PM_i \\ A_{di}^T P & -Q & 0 & 0 \\ \mu^{-1} N_i & 0 & -I & 0 \\ \mu M_i^T P & 0 & 0 & -I \end{bmatrix} < 0. \tag{28}$$

Note that (28) is not linear in μ. Let $\varepsilon := \mu^{-2}$. Pre- and postmultiplying the inequality (28) by $\mathrm{diag}\{I, I, \mu^{-1}I, \mu^{-1}I\}$ yields (25). The proof follows from Theorem 1 immediately. ∎

Remark 3: In Theorem 2, a delay-independent condition is given for the exponential stability (in the mean square) of nonlinear, uncertain time-delay stochastic systems in terms of the solvability of several LMIs via fuzzy-models approach. When $r = 1$, the stability condition is reduced to that of linear stochastic systems with uncertainties and time delay, which is still believed new for stochastic systems. In order to reduce the conservatism, especially when the time delay is known, we should try to establish delay-dependent stability criteria. This would be one of our future research topics.

Remark 4: The stochastic stability-analysis problem for nonlinear stochastic systems has received considerable research interests in the past few decades. Unfortunately, stringent mathematical assumptions, which are not easy to satisfy in practice, have been imposed in the literature in order to establish the stability criteria. As discussed in the introduction, since a large class of nonlinear time-delay stochastic systems could be approximated by a proper T-S model, we aim to examine the exponential stability of a *given* uncertain fuzzy time-delay stochastic system in terms of LMIs, which are very easy to test. We point out that, the results can be exploited to design the fuzzy control laws so that the closed-loop system is exponential stable. Also, the present results can be extended to the systems with multiple time-varying time delays in state.

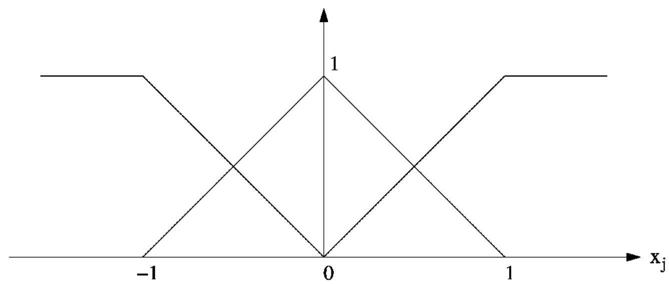

Fig. 1. Membership functions of x_1 and x_2.

IV. Numerical Example

In this section, by means of a simple design example, we illustrate the usefulness of the approach proposed in the previous sections.

Consider a T-S fuzzy stochastic model with time delay and parameter uncertainty (1)–(2), together with the ith rule given as follows:

Rule
1: If $x_1(t)$ is about 0 and $x_2(t)$ is about 0, then

$$\dot{x}(t) = (A_1 + \Delta A_1)x(t) + A_{d1}x(t-h) + J_1 x(t)w(t). \tag{29}$$

Rule
2: If $x_1(t)$ is about 0 and $x_2(t)$ is about ± 1, then

$$\dot{x}(t) = (A_2 + \Delta A_2)x(t) + A_{d2}x(t-h) + J_2 x(t)w(t). \tag{30}$$

Rule
3: If $x_1(t)$ is about ± 1 and $x_2(t)$ is about 0, then

$$\dot{x}(t) = (A_3 + \Delta A_3)x(t) + A_{d3}x(t-h) + J_3 x(t)w(t). \tag{31}$$

The model parameters are given as follows:

$$A_1 = \begin{bmatrix} -3 & 1 \\ -1 & -5 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -0.1 & 0.5 \\ 0.2 & -0.3 \end{bmatrix}$$
$$J_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$
$$M_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad N_1 = [0.2 \quad 0.2], \quad H_1(t) = \sin(t)$$
$$A_2 = \begin{bmatrix} -4 & 0.5 \\ 0.5 & -6 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.1 & -0.5 \\ -0.2 & 0.3 \end{bmatrix}$$
$$J_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix}$$
$$M_2 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}, \quad N_1 = [-0.2 \quad 0.2], \quad H_1(t) = -\sin(t)$$
$$A_3 = \begin{bmatrix} -5 & 1 \\ 2 & -7 \end{bmatrix}, \quad A_{d3} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}$$
$$J_3 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}$$
$$M_3 = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}, \quad N_3 = [0.2 \quad -0.2], \quad H_1(t) = \cos(t).$$

where $\Delta A_i = M_i H_i(t) N_i (i = 1, 2, 3)$, and $h = 0.5$. The membership functions for x_1 and x_2 are shown in Fig. 1.

Next, solving the LMIs (25) for $i = 1, 2, 3$ gives

$$P = \begin{bmatrix} 30.2854 & 31.8281 \\ 31.8281 & 68.0078 \end{bmatrix}$$
$$Q = \begin{bmatrix} 0.8892 & -0.0235 \\ -0.0235 & 0.7343 \end{bmatrix}, \quad \varepsilon = 0.2170.$$

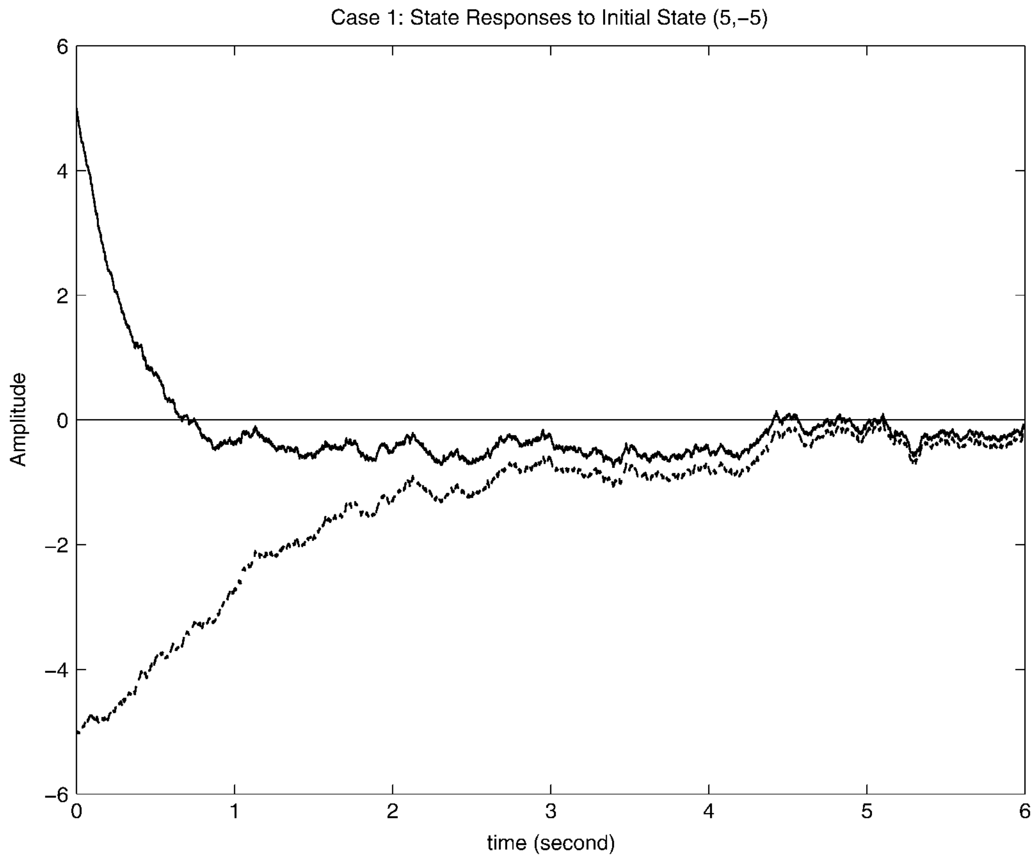

Fig. 2. x_1 (solid), x_2 (dashed).

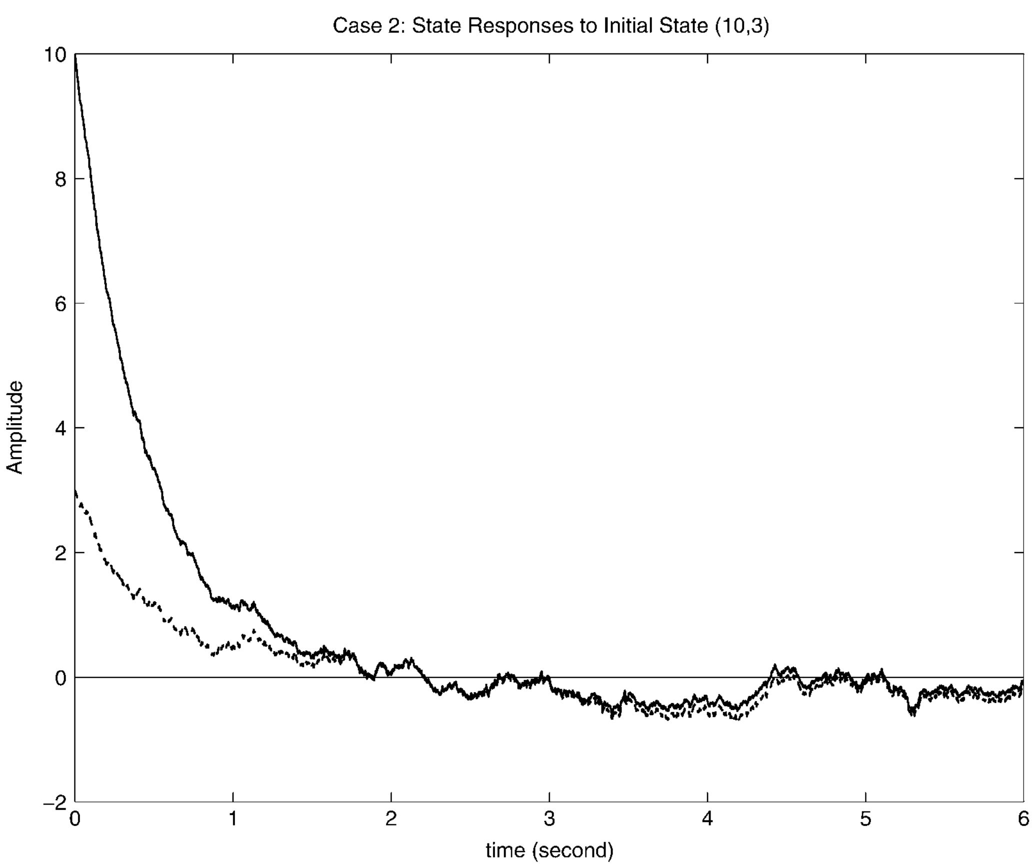

Fig. 3. x_1 (solid), x_2 (dashed).

It then follows from Theorem 2 that the stochastic fuzzy system (13) is exponentially stable in the mean square. Also, the state responses to the initial conditions $(5, -5)$ and $(10, 3)$ are shown in Figs. 2 and 3, respectively. The simulation results verify that the stochastic fuzzy system is indeed exponentially stable in the mean square.

V. Conclusion

This note has introduced an LMI approach to the robust stability-analysis problem for a class of uncertain stochastic continuous time-delay fuzzy systems. We have focused on the derivation of sufficient conditions under which the overall fuzzy system is stochastically exponentially stable in the mean square for all admissible parameter uncertainties, independent of the time delay. These conditions are expressed in terms of the solutions to a set of LMIs, which can be solved efficiently by Matlab LMI toolbox.

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