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A NOTE ON THE SIMILARITY OF A MATRIX AND ITS CONJUGATE TRANSPOSE

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It is well-known that each square matrix A over a field is similar to its transpose A^T and there exists a nonsingular symmetric matrix P for which $PA^T = AP$. The purpose of this note is to show that if A is similar to its conjugate transpose A^* then, under certain conditions, there exists a nonsingular Hermitian matrix Q for which $QA^* = AQ$.

Let f be an automorphism of order 2 on a field F and let K be the fixed field of f. For each x in F, we denote $f(x)$ by $\bar{x}$. If $A = (a_{ij})$ is a matrix over F, let $A^* = (b_{ij})$ where $b_{ij} = \overline{a_{ji}}$. A matrix M is called Hermitian (skew-Hermitian) provided $M^* = M(M^* = -M)$.

Taussky and Zassenhaus [1] have shown that for each square matrix over a field, there exists a nonsingular symmetric matrix which transforms the given matrix into its transpose. Our main result is

THEOREM 1. *Suppose F is an infinite field whose characteristic is different from 2. If a matrix A over F is similar to A^*, there exists a nonsingular Hermitian matrix Q over F for which $QA^* = AQ$.*

We shall utilize the following lemmas in both of which z denotes an element of F which is not in K.

LEMMA 1. *Every element of F can be expressed uniquely in in the form $a + bz$ where both a and b lie in K.*

Proof. If c belongs to F, it is clear that

$$c = c - (c - \bar{c})(z - \bar{z})^{-1}z + (c - \bar{c})(z - \bar{z})^{-1}z$$

since $z \neq \bar{z}$. This is the required form since both

$$a = c - (c - \bar{c})(z - \bar{z})^{-1}z$$

and

$$b = (c - \bar{c})(z - \bar{z})^{-1}$$

lie in K. The uniqueness of the expression follows from the fact that z does not belong to K.

LEMMA 2. *If $c = r + sz$ and $d = t + uz$ with $r, s, t,$ and u in K and $c/\bar{c} = d/\bar{d}$, then $ru = st$.*

Lemma 2 implies that there exists a one-to-one correspondence between K and the set of all elements $c/\bar{c}$ where $c = r + z$ and r ranges over K. If F is infinite, Lemma 1 implies that K is infinite.

Proof of Theorem 1. Suppose $PA^* = AP$ with P nonsingular. Since the characteristic of F is not 2, the matrix P can be expressed as the sum of an Hermitian matrix H and a skew-Hermitian matrix S. Hence $HA^* = AH$ and it remains to show that H may be chosen nonsingular.

Since $(cP)A^* = A(cP)$ for all c in F, we want to choose c so that $M = cP + \bar{c}P^*$ is nonsingular. The matrix M is nonsingular if and only if $-c/\bar{c}$ is distinct from all of the eigenvalues of $P^{-1}P^*$. Since there exist infinitely many values of $-c/\bar{c}$, an element c can be properly chosen and the proof is complete.

In regard to finite fields, we have

THEOREM 2. *Suppose A is a square matrix of order n over a field F whose characteristic is different from 2 and $PA^* = AP$ with P nonsingular. If there exists an element y in F such that y^m does not belong to K for $1 \leqq m \leqq n + 1$, there exists a nonsingular Hermitian matrix Q for which $QA^* = AQ$.*

Proof. Utilizing the same decomposition of P as in the proof of Theorem 1, it is sufficient to show there exists an element c in F such that $cP + \bar{c}P^*$ is nonsingular. For c nonzero, $cP + \bar{c}P^*$ is nonsingular if and only if $-\bar{c}/c$ is not an eigenvalue of $P(P^*)^{-1}$. Hence let $k_1, k_2, \cdots, k_t$ be the distinct eigenvalues of $P(P^*)^{-1}$ in F and let

$$W = \{1, -k_1, -k_2, \cdots, -k_t\} .$$

If for each nonzero x in F there exists k in W such that $\bar{x} = kx$, then k^r belongs to W for all positive integers r since $\bar{x}^r = k^r x^r$. In particular, for the element y mentioned in the hypothesis of the theorem, $\bar{y} = dy$ for some d in W and hence the elements d^i, for $1 \leqq i \leqq n + 2$, all belong to W. Since W contains only $t + 1$ elements and $0 \leqq t \leqq n$, it follows that $d^i = d^j$ for some integers i and j, $i < j$, between 1 and $n + 2$, inclusively. Hence $j - i \leqq n + 1$ and $d^{j-i} = 1$ since $d \neq 0$. Therefore

$$f(y^{j-i}) = d^{j-i}y^{j-i} = y^{j-i}$$

implies y^{j-i} belongs to K. This contradiction shows the existence of some c in F such that $\bar{c} \neq kc$ for all k in W: hence c does not belong to K and $cP + \bar{c}P^*$ is nonsingular as required.

As a simple application of Theorem 2, suppose $F = GF(p^{2s})$ with $p \neq 2$ and let $f(x) = x^{p^s}$ for all x in F. By considering a generator of the multiplicative group of F, one may verify the result for matrices over F of order less than p^s.

Reference

1. Olga Taussky and Hans Zassenhaus, *On the similarity transformation between a matrix and its transpose*, Pacific J. Math. **9** (1959), 893-896.

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