A Note on the Solution of Certain Approximation Problems in Network Synthesis

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IN NETWORK SYNTHESIS

R. M. Fano

Abstract

This paper presents a method of determining
appropriate functions to be used as amplitude character-
istics of networks when certain ideal behaviors are to be
approximated within specified tolerances. This method is
based on the use of suitable transformations in the
complex plane which, by simplifying the geometry of the
problem, permit the direct determination by inspection of
the desired functions. In the simplest cases of practical
interest, this method has the only advantage of presenting
results already known in a more intuitive form. On the
other hand, it yields also alternative solutions satisfy-
ing additional and less conventional requirements and, at
the same time, it indicates an approach likely to be
fruitful in the solution of more complex approximation
problems.

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1. Introduction

The first step in the design of a network is the selection of an
appropriate function to represent the particular frequency characteristic of
interest. Such a function must, first of all, satisfy some necessary and
sufficient conditions to insure the physical realizability of the network.
In addition, it must meet whatever requirements are imposed by the design
specifications. In many practical cases for instance, these design specifi-
cations require the function representing the behavior of the network at
real frequencies* to approximate some simple step function within specified
tolerances.

The approximation procedure described below has been developed[1]
in connection with functions representing the behavior of the magnitude of
the transmission coefficient of a two-terminal-pair network terminated, on
both sides, by pure resistances. However, it can be easily applied, with
obvious modifications, to other types of amplitude responses, such as, for
instance, the magnitude of a reflection coefficient or of a driving-point
impedance. For the sake of simplicity, the discussion will be limited to
the case of transmission coefficients, leaving to the reader the extention
of the approximation procedure to other similar cases.

Any function representing the transmission coefficient t of a
passive two-terminal-pair network must satisfy the following necessary and
sufficient conditions of physical realizability:[2,3]

(1) It must be a real rational function of the complex
frequency variable $\lambda = \sigma + j\omega$.

(2) Its magnitude must satisfy the inequality $|t| \leq 1$ in the
right half and on the $j\omega$-axis of the complex plane.

It can be shown that the second condition is always satisfied if the poles
of t are in the left half-plane and $|t| \leq 1$ on the imaginary axis.

The square of the magnitude of the transmission coefficient
<u>on the imaginary axis</u> is seen to be equal to the value of the analytic
function $T(\lambda) = t(\lambda)t(-\lambda)$ for $\lambda = j\omega$. The distribution of the zeros and
poles of this new function $T(\lambda)$ must be symmetric with respect to the
imaginary axis as well as to the real axis, and for physical realizability,
the inequality $T(j\omega) \leq 1$ must be satisfied. It can be shown that, if $T(\lambda)$
meets these requirements, it is always possible to find at least one cor-
responding function $t(\lambda)$ which satisfies conditions (1) and (2) above. The

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* When the complex variable $\lambda = \sigma + j\omega$ is used, "at real frequencies" means
over the imaginary axis of the λ-plane.

determination of an appropriate function $T(\lambda)$ is just the problem with which this paper is concerned. The reader is referred to the literature[1,2,3] for the rest of the design procedure, namely, obtaining $t(\lambda)$ from $T(\lambda)$ and synthesizing the corresponding network. It must be pointed out, however, that considerations arising from these later stages of the design often play an important part in the selection of the function $T(\lambda)$. On the other hand, these considerations do not alter the main idea in the approximation procedure presented in this paper, and therefore will be disregarded in the following discussion.

2. The Electrostatic Potential Analogy

The formal analogy between static field theory in two dimensions and complex function theory was recognized by the early workers in these fields. As a matter of fact, functions of a complex variable are commonly used in the solution of field problems. The reverse process has also been used successfully in the solution of network problems,[4,5,6,7] because the field analogy provides convenient experimental techniques for the study of networks, and because the physical feeling for electrostatic fields can be very helpful as a guide in the selection of appropriate functions. This physical point of view will form the basis of the approximation procedure presented below.

The analogy between static fields in two dimensions and functions of a complex variable can be stated briefly as follows. The lines of constant real part and the lines of constant imaginary part of a function of a complex variable form in the complex plane two orthogonal families of curves, which can be identified with the equipotential lines and the lines of force of an electrostatic field. In particular, if one is interested in the magnitude of a function $F(\lambda)$, the real part of $\ln F(\lambda)$, that is $\ln |F(\lambda)|$, is identified with the potential; the constant phase lines become then the lines of force. If one turns his attention to the critical points (zeros and poles) of the type of function $F(\lambda)$ encountered in network theory (that is, the ratio of two finite polynomials), it is seen that these critical points become the sources of the corresponding electrostatic field. More precisely, one writes the function $\ln F(\lambda)$ in the form:

$$\ln F(\lambda) = \ln \left[A \frac{\prod\limits_{1}^{n} (\lambda - \lambda_{oi})}{\prod\limits_{1}^{m} (\lambda - \lambda_{pi})} \right] = \ln A + \sum\limits_{1}^{m} \ln \frac{1}{\lambda - \lambda_{pi}} - \sum\limits_{1}^{n} \ln \frac{1}{\lambda - \lambda_{oi}} \quad (1)$$

where the λ_{oi} and the λ_{pi} are, respectively, the zeros and the poles of $F(\lambda)$, and A is a constant. It is clear then that the potential represented by the function $\ln |F(\lambda)|$ is produced, apart from ln A, by positive and negative line charges of unit density located, respectively, at the poles and at the zeros of $F(\lambda)$. Consider now the approximation problem stated in the preceding section from this potential-theory point of view. This problem becomes that of determining a set of unit line charges which produces over an axis (the imaginary axis of the λ-plane) a potential distribution approximating a given function within a prescribed tolerance. An experimental technique for solving this problem has been developed by Hansen[4] using an electrolytic tank. A general theoretical solution, however, is not available at present and the procedure presented below should be considered as an attempt to fill this gap in a number of very simple but important cases. Because of the lack of generality of such a solution, the main idea can be best explained by means of specific examples. The following sections are devoted to such special cases.

3. Uniform Pass-Band Approximation

Suppose it is desired to determine a function $T(\lambda)$ (representing the square of the magnitude of a transmission coefficient) which approximates a constant within a specified maximum deviation over the fraction of the imaginary axis corresponding to $-1 < \omega < 1$, and which approaches zero monotonically when ω approaches infinity. The reader will recognize these specifications as typical of low-pass filters from which corresponding high-pass, band-pass, and band-elimination filters can be obtained without difficulty.

Let then $T(\lambda)$ be the $F(\lambda)$ of the preceding section. Since $T(j\omega)$ is, by definition, real and positive, $\ln T(j\omega)$ is real. It will be remembered that $T(\lambda)$ is related to the transmission coefficient $t(\lambda)$ by $T(\lambda) = t(\lambda)t(-\lambda)$. Following the line of thought discussed in the previous section, the problem at hand becomes that of determining a distribution of line charges which produces a potential function meeting the above specifications. The solution of this problem is made difficult by the fact that the potential is supposed to approximate a constant over only a portion of the imaginary axis. The problem would be quite simple, on the other hand, if the region of approximation extended over the whole axis.

The above remarks suggest immediately a method of approach which is currently used in field problems, namely that of simplifying the geometry of the problem by means of an appropriate transformation of the complex λ-plane. More specifically, in the case being considered, one wishes to find a function $\lambda = f(z)$, where $z = x + jy$, which transforms

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the segment $-1 < \omega < 1$ of the imaginary axis of the λ-plane into the whole imaginary axis of the z plane. It is readily recognized that the function $\lambda = \sinh z$ satisfies just this requirement. The straight lines $x = $ constant become in the λ-plane a family of confocal ellipses, and the lines $y = $ constant, a family of confocal hyperbolas. Figures 1 and 2 illustrate
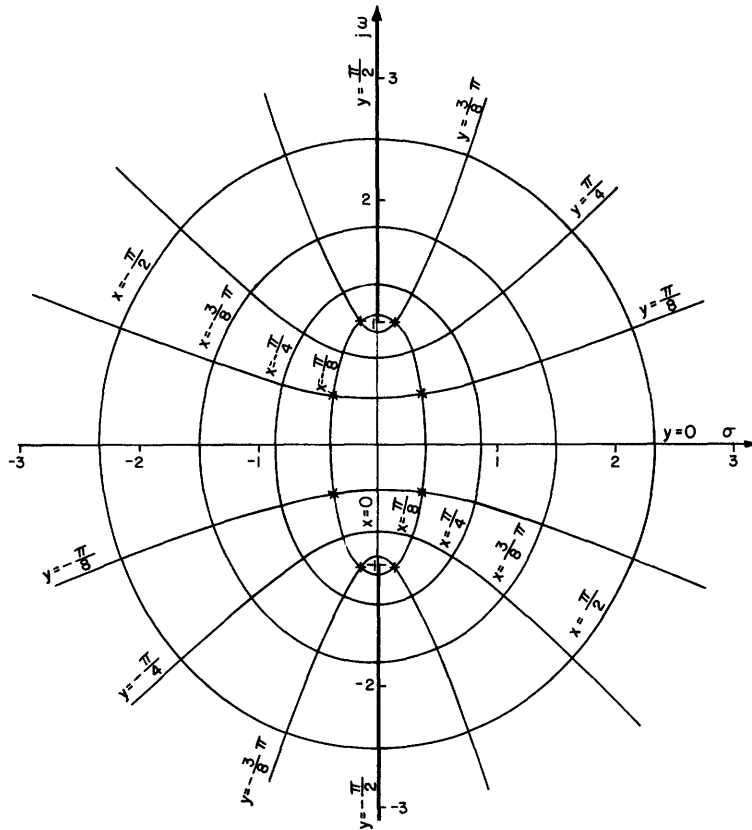

Fig. 1.   The transformation $\lambda = \sinh z$ in the λ-plane. The points indicated with crosses correspond to the points indicated in a similar manner in the z-plane (Fig. 2).

this transformation. It must be carefully noted, in this regard, that z is a multivalued function of λ, so that the whole λ-plane corresponds in the z plane to an infinite number of regions, each one of which is limited by the straight lines $y = k\pi/2$ and $y = (k+2)\pi/2$, where k is any positive or negative odd integer. The heavy lines in Fig. 1 are the branch lines of the function.

The solution of the approximation problem in the z-plane does not present any difficulty. Any uniform distribution of critical points over lines parallel to the imaginary axis will yield a function T with the desired type of oscillatory behavior over the y-axis. One must note, however, that, since $T(\lambda) \leq 1$ for $\lambda = j\omega$, no pole can be placed on any of the lines $y = k\pi/2$, (k is again any positive or negative odd integer),
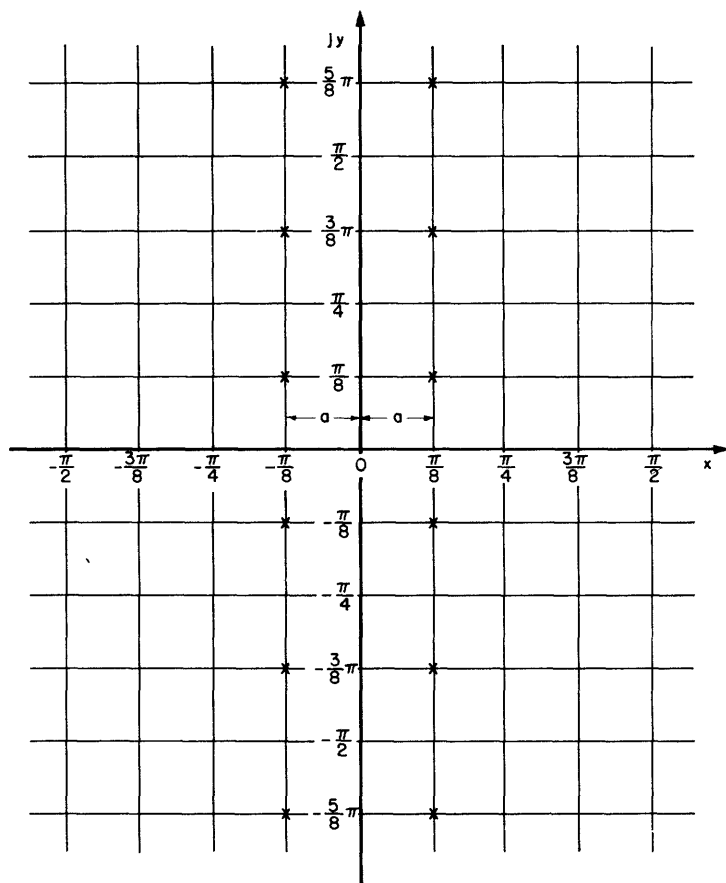
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Fig. 2. The z-plane corresponding to the λ-plane of Fig. 1.

because these lines are transformed, in the λ-plane into the part of the imaginary axis corresponding to $|w| > 1$. The simplest type of uniform distribution of critical points consists of two rows of simple poles parallel to the y-axis and symmetrically located with respect to it. If n is the number of poles in each row between any pair of lines $y = k\pi/2$ and $y = (k+2)\pi/2$, one obtains, for n even:

$$\frac{A}{T(\lambda(z))} = \prod_{m=-\infty}^{\infty}\left(z + a + j\frac{\pi(1+2m)}{2n}\right)\left(z - a + j\frac{\pi(1+2m)}{2n}\right) =$$

(2)

$$= \cosh n(z+a)\ \cosh n(z-a) = \sinh^2 na + \cosh^2 nz$$

and for n odd:

$$\frac{A}{T(\lambda(z))} = -\prod_{m=-\infty}^{\infty}(z + a + j\frac{m\pi}{n})(z - a + j\frac{m\pi}{n}) =$$

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$$= -\sinh n(z+a) \sinh n(z-a) = \sinh^2 na - \sinh^2 nz \qquad (3)$$

in which A is an arbitrary constant and $\underline{a}$ is the distance of the two rows of poles from the imaginary axis. The special case of $n = 4$, $a = \pi/8$ is illustrated in Figs. 1 and 2.

Transforming back into the λ-plane, one obtains from Eqs. (2) and (3),

$$T(\lambda) = \frac{A}{\sinh^2 na + \cosh^2(n \sinh^{-1}\lambda)} \qquad \text{for } n \text{ even} \qquad (4)$$

$$T(\lambda) = \frac{A}{\sinh^2 na - \sinh^2(n \sinh^{-1}\lambda)} \qquad \text{for } n \text{ odd.} \qquad (5)$$

Both of these expressions yield on the imaginary axis of the λ-plane:

$$T(j\omega) = \frac{A}{\sinh^2 na + \cos^2(n \cos^{-1}\omega)} = \frac{A}{\sinh^2 na + T_n^2(\omega)} \qquad (6)$$

where $T_n(\omega) = \cos(n \cos^{-1}\omega)$ is the Tchebycheff polynomial[8] of the first kind and of order n. It is clear from this equation that $T(j\omega)$ oscillates n times between the extreme values $A/\sinh^2 na$ and $A/\cosh^2 na$ when ω varies from -1 to 1, and approaches zero when $|\omega|$ approaches infinity. The constants A and a are so selected as to make these values agree with the design specifications. The order n of the function depends on the desired sharpness of cut-off and on other design considerations that are beyond the scope of this paper.

The location of the poles of $T(\lambda)$ in the λ-plane is obtained from the location of the poles in the z-plane. One has

$$\lambda_p = \sinh z_p = \begin{cases} \sinh\left(\pm a + j\frac{\pi(1+2m)}{2n}\right) & n \text{ even} \\[2em] \sinh\left(\pm a + j\frac{m\pi}{n}\right) & n \text{ odd} \end{cases} \qquad (7)$$

where m is any integer of a consecutive series of n integers. It is clear that the λ_p are located on an ellipse centered at the origin and of semi-axes equal to $\sinh a$ and $\cosh a$.

The distribution of poles discussed above is the simplest distribution of critical points which yields an oscillatory behavior over the imaginary axis of the z-plane. The corresponding approximation function $T(\lambda)$ of Eq. (6), has been used by Darlington[2] and others[1,3] as the magnitude squared of the transmission coefficient of low-pass filters. The function $-10 \log_{10} T(j\omega)$ is plotted in Fig. 3 for $n = 4$, $A = \sinh 4a$, and $a = 0.275$.
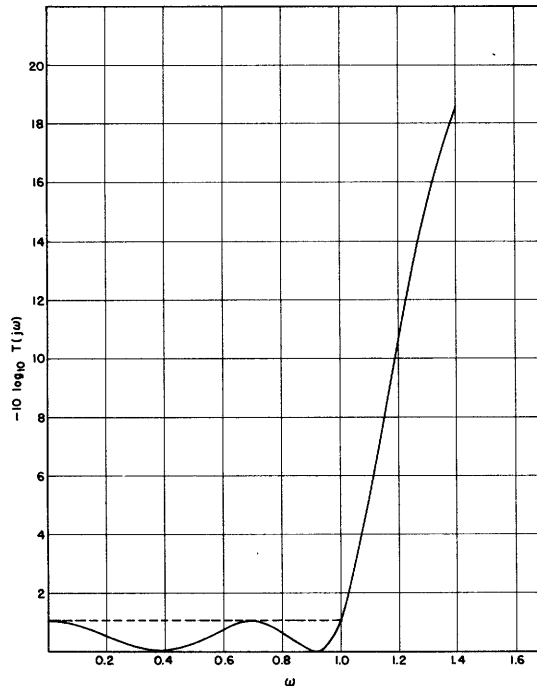

Fig. 3. The insertion loss of a filter with uniform pass-band approximation.

Approximation functions suitable for representing other network characteristics as well as reflection coefficients can be obtained by means of more complex distributions of critical points. One could use, for instance, more than one pair of rows of simple poles or multiple poles, and, in addition, one or more pairs of rows of zeros. Each pair of rows in the z-plane will transform, of course, into an ellipse in the λ-plane, so that all the critical points in the λ-plane will be located on confocal ellipses. The type of distribution of critical points that is to be selected in each particular case depends on the design specifications, on the form of network desired, and on the network characteristic that $F(\lambda)$ has to represent. For instance, if $F(\lambda)$ is to be the magnitude squared $|\rho|^2$ of the reflection coefficient of a low-pass filter, the number of zeros must be equal, at least, to the number of poles because $F(\lambda)$ must not vanish at infinity. Such a function can be obtained readily from the $T(j\omega)$ of Eq. (6) by means of the well-known relation[1,2,3]

$$|\rho|^2 = 1 - |t|^2 = 1 - T(j\omega). \tag{8}$$

One has then

$$|\rho|^2 = 1 - \frac{A}{\sinh^2 na + \cos^2(n \cos^{-1}\omega)} = \frac{\sinh^2 nb + \cos^2(n \cos^{-1}\omega)}{\sinh^2 na + \cos^2(n \cos^{-1}\omega)} \tag{9}$$

where $A \leq \sinh^2 na$, and $\sinh^2 nb = \sinh^2 na - A$. The zeros of $|\rho|^2$ are located on an ellipse of semiaxes equal to sinh b and cosh b.[1]

The use of several rows of poles or of multiple poles is indicated when the desired behavior of a lossless network has to be predistorted to counterbalance the effect of incidental dissipation.[2,3] The amount of dissipation for which such a predistortion is possible is limited by the distance from the imaginary axis of the pole closest to it. (The same reasoning applies to zeros when they are required to lie in the left half-plane as in the case of impedance functions.) This limitation results, in turn, in a limitation on the number of singularities on each ellipse, as indicated by Figs. 1 and 2. When such a situation arises the required pass-band tolerance of approximation and the required sharpness of cut-off can be obtained by increasing the multiplicity of the poles or the number of rows instead of the number of poles in each row. It must be noted, however, that this procedure results in a less effective use of the degrees of freedom available.

A last remark is in order regarding high-pass, band-pass, and band-elimination approximation problems. The well-known transformations from low-pass behavior to high-pass, band-pass, and band-elimination behavior can be used in these cases to obtain the desired functions. However, these transformations are more conveniently performed directly on the network rather than on the function.[3]

4. Uniform Approximation in the Pass-Band and in the Rejection Band

In many filter-design problems the requirements on the sharpness of cut-off and on the minimum rejection-band attenuation cannot be efficiently met by approximation functions of the type discussed in the previous section. In these cases, it is desirable to use, for the magnitude squared of the transmission coefficient $T(j\omega)$, a function which approximates in an oscillatory fashion a small constant in the rejection band, in addition to approximating a larger constant (smaller than unity) in the pass band. In most cases these two constants are made equal respectively to zero and one.

One has now two distinct regions of approximation on the imaginary axis of the λ-plane. The pass-band region corresponding to $|\omega| \leq 1$ and the rejection-band region corresponding to $|\omega| \geq 1/k$, where the constant $k < 1$ is specified by the requirement on the sharpness of cut-off. The change of complex variable $\lambda = f(z)$ desired in this case must transform the pass-band and the rejection-band regions of the $j\omega$-axis into parallel straight lines in the z-plane. Closer consideration of this situation indicates that the transformation function $\lambda = f(z)$ must be periodic with respect to both x and y, while in the previous case sinh z was periodic only with respect to y. It is clear then that the whole λ-plane will correspond, in the z-plane, to an infinite number of equal rectangular regions such as the one limited by the two pairs of straight lines $x = \pm K$ and $y = \pm K'$, K and K' being two constants. It is convenient in this case to make the pass-band region of the imaginary axis of the λ-plane correspond to the real rather than to the imaginary axis of the z-plane. In this way the imaginary unit disappears from many equations. (The same situation would have occurred in Sec. 3 if the notation sinh z were not in common use and the notation $j \sin(-jz)$ had to be used instead.) It will also be convenient for the same reason to use the variable $w = u + jv = -j\lambda$ instead of λ.

The solution of the transformation problem stated above can be found in the literature or can be obtained without difficulty by means of the Schwarz-Christoffel transformation. The desired function $w = f(z)$ is the Jacobian elliptic function[9,10,11] of modulus k:

$$w = \text{sn}(z,k) = \sin \varphi \tag{10}$$

where

$$z = \int_0^\varphi \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} = \int_0^w \frac{dt}{\sqrt{1 - t^2}\sqrt{1 - k^2 t^2}} \quad . \tag{11}$$

The constants K and K' are the complete elliptic integrals

$$K = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} \tag{12}$$

and

$$K' = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 - (k')^2 \sin^2 \psi}} \tag{13}$$

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where the complementary modulus k' is given by

$$k' = \sqrt{1 - k^2} \,. \tag{14}$$

This transformation is illustrated in Figs. 4 and 5. The heavy lines in Fig. 4 are the branch lines of the function.
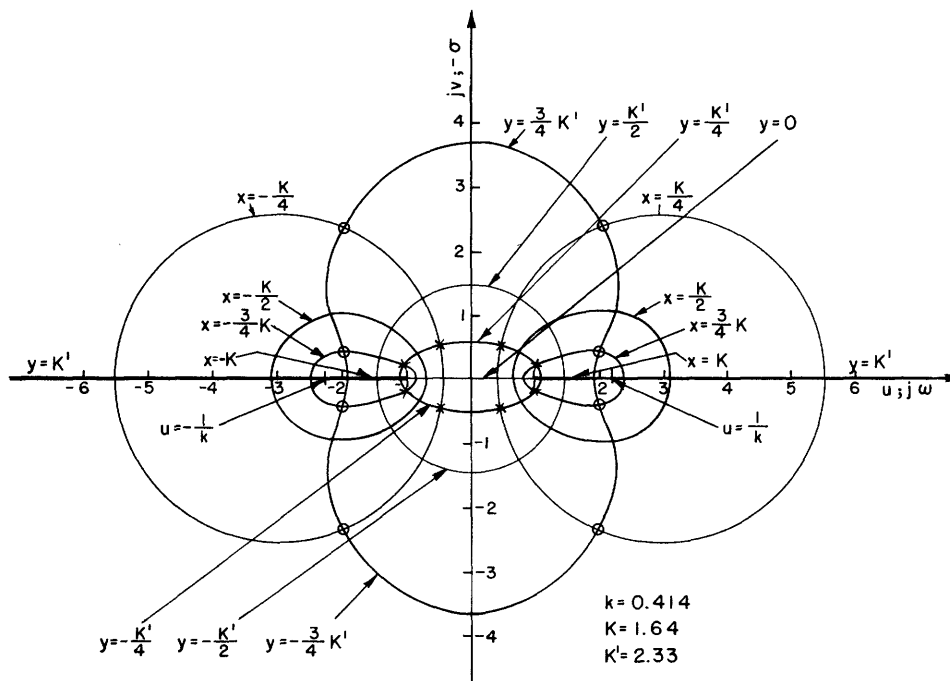

Fig. 4. The transformation w = sn(z,k) in the w-plane. The points indicated with crosses and circles correspond to the points indicated in the same manner in the z-plane (Fig. 5).

By following a reasoning similar in principle to that used in Sec. 3, it can be seen that the critical points in the z-plane must be distributed uniformly along straight lines parallel to the axis or axes over which oscillatory approximation is desired. In this case, however, one has to use an infinite number of rows of poles and zeros because the transformation w = sn z is doubly periodic. It will be noted that zeros as well as poles are required because the function $T(\lambda)$ must approximate zero or a small positive constant for $|\omega| \geq 1/k$. The simplest type of distribution of critical points satisfying these requirements consists of an infinite number of rows of poles and of zeros located in pairs at distances a and b, respectively, from the straight lines y = 2mK'. In most filter designs one makes b = K' in which case the zeros of $T(\lambda)$ will be located on the imaginary axis of the λ-plane. The poles of $T(\lambda)$, on the other hand, must not lie on the jω-axis and therefore must not lie on the lines x = $\pm$(2m+1)K in the z-plane
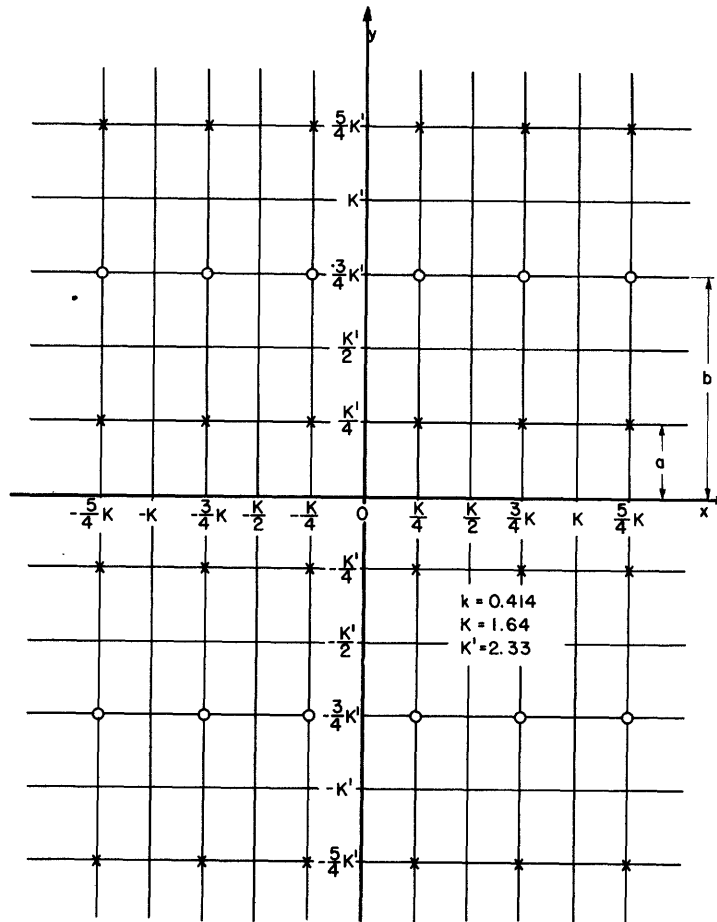
-10-

Fig. 5. The z-plane corresponding to the w-plane of Fig. 4.

where m is any integer. Figure 5* illustrates the case in which the critical points in the same row are spaced a distance equal to 2K/4. Figure 4 shows the location of the corresponding critical points in the w- and λ-planes, obtained from the relation

$$w_1 = -j\lambda_1 = sn(z_1,k)$$ (15)

where z_1, w_1, and λ_1 are the critical points in the z-, w-, and λ-planes, respectively.

To express in closed form the function whose critical points satisfy the above requirements, one observes first that the distribution of critical points for the function $sn^2(C_1 z,k_1)$, where C_1 is a constant, consists of a rectangular array of double zeros and a rectangular array of double poles. Both arrays have spacings along the $C_1 x$ and $C_1 y$ axes equal, respectively, to

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* The values of the parameters used in this figure (k = 0.414, a = K'/4, b = 3K'/4) are not representative of a practical design but lead to simpler and clearer graphical representations.

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$2K_1$ and $2K_1'$, one zero being located at the origin and one pole at the point $C_1 z = jK_1'$. The constants K_1 and K_1' are the complete elliptic integrals of modulus equal respectively to k_1 and $k_1' = \sqrt{1 - k_1^2}$. This distribution of critical points is illustrated in Fig. 6. One observes further that the
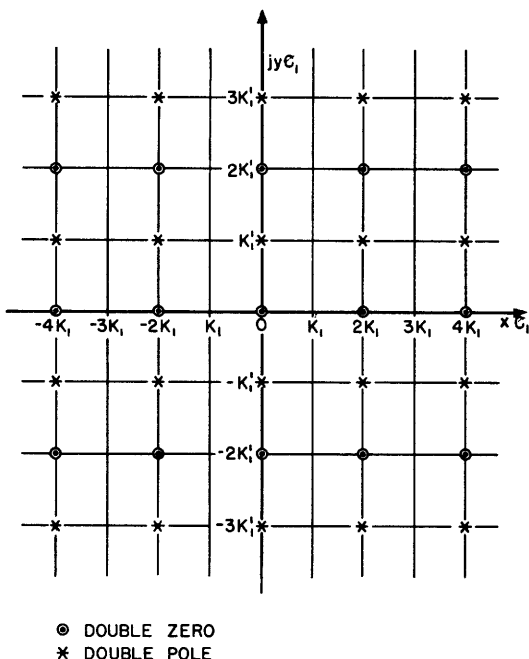

⊚ DOUBLE ZERO
✶ DOUBLE POLE

Fig. 6. Zeros and poles of the function $sn^2(C_1 z, k_1)$.

function considered has imaginary values over the lines $C_1 x = mK_1$, where m is any integer. It follows that the function

$$- sn^2(jC_1 a, k_1) + sn^2(C_1 z, k_1) \quad , \qquad (16)$$

where a is a positive constant smaller than K_1' , has the distribution of critical points illustrated in Fig. 7. It will be noted that the addition of a positive constant splits the rows of double zeros parallel to the real axis into rows of simple zeros displaced a distance $C_1 a$ along the imaginary axis on both sides of the original position. The location of the poles, on the contrary, does not change.

Consider finally the function

$$T\left[\lambda(z)\right] = A\left[\frac{- sn^2(jC_1 b, k_1) + sn^2(C_1 z, k_1)}{- sn^2(jC_1 a, k_1) + sn^2(C_1 z, k_1)}\right] \qquad (17)$$

where A is a constant. The poles of the numerator cancel with the poles of the denominator and the zeros of the numerator are arranged in pairs of rows
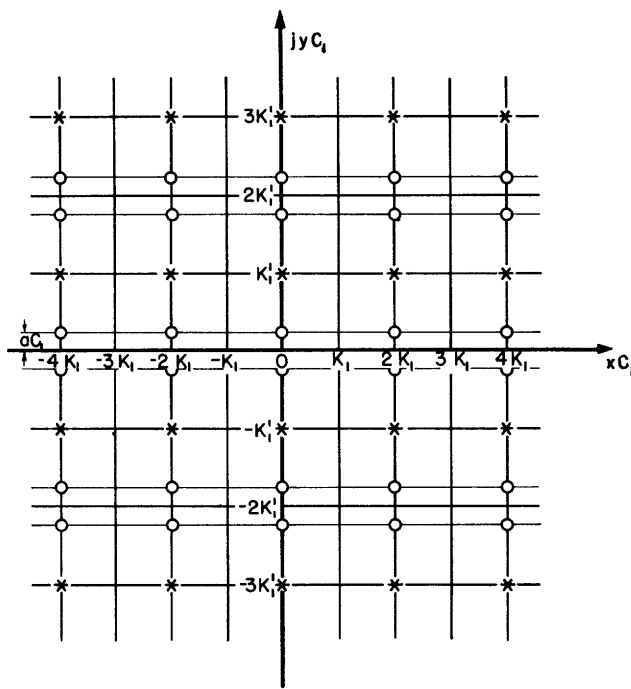
O SINGLE ZERO

✱ DOUBLE POLE

Fig. 7. Zeros and poles of the function (23).

parallel to the real axis and displaced a distance $C_1 b$ from the lines $C_1 y = 2m K_1'$ where m is any integer. This distribution of critical points is similar to that illustrated in Fig. 5, for $a = K'/4$ and $b = 3K'/4$. To identify completely these two distributions one must make

$$\left. \begin{array}{c} \dfrac{K}{n} = \dfrac{K_1}{C_1} \\[4mm] K' = \dfrac{K_1'}{C_1} \end{array} \right\} \tag{18}$$

and in addition, if the number n of singularities in any length 2K of each row is odd, the variable z must be changed into $z + K/n$. The computation of C_1 and k_1 (K_1 and K_1' are given by tables[11] as functions of k_1) from the known values of K, K', and n is carried out as follows, by using the relations[12]

$$q_1 = e^{-\pi K_1'/K_1} = e^{-n\pi K'/K} \tag{19}$$

$$k_1^{\frac{1}{2}} = \frac{\vartheta_2}{\vartheta_3} = 2q_1^{\frac{1}{4}} \frac{1 + q_1^2 + q_1^6 + q_1^{12} + \cdots}{1 + 2q_1 + 2q_1^4 + 2q_1^9 + \cdots} \tag{20}$$

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$$C_1 = \frac{K_1'}{K'} = n\frac{K_1}{K} \qquad (21)$$

where ϑ_2 and ϑ_3 are theta functions.

The next task is the determination of the tolerances of approximation in terms of the constants a, b, and A. For this purpose, one considers first the behavior of $T[\lambda(z)]$ on the real axis of the z-plane which corresponds to the segment $-1 < \omega < 1$ of the imaginary axis of the λ-plane. One observes that the function $sn(C_1 x, k_1)$ is real and oscillates between the values 1 and -1, and that, if $b > a$, $-sn^2(jC_1 b, k_1) > -sn^2(jC_1 a, k_1)$. Therefore the function $T[\lambda(x)]$ oscillates between the maximum value

$$\left(T_{max}\right)_{|\omega| < 1} = A\frac{sn^2(jn\frac{K_1}{K}b, k_1)}{sn^2(jn\frac{K_1}{K}a, k_1)} \qquad (22)$$

and the minimum value

$$\left(T_{min}\right)_{|\omega| < 1} = A\frac{1 - sn^2(jn\frac{K_1}{K}b, k_1)}{1 - sn^2(jn\frac{K_1}{K}a, k_1)} = A\frac{cn^2(jn\frac{K_1}{K}b, k_1)}{cn^2(jn\frac{K_1}{K}a, k_1)} \qquad (23)$$

where $cn(z, k_1) = \cos\varphi$, φ being given by Eq. (11). Standard trigonometric relations can be used to transform the above expressions into expressions more convenient for actual computations. It is to be noted in this regard that the tabulations available[11] give z as a function of φ for real values of φ.

The tolerance of approximation on that part of the imaginary axis of the λ-plane corresponding to $|\omega| > 1/k$ can be obtained by considering the behavior of T over the line $y = K'$. Then, using the relation[9]

$$sn(z + jK', k) = \frac{1}{k \, sn(z, k)} \qquad (24)$$

one obtains immediately that the function T oscillates between a maximum value

$$\left(T_{max}\right)_{|\omega| > 1/k} = A\frac{1 - k_1^2 sn^2(jn\frac{K_1}{K}b, k_1)}{1 - k_1^2 sn^2(jn\frac{K_1}{K}a, k_1)} \qquad (25)$$

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and a minimum value

$$\left(T_{min}\right)_{|\omega|>1/k} = A .$$ (26)

If one wishes to make $\left(T_{min}\right)_{|\omega|>1/k} = 0$ as is usually the case, the above
formulas lead to a confusing limiting process which can be avoided by
simply setting the numerator of Eq. (17) equal to one. This modification
amounts to using the double poles of the denominator as zeros of the function.
Equation (17) becomes then, for n odd,

$$T\left[\lambda(z)\right] = \frac{A}{- sn^2(jn\frac{K_1}{K}a,k_1) + sn^2(n\frac{K_1}{K}z,k_1)} .$$ (27)

For n even, z is changed into z + K/n. The limits of the two oscillations
become

$$\left(T_{max}\right)_{|\omega|<1} = \frac{A}{- sn^2(jn\frac{K_1}{K}a,k_1)}$$ (28)

$$\left(T_{min}\right)_{|\omega|<1} = \frac{A}{1 - sn^2(jn\frac{K_1}{K}a,k_1)} = \left[\frac{1}{A} + \frac{1}{\left(T_{max}\right)_{|\omega|<1}}\right]^{-1}$$ (29)

$$\left(T_{max}\right)_{|\omega|>1/k} = \frac{k_1^2 A}{1 - k_1^2 sn^2(jn\frac{K_1}{K}a,k_1)} = \left[\frac{1}{k_1^2 A} + \frac{1}{\left(T_{max}\right)_{|\omega|<1}}\right]^{-1}$$ (30)

$$\left(T_{min}\right)_{|\omega|>1/k} = 0 .$$ (31)

The zeros and poles of $T(\lambda)$ are obtained directly from the corresponding
critical points in the z-plane by means of Eq. (15). One has then for the
poles

$$\lambda_p = j\ sn\left\{\left[(2m + 1)\frac{K}{n} \pm ja\right],k\right\} \quad n\ even$$

$$\lambda_p = j\ sn\left\{\left[2m\frac{K}{n} \pm ja\right],k\right\} \quad n\ odd$$ (32)

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and for the zeros

$$\lambda_0 = j \, sn\left\{\left[(2m + 1)\frac{K}{n} \pm jb\right], k\right\} \qquad n \text{ even}$$

$$\lambda_0 = j \, sn\left\{\left[2m\frac{K}{n} \pm jb\right], k\right\} \qquad n \text{ odd.}$$

(33)

where m is any integer of a consecutive series of n integers. When $\left(T_{min}\right)_{|\omega| > 1/k} = 0$, $b = K'$ and therefore, using Eq. (24),

$$\lambda_0 = j \, \frac{1}{k \, sn\left[(2m+1)\frac{K}{n}, k\right]} \qquad n \text{ even}$$

$$\lambda_0 = j \, \frac{1}{k \, sn\left[2m\frac{K}{n}, k\right]} \qquad n \text{ odd.}$$

(34)

The function $T(\lambda)$ can be written in terms of its critical points as follows:

$$T(\lambda) = B \frac{(\lambda^2 - \lambda_{o1}^2)(\lambda^2 - \lambda_{o2}^2)\dots(\lambda^2 - \lambda_{on}^2)}{(\lambda^2 - \lambda_{p1}^2)(\lambda^2 - \lambda_{p2}^2)\dots(\lambda^2 - \lambda_{pn}^2)}$$

(35)

where each factor represents two critical points with opposite signs. The value of B is determined by considering the behavior of the function when λ approaches infinity and therefore z approaches $\pm jK'$. One obtains without difficulty

$$B = \left(T_{min}\right)_{|\omega| > 1/k} \qquad n \text{ odd}$$

$$B = \left(T_{max}\right)_{|\omega| > 1/k} \qquad n \text{ even}$$

(36)

where the T's are given by Eqs. (25) and (26) or (30). However, for n odd and $b = K_1'$ Eq. (36) is meaningless because $T(\lambda)$ vanishes at infinity. One observes then that $T(\lambda)$ vanishes as B/λ^2 when λ approaches infinity (because one of the factors of the numerator of Eq. (35) is missing) and that $sn(z+jK', k)$ approaches infinity as $1/kz$ when z approaches zero. A little manipulation of Eq. (27) shows that

$$\lim_{\lambda \to \infty} T(\lambda) = \lim_{\lambda \to \infty} \frac{B}{\lambda^2} = \lim_{\lambda \to \infty} \left[-\left(\frac{k_1}{k} \frac{K_1'}{K'}\right)^2 \frac{A}{\lambda^2} \right] . \qquad (37)$$

Therefore

$$B = -A \left(\frac{k_1}{k} \frac{K_1'}{K'}\right)^2 . \qquad (38)$$

It can be shown that Eq. (27) (for n even substitute $z + \frac{K}{n}$ for z) together with the transformation $\lambda = j \, sn(z,k)$ yields the approximation function used by many authors, for instance by Darlington.[2] Figure 8
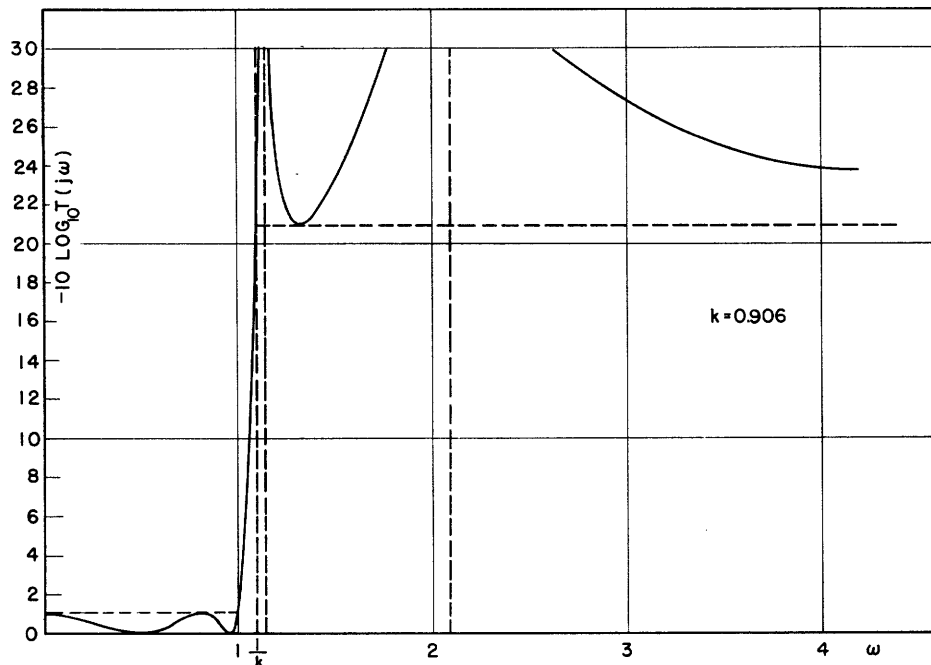

Fig. 8. The insertion loss of a filter with uniform approximation in the pass band and in the rejection band.

illustrates the behavior of the function $-10 \log_{10} T(j\omega)$ when n = 4, $\left(T_{min}\right)_{|\omega|>1/k} = 0$, $\left(T_{max}\right)_{|\omega|<1} = 1$, and $\left(T_{min}\right)_{|\omega|<1} = 0.8$. To the knowledge of the author, the more general function of Eq. (17) has not been previously presented. Other more complex functions involving critical points of higher order or additional arrays of critical points can be constructed as in the case discussed in the previous section. It is felt that a detailed analysis of these functions will not add materially to the understanding of the design procedure.

5. Conclusions

The examples presented in the preceding sections should be sufficient to make clear the ideas on which the approximation procedure is based. The field of application of such a procedure, on the other hand, is not limited to the examples discussed above although other cases have not been worked out in detail by the author. The usefulness of certain unconventional types of approximation functions in connection with the compensation of the effects of incidental dissipation were pointed out in Sec. 3.

A discussion of the synthesis procedure following the selection of the approximation function is beyond the scope of this paper. The reader is referred to Refs. 2 and 3 for such a discussion, and to Ref. 1 for a practical application of the approximation procedure presented above.

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