Department
of
APPLIED MATHEMATICS

A NOTE ON THE STABILITY OF STEADY
INVISCID HELICAL GAS FLOWS

by

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Report No.61 January 1976

UNIVERSITY OF BERGEN
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Abstract.

The stability of steady shear flow of an inviscid compressible fluid rotating between two coaxial sylinders is investigated. We consider flows where the streamlines are helical paths depending on the radial distance only. A necessary condition for stability is obtained.

The method applied is based on a study of the transport equations similar to those appearing in geometrical acoustics.

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Introduction.

The stability of rotating fluids has earlier been investigated by several authors (see the references quoted in [3]).

Most of the earlier work has considered incompressible fluids. The literature on compressible fluids is in fact rather small.

In this paper we study the stability properties of steady shear flow of an inviscid compressible fluid rotating between two coaxial sylinders. We consider flows where the streamlines are helical paths depending on the radial distance only (i.e. the rotating fluid has an axial velocity depending on the radial distance). A necessary condition for stability is obtained.

The method applied is developed by Eckhoff [1] and is based on a study of the transport equations similar to those appearing in geometrical acoustics.

The stability problem studied in this paper has also been investigated by Warren [5]. By a different method Warren deduced a sufficient condition for stability.

In the last section our results are compared with Warren's criterion It is proved, as should be expected, that his criterion in general is more restrictive than ours. For some velocity profiles the criteria coincide ; an equation determining these profiles is found.

Introduction.

The stability of rotating fluids has earlier been investigated by several authors (see the references quoted in [1]).

Most of the earlier work has considered incompressible fluids. The literature on compressible fluids is in fact rather small.

In this paper we study the stability properties of steady shear flow of an inviscid compressible fluid rotating between two coaxial cylinders. We consider flows where the streamlines are helical paths depending on the radial distance only (i.e. the rotating fluid has an axial velocity depending on the radial distance). A necessary condition for stability is derived.

The method applied is developed by Howard [1] and is based on a study of the [illegible] equations which [illegible] geometrical properties.

The stability problem studied in this paper has also been investigated by Warren [2]. By a different method he deduced a sufficient condition for stability.

In the last section our results are compared with Warren's [illegible]. In this respect we should be reminded, that the criterion in [illegible] is more restrictive than ours. For some velocity profiles the [illegible] equation [illegible]

1. Formulation of the problem.

We consider an ideal compressible adiabatic fluid. The basic equations are:

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = - \frac{1}{\rho} \nabla p + \nabla V$$

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho + \rho \nabla \cdot \underline{v} = 0 \qquad (1.1)$$

$$\frac{\partial}{\partial t} (p\rho^{-\gamma}) + v \cdot \nabla (p\rho^{-\gamma}) = 0$$

Here $\underline{v}$, ρ, p, t and V denote velocity, density, pressure, time and potential for the external forces, respectively, and γ is a constant.

We want to investigate the stability properties of a fluid (gas) rotating between two coaxial sircular sylinders (radii r_1 and r_2, $0 < r_1 < r_2$). Like Warren [5] we assume steady flows where the velocity is of the form

$$\underline{U} = v_0(r)\underline{e}_\phi + w_0(r)\underline{e}_z \qquad (1.2)$$

and where the density, pressure and potential also are functions of the radius only, i.e.

$$\rho_0 = \rho_0(r), \quad p_0 = p_0(r), \quad V = V(r) \qquad (1.3)$$

Here (r,ϕ,z) are sylindrical coordinates and $\underline{e}_r$, $\underline{e}_\phi$, $\underline{e}_z$ are the unit vectors in these coordinates.

It follows from (1.2) that the streamlines are sircular helixes depending on the radial distance r.

It is readily seen that (1.2,1.3) is a solution of (1.1) for arbitrary functions v_0, w_0, ρ_0 and V if and only if the function p_0 satisfies the equation:

$$\frac{dp_0}{dr} = \rho_0\left(\frac{v_0^2}{r} + \frac{dV}{dr}\right) \qquad (1.4)$$

For given v_0, ρ_0 and V this equation determines the function p_0 up to an additive constant.

In order to study the stability properties of the solution $\underline{U}$, ρ_0, p_0, we perturbate it by introducing into (1.1) the following expressions:

$$\underline{v} = \underline{U} + \frac{1}{F}\underline{w} \qquad p = p_0 + \frac{\rho_0 c}{F} s_2$$
$$\rho = \rho_0 + \frac{F}{\rho_0 Ec} s_1 + \frac{\rho_0}{Fc} s_2 \tag{1.5}$$

Here F and E are arbitrary weightfunctions depending on r, we only assume that $F \neq 0$, $E \neq 0$ everywhere. The quantity c is the local sound speed given by:

$$c = \sqrt{\frac{\gamma p_0}{\rho_0}} \tag{1.6}$$

The transformation (1.5) is essentially the one applied in [3] & [4]. The linearized version of the perturbation equations (given in [3] & [4]) is a system of linear symmetric hyperbolic p.d.e.'s.

We now write

$$\underline{w} = w_r \underline{e}_r + w_\phi \underline{e}_\phi + w_z \underline{e}_z \tag{1.7}$$

and we introduce the 5-dimensional vector

$$\omega = (w_r, w_\phi, w_z, s_1, s_2) \ . \tag{1.8}$$

The perturbation equations in sylinderical coordinates can be written:

$$\frac{\partial\omega}{\partial t} + A^1 \frac{\partial\omega}{\partial r} + A^2 \frac{\partial\omega}{\partial\phi} + A^3 \frac{\partial\omega}{\partial z} + B\omega = 0 \tag{1.9}$$

Here A^1, A^2, A^3 and B are 5×5 matrices and ω is treated as a column vector. These matrices are calculated from (1.2, 1.3 & 1.5) and the perturbation equations to be:

For given v_0, ρ_0 and V this equation determines the function p_0 up to an additive constant.

In order to study the stability properties of this solution $\{v_0, p_0, \rho_0\}$ we perturbate it by introducing into (1.1) the following expressions:

$$\underline{v} = \underline{v}_0 + \rho_0^{-\frac{1}{2}}\underline{u}, \quad p = p_0 + \frac{1}{\gamma}\rho_0^{\frac{1}{2}} c\, s_4$$

$$\rho = \rho_0 + \frac{\rho_0^{\frac{1}{2}}}{c}\, s_4 + \frac{\rho_0^{\frac{1}{2}}}{\gamma}\, s_5 \qquad (2.3)$$

Here V and B are arbitrary weightfunctions depending on t; we only assume that $V > 0$, $B \neq 0$ everywhere. The quantity c is the local sound speed given by:

$$c = \sqrt{\frac{\gamma p_0}{\rho_0}} \qquad (2.4)$$

The transformation (2.3) is essentially the one applied in [1] & [4]. The linearized system of the perturbation equations (based on (1.1) & (2.3)) is a system of linear symmetric hyperbolic p.d.e.'s.

We now write

$$\underline{u} = u_1 \underline{e}_1 + u_2 \underline{e}_2 + u_3 \underline{e}_3 \qquad (2.5)$$

and we introduce the 5-dimensional vector

$$w = (u_1, u_2, u_3, s_4, s_5) \qquad (2.6)$$

The perturbation equations in Cartesian coordinates can be written:

$$\frac{\partial w}{\partial t} + A^x \frac{\partial w}{\partial x} + A^y \frac{\partial w}{\partial y} + A^z \frac{\partial w}{\partial z} + Bw = 0 \qquad (2.7)$$

where A^x, A^y, A^z and B are 5x5 matrices and w is treated as a column vector. These matrices are calculated from (1.1, 1.2, & 2.3) and the perturbation equations to be:

$$A^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} \frac{1}{r}v_o & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{r}v_o & 0 & 0 & \frac{1}{r}c \\ 0 & 0 & \frac{1}{r}v_o & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r}v_o & 0 \\ 0 & \frac{1}{r}c & 0 & 0 & \frac{1}{r}v_o \end{bmatrix} \quad \text{(1.1}$$

$$A^3 = \begin{bmatrix} w_o & 0 & 0 & 0 & 0 \\ 0 & w_o & 0 & 0 & 0 \\ 0 & 0 & w_o & 0 & c \\ 0 & 0 & 0 & w_o & 0 \\ 0 & 0 & c & 0 & w_o \end{bmatrix}, B = \begin{bmatrix} 0 & -2\frac{v_o}{r} & 0 & \beta & H \\ \frac{v_o}{r} + \frac{dv_o}{dr} & 0 & 0 & 0 & 0 \\ \frac{dw_o}{dr} & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 \\ G & 0 & 0 & 0 & 0 \end{bmatrix},$$

where the quantities α, β, G and H are:

$$\left.\begin{aligned} \alpha &= \frac{E\rho_o}{F} c\left[\frac{d\rho_o}{dr} - \frac{1}{c^2}\frac{dp_o}{dr}\right] = \frac{E\rho_o^{\,2}}{F} c\left[\frac{1}{\rho_o}\frac{d\rho_o}{dr} - \frac{1}{c^2}\left(\frac{v_o^{\,2}}{r} + \frac{dV}{dr}\right)\right] \\ \beta &= -\frac{F^2}{E\rho_o^{\,3}}\frac{1}{c}\frac{dp_o}{dr} = -\frac{F^2}{E\rho_o^{\,2}}\frac{1}{c}\left(\frac{v_o^{\,2}}{r} + \frac{dV}{dr}\right) \end{aligned}\right\} \quad (1.11)$$

$$\left.\begin{aligned} G &= \frac{1}{c}\left(\frac{\gamma}{2} - 1\right)\left[\frac{v_o^{\,2}}{r} + \frac{dV}{dr}\right] + \frac{c}{2\rho_o}\frac{d\rho_o}{dr} - \frac{c}{F}\frac{dF}{dr} \\ H &= \frac{c}{r} + \frac{1}{c}\left(\frac{v_o^{\,2}}{r} + \frac{dV}{dr}\right) - \frac{c}{F}\frac{dF}{dr} \end{aligned}\right\} \quad (1.12)$$

The characteristic equation associated with (1.9) is:

$$\det(\xi^1 A^1 + \xi^2 A^2 + \xi^3 A^3 - \lambda I) = \left[\frac{1}{r}\,\xi^2 v_o + \xi^3 w_o\right]\left[\left(\frac{1}{r}\,\xi^2 v_o + \xi^3 w_o\right)^2 - c^2 k^2\right] = 0 \tag{1.13}$$

where

$$k = \sqrt{\left(\xi^1\right)^2 + \left(\frac{1}{r}\,\xi^2\right)^2 + \left(\xi^3\right)^2} \tag{1.14}$$

The characteristic roots and the associated orthonormal eigenvectors are easily obtained (cf. [3]) from (1.10) and (1.13):

$$\left.\begin{aligned} \Omega^1 &= \frac{1}{r}\,\xi^2 v_o + \xi^3 w_o \quad \text{(triple root)} \\ \Omega^2 &= \frac{1}{r}\,\xi^2 v_o + \xi^3 w_o + ck \quad \text{(simple root)} \\ \Omega^3 &= \frac{1}{r}\,\xi^2 v_o + \xi^3 w_o - ck \quad \text{(simple root)} \end{aligned}\right\} \tag{1.15}$$

$$\left.\begin{aligned} r^{11} &= \frac{1}{k}\left(\frac{1}{r}\,\xi^2, -\xi^1, 0, \xi^3, 0\right) \\ r^{12} &= \frac{1}{k}\left(0, \xi^3, -\frac{1}{r}\,\xi^2, \xi^1, 0\right) \\ r^{13} &= \frac{1}{k}\left(-\xi^3, 0, \xi^1, \frac{1}{r}\,\xi^2, 0\right) \end{aligned}\right\} \tag{1.16}$$

$$\left.\begin{aligned} r^{21} &= \frac{1}{\sqrt{2}\,k}\left(\xi^1, \frac{1}{r}\,\xi^2, \xi^3, 0, k\right) \\ r^{31} &= \frac{1}{\sqrt{2}\,k}\left(\xi^1, \frac{1}{r}\,\xi^2, \xi^3, 0, -k\right) \end{aligned}\right\} \tag{1.17}$$

The stationary solution $\underline{U}, p_o, \rho_o$ of (1.1) corresponds to the null solution $\omega = 0$ of (1.9). In this paper we are going to study the stability properties of the latter by the method developed in [1]. The discussion of the acoustic waves corresponding to the simple roots Ω^2 and Ω^3 is completely analogues to the discussion in [2]. In our problem these waves never give rise to any instabilities for realistic physical flows (cf. [3]). Thus we shall limit our further discussion to the mass waves ("gravity" waves) corresponding to the triple root Ω^1.

2. The transport equations associated with Ω^1.

For the mass waves the stability equations are identical with the transport equations associated with Ω^1 (cf. [1], [3] & [4]). Therefore we have to study the stability properties of these equations. Applying the general formulas given in [1] we deduce the transport equations from (1.10, 1.15, 1.16) to be:

$$\frac{dr}{dt} = 0 \ , \quad \frac{d\phi}{dt} = \frac{1}{r} v_o \ , \quad \frac{dz}{dt} = w_o$$

$$\frac{d\xi^1}{dt} = \frac{1}{r} \xi^2\left(\frac{v_o}{r} - \frac{dv_o}{dr}\right) - \xi^3 \frac{dw_o}{dr} \tag{2.1}$$

$$\frac{d\xi^2}{dt} = 0 \ , \quad \frac{d\xi^3}{dt} = 0$$

$$\begin{aligned}\frac{d\sigma_1^1}{dt} = {} & \frac{1}{k^2}\left[-\frac{1}{r}\xi^1\xi^2\left(\frac{v_o}{r} - \frac{dv_o}{dr}\right) - \frac{1}{r}\xi^2\xi^3(\alpha+\beta)\right]\sigma_1^1 \\ & + \frac{1}{k^2}\left[-\frac{1}{r}\xi^1\xi^2\beta + \frac{1}{r}\xi^2\xi^3\left(\frac{v_o}{r} + \frac{dv_o}{dr}\right) + \left(\xi^3\right)^2 \frac{dw_o}{dr}\right]\sigma_2^1 \\ & + \frac{1}{k^2}\left[-\xi^1\xi^3\left(\frac{v_o}{r} + \frac{dv_o}{dr}\right) - \left(\frac{1}{r}\xi^2\right)^2\beta + \left(\xi^3\right)^2\alpha\right]\sigma_3^1\end{aligned} \tag{2.2}$$

$$\begin{aligned}\frac{d\sigma_2^1}{dt} = {} & \frac{1}{k^2}\left[-\frac{1}{r}\xi^1\xi^2\alpha + \left(\frac{1}{r}\xi^2\right)^2 \frac{dw_o}{dr} - \frac{1}{r}\xi^2\xi^3\, 2\frac{dv_o}{dr} - \left(\xi^3\right)^2 \frac{dw_o}{dr}\right]\sigma_1^1 \\ & + \frac{1}{k^2} \cdot 0 \\ & + \frac{1}{k^2}\left[\xi^1\xi^3\alpha + \left(\frac{1}{r}\xi^2\right)^2\left(\frac{v_o}{r} - \frac{dv_o}{dr}\right) - \frac{1}{r}\xi^2\xi^3\, 2\frac{dw_o}{dr} + \left(\xi^3\right)^2\left(\frac{v_o}{r} + \frac{dv_o}{dr}\right)\right]\sigma_3^1\end{aligned} \tag{2.3}$$

$$\frac{d\sigma_3^1}{dt} = \frac{1}{k^2}\Big[-\frac{1}{r}\,\xi^1\xi^2\,\frac{dw_o}{dr} + \xi^1\xi^3\;2\,\frac{v_o}{r} - \left(\frac{1}{r}\,\xi^2\right)^2\alpha\;\;\left(\xi^3\right)^2\beta\Big]\sigma_1^1$$

$$+ \frac{1}{k^2}\Big[\xi^1\xi^3\beta - \left(\frac{1}{r}\,\xi^2\right)^2\left(\frac{v_o}{r} - \frac{dv_o}{dr}\right) + \frac{1}{r}\,\xi^2\xi^3\,\frac{dw_o}{dr} + \left(\xi^3\right)^2\cdot 2\,\frac{v_o}{r}\Big]\sigma_2^1 \qquad (2.4)$$

$$+ \frac{1}{k^2}\Big[\xi^1\xi^3\,\frac{dw_o}{dr} + \frac{1}{r}\,\xi^2\xi^3(\alpha+\beta)\Big]\sigma_3^1$$

The bicharacteristic equations (2.1) can be integrated directly, the solutions are:

$$r = r_o\;,\quad \phi = \phi_o + \frac{v_o(r_o)}{r_o}\,t\;,\quad z = z_o + w_o(r_o)t$$

$$\xi^1 = \xi_o^1 + \{\frac{1}{r_o}\,\xi_o^2\left(\frac{v_o}{r_o} - \frac{dv_o}{dr}\right) - \xi_o^3\,\frac{dw_o}{dr}\}\,t\;,\qquad \xi^2 = \xi_o^2\;,\quad \xi^3 = \xi_o^3 \qquad (2.5)$$

where r_o, ϕ_o, z_o, ξ_o^1, ξ_o^2, ξ_o^3 are initial values at $t = 0$.

Substituting (2.5) into the amplitude equations (2.2, 2.3, 2.4) we obatain a closed linear system of ordinary differential equations for the amplitudes σ_1^1, σ_2^1, σ_3^1 of the mass waves:

$$\frac{d\sigma}{dt} = A(t)\sigma \qquad (2.6)$$

where $A(t)$ is a 3×3 matrix and σ is a 3-dimensional column vector with components σ_1^1, σ_2^1, σ_3^1, respectively.

3. Discussion of stability.

According to the theory of stability developed in [1], the steady flow $\underline{U}$, ρ_0, p_0 is stable only if (2.6) is stable at the origin (i.e. $\sigma = 0$ is stable) for all possible values of the parameters r_0, ϕ_0, z_0, ξ_0^1, ξ_0^2, ξ_0^3. Thus we have to study the stability properties of the null solution of (2.6) for different choices of the parameters.

It is easily seen from (2.5) that (2.6) is autonomous if and only if:

$$\frac{1}{r_0}\, \xi_0^2 \left(\frac{v_0}{r_0} - \frac{dv_0}{dr}\right) - \xi_0^3\, \frac{dw_0}{dr} = 0 \qquad (3.1)$$

For all possible values of r_0, ξ_0^2 ξ_0^3 which satisfy (3.1), the stability properties of (2.6) at the origin are thus determined by the eigenvalues of the coefficient matrix A. These eigenvalues are found to be:

$$\lambda_1 = 0\ , \quad \lambda_2 = i\, \frac{1}{k_0}\, D\ , \quad \lambda_3 = -\lambda_2 \qquad (3.2)$$

where i is the imaginary unit,

$$k_0 = \sqrt{\left(\xi_0^1\right)^2 + \left(\frac{1}{r_0}\, \xi_0\right)^2 + \left(\xi_0^3\right)^2} \qquad (3.3)$$

and

$$D^2 = -\left(\frac{1}{r_0}\, \xi_0^2\right)^2 \alpha\beta - \frac{1}{r_0}\, \xi_0^2 \xi_0^3\; 2\, \frac{v_0}{r_0} \cdot \frac{dw_0}{dr} - \left(\xi_0^3\right)^2 \left[\alpha\beta - 2\, \frac{v_0}{r_0}\left(\frac{v_0}{r_0} + \frac{dv_0}{dr}\right)\right] \qquad (3.4)$$

From (3.2) we conclude that a necessary condition for (2.6) to be stable at the origin for all values of r_0, ϕ_0, z_0, ξ_0^1, ξ_0^2, ξ_0^3 is that D is a real quantity when (3.1) is satisfied. Thus a necessary condition for stability is that the quadratic form D^2 given by (3.4) is non-negative. The general discussion of the quadratic form is done below in connection with the non-autonomous case. Here we only consider the marginal case $D^2 = 0$. Since $D^2 = 0$ implies that $\lambda = 0$ is a triple eigenvalue, stability of

3. Discussion of stability

According to the theory of stability developed in [4], the steady flow $\underline{U}$, ρ_0, p_0 is stable only if (2.6) is stable at the origin (i.e. $\delta^* = 0$ is stable) for all possible values of the parameters r_0, Ω_0, ω_0, ζ_0^1, ζ_0^2, ζ_0^3. Thus we have to study the stability properties of the null solution of (2.6) for different choices of the parameters.

It is easily seen from (2.5) that (2.6) is autonomous if and only if:

$$\frac{1}{r_0}\zeta_0^2\left(\frac{v}{r_0}-\frac{dv}{dr}\right) - \zeta_0^1\frac{dw}{dr} = 0 \qquad (3.1)$$

For all possible values of r_0, ζ_0^2, ζ_0^3 which satisfy (3.1), the stability properties of (2.6) at the origin are thus determined by the eigenvalues of the coefficient matrix A. These eigenvalues are found to be:

$$\lambda_1 = 0\ ,\quad \lambda_2 = i\,\frac{1}{\lambda_0}D\ ,\quad \lambda_3 = -\lambda_2 \qquad (3.2)$$

where i is the imaginary unit,

$$\lambda_0 = \sqrt{\left(\zeta_0^1\right)^2 + \left(\frac{1}{r_0}\zeta_0^2\right)^2 + \left(\zeta_0^3\right)^2} \qquad (3.3)$$

and

$$D^2 = \left(\frac{1}{r_0}\zeta_0^2\right)^2\omega^2 - \frac{1}{r_0}\zeta_0^1\zeta_0^2\,2\frac{v}{r_0}\frac{dw}{dr} - \left(\zeta_0^1\right)^2\left[\omega^2 - 2\frac{v_0}{r_0}\left(\frac{v}{r_0}+\frac{dv}{dr}\right)\right] \qquad (3.4)$$

From (3.2) we conclude that a necessary condition for (2.6) to be stable at the origin for all values of r_0, Ω_0, ω_0, ζ_0^1, ζ_0^2, ζ_0^3 is that D is a real quantity when (3.1) is satisfied. Thus a necessary condition for stability is that the quadratic term D^2 given by (3.4) is non-negative. The general discussion of the quadratic form is done below in connection with the non-autonomous case. Here we only consider the marginal case $D^2 = 0$. Since $D^2 = 0$ implies that $\lambda = 0$ is a triple eigenvalue, stability of

(2.6) at the origin requires the coefficient matrix A to be zero. If $\xi_0^2 = \xi_0^3 = 0$, it is easily seen from the amplitude equations that $A = 0$. On the other hand, if $\left(\xi_0^2\right)^2 + \left(\xi_0^3\right)^2 \neq 0$, $A = 0$ for all possible values of the parameters only if

$$\alpha = \beta = v_o = \frac{dv_o}{dr} = \frac{dw_o}{dr} = 0 \tag{3.5}$$

Then we consider all values of r_o, ξ_o^2, ξ_o^3 such that (3.1) is not satisfied. In this case the system (2.6) is seen to be a non-autonomous system where all the coefficients tend to zero as $t \to +\infty$. Introducing $\tau = \ln t$ (ln = nat. log) as a new variable in (2.6) (cf. [2]), this system is transformed into an equivalent system with respect to stability at the origin. Asymptotically as $\tau \to +\infty$ the transformed system tends to a system with constant coefficients. The coefficient matrix of the latter is seen to be:

$$B_o = \lim_{t\to+\infty} t\,A(t) = \frac{1}{e}\begin{bmatrix} \frac{1}{r_o}\,\xi_o^2\left(\frac{v_o}{r} - \frac{dv_o}{dr}\right) & -\frac{1}{r_o}\,\xi_o\,\beta & -\xi_o^3\left(\frac{v_o}{r} + \frac{dv_o}{dr}\right) \\ \frac{1}{r_o}\,\xi_o^2\,\alpha & 0 & \xi_o^3\,\alpha \\ -\frac{1}{r_o}\,\xi_o^2\,\frac{dw_o}{dr} + \xi_o^3\,2\,\frac{v_o}{r} & \xi_o^3\,\beta & \xi_o^3\,\frac{dw}{dr} \end{bmatrix} \tag{3.6}$$

where

$$e = \frac{1}{r_o}\,\xi_o^2\left(\frac{v_o}{r} - \frac{dv_o}{dr}\right) - \xi_o^3\,\frac{dw_o}{dr} \neq 0 \tag{3.7}$$

The eigenvalues are:

$$\lambda_1 = 0\,,\quad \lambda_2 = -\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{e^2}\,D^2}$$
$$\lambda_3 = -\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{e^2}\,D^2} \tag{3.8}$$

From results proved in the appendix of [2], it follows that (2.6) is stable at the origin (when $e \neq 0$) if the autonomous system with coefficient matrix B_o is stable at the origin. Thus it is seen from (3.8) that (2.6) must be stable at the origin for all

(2.6) at the origin requires the coefficient matrix A to be zero. If $\xi_o^2 = \xi_o^3 = 0$, it is easily seen from the amplitude equations that $A = 0$. On the other hand, if $\left(\xi_o^2\right)^2 + \left(\xi_o^3\right)^2 \neq 0$, $A = 0$ for all possible values of the parameters only if

$$a = r = v_o = \frac{dv_o}{dr} = \frac{dw_o}{dr} = 0 \tag{3.5}$$

Then we consider all values of r, ξ_o^2, ξ_o^3 such that (3.1) is not satisfied. In this case the system (2.6) is seen to be a non-autonomous system where all the coefficients tend to zero at $t \to +\infty$. Introducing $\tau = \ln t$ (ln = nat. log) as a new variable in (2.6) (cf. [3]), this system is transformed into an equivalent system with respect to stability at the origin. Asymptotically as $\tau \to +\infty$ the transformed system tends to a system with constant coefficients. The coefficient matrix of the latter is seen to be:

$$B_o = \lim_{t \to +\infty} t\,A(t) = \frac{1}{a}\begin{bmatrix} \left(\frac{v_o}{r} + \frac{dv_o}{dr}\right) & [illegible] & \left(\frac{dw_o}{dr} - \frac{v_o}{r}\right)\frac{1}{\xi_o} \\ \xi_o^2\, a & 0 & \frac{1}{\xi_o}\,\xi_o^2\, a \\ [illegible] & [illegible] & [illegible] \end{bmatrix} \tag{3.6}$$

where

$$a = \frac{1}{\xi_o}\,\xi_o^2\left(\frac{dw_o}{dr} - \frac{v_o}{r}\right) + \xi_o^3\,\frac{dw_o}{dr} \neq 0 \tag{3.7}$$

The eigenvalues are:

$$\lambda_1 = 0\,, \quad \lambda_2 = -\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{a^2}\,D}$$

$$\lambda_3 = -\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{a^2}\,D} \tag{3.8}$$

From results proved in the section of [3], it follows that (2.6) is stable at the origin (when $\tau \to +\infty$) if the autonomous system with coefficient matrix B_o is stable at the origin. Thus it is seen from (3.8) that (2.6) must be stable at the origin for all possible values of r_o, ϕ_o, z_o, ξ_o^1, ξ_o^2, ξ_o^3 when the quadratic form D^2 is positive definite. Consequently, we are not led to any additional conditions for stability considering values r_o, ξ_o^2, ξ_o^3 such that $e \neq 0$.

Combining the results from the autonomous and non-autonomous case, we conclude that the linear system (2.6) is stable at the origin for all possible values of r_o, ϕ_o, z_o, ξ_o^1, ξ_o^2, ξ_o^3 if and only if D^2 is positive definite or (3.5) is satisfied.

Finally we discuss the definite properties of the quadratic form. Obviously D^2 can be transformed to a diagonal form

$$D^2 = \varkappa_1 x^2 + \varkappa_2 y^2 \tag{3.9}$$

by an orthogonal transformation $\left(\xi_o^1, \xi_o^2\right) \rightarrow (x,y)$. The coefficients $\varkappa_1$ and $\varkappa_2$ are the eigenvalues of the symmetric matrix associated with the quadratic form and are found to be:

$$\varkappa_n = -\alpha\beta + \frac{v_o}{r_o}\left(\frac{v_o}{r_o} + \frac{dv_o}{dr}\right) + (-1)^n \sqrt{\left(\frac{v_o}{r_o}\right)^2 \left\{\left(\frac{v_o}{r_o} \quad \frac{dv_o}{dr}\right)^2 + \frac{dw_o}{dr}\right\}} \tag{3.10}$$

$$n = 1,\ 2$$

For D^2 to be positive definite everywhere it is necessary and sufficient that $\varkappa_1 > 0$ and $\varkappa_2 > 0$ for all values of r_o ($\alpha, \beta, v_o, \frac{dv_o}{dr}, \frac{dw_o}{dr}$ are all functions of r_o). This is satisfied if

$$\alpha\,\beta < \frac{v_o}{r_o}\left(\frac{v_o}{r_o} + \frac{dv_o}{dr}\right) - \sqrt{\left(\frac{v_o}{r_o}\right)^2 \left\{\left(\frac{v_o}{r_o} + \frac{dv_o}{dr}\right)^2 + \left(\frac{dw_o}{dr}\right)^2\right\}} \tag{3.11}$$

for all values of r_o.

According to the general stability theory [1], we now can summarize our results as follows (using the relations (1.11):

possible values of r_o, ω_o, a_o^1, ξ_o^1, ξ_o^2, ζ_o^1 when the quadratic form D^2 is positive definite. Consequently we assume no [?] additional conditions for stability considering values r_o, ξ_o^1, ξ_o^2 such that $e = 0$.

Combining the results from the autonomous and non-autonomous cases, we conclude that the linear system (2.9) is stable at the origin for all possible values of r_o, ω_o, a_o, ξ_o^1, ξ_o^2, ζ_o^2 if and only if D^2 is positive definite or (3.5) is satisfied.

Finally we discuss the definite properties of the quadratic form. Obviously D^2 can be transformed to a diagonal form

$$D^2 = \kappa_1 x^2 + \kappa_2 y^2 \qquad (3.9)$$

by an orthogonal transformation $(\xi_o^1, \xi_o^2) \to (x, y)$. The coefficients κ_1 and κ_2 are the eigenvalues of the symmetric matrix associated with the quadratic form and are found to be:

$$\kappa_n = -\alpha\beta + \frac{\beta}{2}\left(\frac{v_o}{r_o} + \frac{dv_o}{dr}\right) + (-1)^n \sqrt{\left(\frac{v_o}{r_o}\right)^2\left(\frac{v_o}{r_o} - \frac{dv_o}{dr}\right)^2 + \left(\frac{dw_o}{dr}\right)^2} \qquad (3.10)$$

$$n = 1, 2$$

For D^2 to be positive definite everywhere it is necessary and sufficient that $\kappa_1 > 0$ and $\kappa_2 > 0$ for all values of r_o ($\alpha, \beta, v_o, \frac{dv_o}{dr}, \frac{dw_o}{dr}$ are all functions of r_o). This is satisfied if

$$\alpha\beta < \frac{\beta}{2}\left(\frac{v_o}{r_o} + \frac{dv_o}{dr}\right) - \sqrt{\left(\frac{v_o}{r_o}\right)^2\left(\frac{v_o}{r_o} - \frac{dv_o}{dr}\right)^2 + \left(\frac{dw_o}{dr}\right)^2} \qquad (3.11)$$

for all values of r_o.

According to the general stability theory [1], we now can summarize our results as follows (using the relations (1.11)):

Theorem.

A necessary condition for stability of the flow (1.2, 1.3) is that

$$\left(\frac{V_o^2}{r}+\frac{dV}{dr}\right)\left[\frac{1}{\rho_o}\frac{d\rho_o}{dr}-\frac{1}{c^2}\left(\frac{V_o^2}{r}+\frac{dV}{dr}\right)\right] \geq$$

$$-\frac{v_o}{r}\left(\frac{v_o}{r}+\frac{dv_o}{dr}\right)+\sqrt{\left(\frac{v_o}{r}\right)^2\left\{\left(\frac{v_o}{r}+\frac{dv_o}{dr}\right)^2+\left(\frac{dw_o}{dr}\right)^2\right\}} \tag{3.12}$$

holds everywhere in the fluid.

If equality holds in (3.12) on some set of positive measure, it is necessary for stability that

$$\frac{d\rho_o}{dr}=\frac{dV}{dr}=0 \quad \& \quad v_o=\frac{dv_o}{dr}=\frac{dw_o}{dr}=0 \tag{3.13}$$

hold almost everywhere on this set.

Remarks.

It follows from (3.12) that the shear in the axial velocity (i.e. $\frac{dw_o}{dr}$) may give rise to instabilities. Stability is obtainable only if this shear is not too large.

If $\frac{dw_o}{dr}=0$ everywhere in the fluid, we get a special case treated in [3]. The condition (3.12) then simplifies to

$$N^2 \geq \begin{cases} -2\,\frac{v_o}{r}\left(\frac{v_o}{r}+\frac{dv_o}{dr}\right) & \text{if} \quad \frac{v_o}{r}\left(\frac{v_o}{r}+\frac{dv_o}{dr}\right)<0 \\ 0 & \text{if} \quad \frac{v_o}{r}\left(\frac{v_o}{r}+\frac{dv_o}{dr}\right)\geq 0 \end{cases} \tag{3.14}$$

here N is the analogue of the local Brunt-Väisälä frequency

which is determined by

$$N^2 = -\alpha\beta = \left(\frac{V_o^2}{r} + \frac{dV}{dr}\right)\{\frac{1}{\rho_o}\frac{d\rho_o}{dr} - \frac{1}{c^2}\left(\frac{V_o^2}{r} + \frac{dV}{dr}\right)\}$$
$$= \left(\frac{V_o^2}{r} + \frac{dV}{dr}\right)\{\frac{1}{\rho_o}\frac{d\rho_o}{dr} - \frac{1}{c^2\rho_o}\frac{dp_o}{dr}\} \tag{3.15}$$

The condition (3.14) agrees with the results obtained in [3].

4. A comparison with Warren's stability criterion.

Introducing the Brunt-Väisälä frequency given by (3.14), our necessary stability condition (3.12) can be written

$$N^2 + \frac{v_o}{r}\left(\frac{v_o}{r} + \frac{dv_o}{dr}\right) \geq \sqrt{\left(\frac{v_o}{r}\right)^2 \left\{\left(\frac{v_o}{r} + \frac{dv_o}{dr}\right)^2 + \left(\frac{dw_o}{dr}\right)^2\right\}} \tag{4.1}$$

If $N = 0$ on some set of positive measure, it is necessary that (3.13) is satisfied. At the points where $N \neq 0$, (4.1) has to be a strict inequality almost everywhere.

Now let us compare our results with the stability criterion of Warren [5]. His sufficient condition for stability of the flow (1.2, 1.3) is that

$$N^2 > \frac{1}{4}\{\left(\frac{dv_o}{dr} - \frac{v_o}{r}\right)^2 + \left(\frac{dw_o}{dr}\right)^2\}$$

or equivalently

$$N^2 + \frac{v_o}{r}\left(\frac{v_o}{r} + \frac{dv_o}{dr}\right) > \frac{1}{4}\left\{\left(\frac{v_o}{r} + \frac{dv_o}{dr}\right)^2 + \left(\frac{dw_o}{dr}\right)^2\right\} + \left(\frac{v_o}{r}\right)^2 \tag{4.2}$$

everywhere in the fluid.

Let us denote the right hand side of (4.1) and (4.2), A and B, respectively. Both A and B are non-negative and

$$B^2 - A^2 = \left[\frac{1}{4}\left\{\left(\frac{v_o}{r} + \frac{dv_o}{dr}\right)^2 + \left(\frac{dw_o}{dr}\right)^2\right\} - \left(\frac{v_o}{r}\right)^2\right]^2 \geq 0 \qquad (4.3)$$

This implies that (4.1) and (4.2) become exactly the same relation if and only if

$$\left(\frac{v_o}{r} + \frac{dv_o}{dr}\right)^2 + \left(\frac{dw_o}{dr}\right)^2 = 4\left(\frac{v_o}{r}\right)^2 \qquad (4.4)$$

For all the solutions (1.2, 1.3) not satisfying (4.4), it follows from (4.3) that (4.2) is a more restrictive condition than (4.1). This conclusion is of course not unexpected since Warren's criterion is sufficient for stability.

Finally let us consider velocity profiles satisfying (4.4), i.e. profiles where the criteria coincide

(i) If the shear of the axial velocity vanishes (i.e. $\frac{dw_o}{dr} = 0$ everywhere) the equation (4.4) has two classes of solutions

$$v_o = c_o \frac{1}{r^3} \text{ and } v_o = c_o' r \quad (c_o,\ c_o' \text{ real constants}).$$

(ii) If the axial velocity is of the form

$$w_o = c_1 r^n + c_2, \quad (c_1,\ c_2,\ n \text{ real constants}),$$

the equation (4.4) determines v_o and n to be

$$v_o = c_o r^n, \quad n = \begin{cases} -q + \sqrt{q^2 + 3} \\ -q - \sqrt{q^2 + 3} \end{cases} \quad \text{or}$$

where c_o is a real constant and $q = \dfrac{c_o^2}{c_o^2 + c_1^2}$

which denote the right hand side of (4.1) and (4.2), as A and B, respectively. Both A and B are nonnegative and

$$B^2 - A^2 = \frac{1}{4}\left[\left(\frac{v_\theta}{r} - \frac{dv_\theta}{dr}\right)^2 + \left(\frac{dw}{dr}\right)^2 - \left(\frac{v_\theta}{r}\right)^2\right] \geq 0 \qquad (4.3)$$

This implies that (4.1) and (4.2) become exactly the same condition if and only if

$$\left(\frac{v_\theta}{r}\right)^2 = \left(\frac{dw}{dr}\right)^2 + \left(\frac{v_\theta}{r} - \frac{dv_\theta}{dr}\right)^2 \qquad (4.4)$$

For all the solutions (1.2, 1.3) not satisfying (4.4), it follows from (4.3) that (4.2) is a more restrictive condition than (4.1). This conclusion is of course not unexpected since Warren's criterion is sufficient for stability.

Finally let us consider velocity profiles satisfying (4.4), i.e. profiles where the criteria coincide.

(i) If the shear of the axial velocity vanishes (i.e. $\frac{dw}{dr} = 0$ everywhere) the equation (4.4) has two classes of solutions

$v_\theta = c_0 \frac{1}{r}$ and $v_\theta = c_0 r$ (c_0, c_0 real constants).

(ii) If the axial velocity is of the form

$w_0 = c_1 r^n + c_2$ (c_1, c_2, n real constants),

the equation (4.4) determines v_θ and n reads

$$v_\theta = c_0 r^n, \quad n = \left.\begin{matrix}\sqrt{1+\beta^2} + \beta \\ \sqrt{1+\beta^2} - \beta\end{matrix}\right\} \text{ or}$$

where c_0 is a real constant and $\frac{c_1^2}{1 + c_0^2} = \beta$

(iii) Similarly, if $v_o = c_o r^n$ (c_o, n real constants) the equation (4.4) determines w_o and n to be

$$w_o = c_1 r^n + c_2, \quad n = \begin{cases} -q + \sqrt{q^2+3} \\ -q - \sqrt{q^2+3} \end{cases}$$

where c_1 and c_2 are real constants and $q = \frac{c_o^2}{c_o^2 + c_1}$

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REFERENCES

[1] K.S. Eckhoff:
Stability problems for linear hyperbolic systems.
Report No. 54, Dept. of Appl. Math., Univ. of Bergen (1975).

[2] K.S. Eckhoff:
On stability in ideal compressible hydrodynamics.
Report No. 55, Dept. of Appl. Math., Univ. of Bergen (1975).

[3] K.S. Eckhoff and L. Storesletten:
On the stability of shear flow in a rotating compressible and inviscid fluid.
Report No. 57, Dept. of Appl. Math., Univ. of Bergen (1975).

[4] L. Storesletten:
A note on the stability of horizontal shear flow of an inviscid compressible fluid.
Report No. 56, Dept. of Appl. Math., Univ. of Bergen (19

[5] F.W. Warren:
A comment on Gan's stability criterion for steady inviscid helical gas flows.
J. Fluid Mech. 68, pp 413 - 415 (1975).

References

[1] K.S. Eckhoff:
Stability problems for linear hyperbolic systems.
Report No. 54, Dept. of Appl. Math., Univ. of Bergen (1975).

[2] K.S. Eckhoff:
On stability in ideal compressible hydrodynamics.
Report No. 55, Dept. of Appl. Math., Univ. of Bergen (1975).

[3] K.S. Eckhoff and L. Storesletten:
On the stability of shear flow in a rotating compressible
and inviscid fluid.
Report No. 57, Dept. of Appl. Math., Univ. of Bergen (197

[4] L. Storesletten:
A note on the stability of horizontal shear flow of an
inviscid compressible fluid.
Report No. 56, Dept. of Appl. Math., Univ. of Bergen (19

[5] F.W. Warren:
A comment on Oan's stability criterion for steady inviscid
helical gas flows.
J. Fluid Mech. 68, pp 413 - 415 (1975).