A NOTE ON THE STABILITY OF TRAVELING-WAVE SOLUTIONS TO A CLASS OF REACTION-DIFFUSION SYSTEMS*

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Abstract. Many classes of reaction-diffusion systems have been shown to have traveling-wave solutions. For a class of such systems for which a comparison theorem can be used, we establish a wave stability result which roughly states that these wave solutions are asymptotically stable to perturbations which lie in some weighted L_p-space if their speeds are sufficiently large. We then apply this result to some excitable systems, namely a model of the Belousov-Zhabotinskii reaction, a substrate-inhibition biochemical model, and a class of models recently studied by Fife.

1. Introduction. We are concerned in this paper with showing the stability of traveling wave solutions to systems of the form

$$\mathbf{u}_t = D\mathbf{u}_{xx} + \mathbf{f}(\mathbf{u}) \qquad \mathbf{u} \in \mathbb{R}^n \tag{1.1}$$

where $|x| < \infty$, $t > 0$, D is a positive diagonal matrix and $\mathbf{f}$ is C^1. Traveling-wave solutions for (1.1) are solutions of the form $\mathbf{u}(x, t) = \boldsymbol{\phi}(x + ct) = \boldsymbol{\phi}(\xi)$, for some constant c. Wave solutions to such systems arise in nerve conduction [1, 11, 27], gene propagation [12, 60], flame propagation [10, 16, 17], chemical reactions [8, 13], shock waves [5, 14, 26], etc. When such solutions exist for reaction-diffusion systems, there may not be uniqueness and so the question of their stability arises.

Our notion of stability is the following. If $\mathbf{u}(x, t) = \boldsymbol{\phi}(x + ct) + \mathbf{U}(x, t)$, then $\boldsymbol{\phi}$ is stable if $\lim_{t\to\infty} |e^{-c\xi/2}U_j(\xi, t)| = 0$ for each component U_j of $\mathbf{U}$. We will require $\mathbf{U}(x, 0)$ to live in some "weighted" L_p-space ($p \geq 1$), a notion Sattinger [28, 29] used in his stability work. Unlike Sattinger, we will not need to compute the spectrum of the operator which is the linearized part of the right-hand side of (1.1). On the other hand, we will require the presence of a bounded invariant set for our initial data to live in, though we will not require them to satisfy any monotonicity conditions. We will also require a rather restrictive condition on the initial perturbation $\mathbf{U}(x, 0)$ in the neighborhood of $-\infty$, but no such condition at $+\infty$. This is because of the use of Sattinger's weighted norms. Our approach can be considered a generalization of Moet's work [25] where he deals with stability of traveling wave solutions to a scalar equation, namely the well-known Fisher equation.

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Although our tools, principally a comparison theorem, restrict the class of systems we can consider, there are a few interesting systems we can apply our result to. Namely, in Sec. 3 we consider a model proposed by Murray [22] for the well-known Belousov-Zhabotinskii reaction. This model satisfies the conditions necessary to apply the theory given in Sec. 2. This system models a reaction between bromate and cesium which is known to give various chemical waves. We next discuss a class of systems proposed by Fife [7] to study wave motion. It can be demonstrated that this class of systems can have traveling-wave solutions and their stability is established by our arguments. This class of systems involves properties which are possessed by some nerve conduction models and other chemical reaction models. The last example is biochemical in nature. It is a substrate inhibition model studied by Thomas [31] (see also [3, 18, 23, 24, 30]) to examine threshold mechanisms, pattern formation, etc. The particular system models a uricase enzyme catalyzed reaction between oxygen and uric acid.

2. Stability of wave solutions. We now consider (1.1), where $\mathbf{f}$ is C^1 and each of its components f_j, $j = 1, \ldots, n$, is nondecreasing in each argument u_i, $i = 1, \ldots, n$, for $i \neq j$. These conditions guarantee that we can apply the following comparison theorem, which is a restricted form proved by Fife [8] but adequate for our purposes.

THEOREM (Fife). Let $\mathbf{F}(x, t, \mathbf{u})$ be defined on $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^n$, be continuous in x and t, and satisfy the same conditions in $\mathbf{u}$ as mentioned above for $\mathbf{f}$. Let $\mathbf{u}$, $\mathbf{v}$ be continuous functions from $\mathbb{R} \times \mathbb{R}^+$ to $\mathbb{R}^n$, bounded and C^2 on their domain, and satisfying (component-wise)

$$\begin{aligned} \mathbf{u}_t - D\mathbf{u}_{xx} - \mathbf{F}(x, t, \mathbf{u}) \geq \mathbf{v}_t - D\mathbf{v}_{xx} - \mathbf{F}(x, t, \mathbf{v}) \quad &\text{in} \quad \mathbb{R} \times \mathbb{R}^+, \\ \mathbf{u} \geq \mathbf{v} \quad &\text{in} \quad \mathbb{R} \times \{t = 0\}. \end{aligned}$$

Then $\mathbf{u} \geq \mathbf{v}$ for all $(x, t) \in \mathbb{R} \times \mathbb{R}^+$.

We will use the standard notation throughout that $\mathbf{u} \geq \mathbf{v}$ means $u_i \geq v_i$, $i = 1, 2, \ldots, n$.

Besides assuming the conditions on $\mathbf{f}$, we also require that (1.1) has a bounded, invariant region $S \subset \mathbb{R}^n$ (see Chueh et al. [4] or Weinberger [33]). Thus, for all x, if $u(x, 0) \in S$, then $u(x, t) \in S$ for all $(x, t) \in \mathbb{R} \times \mathbb{R}^+$. This also guarantees the global existence of a solution for (1.1). We are interested in the stability of solutions to (1.1) of the form $\mathbf{u}(x, t) = \boldsymbol{\phi}_c(x + ct) = \boldsymbol{\phi}_c(\xi)$, where c is a positive constant. Let

$$\mathbf{u}(x, 0) = \boldsymbol{\phi}_c(x) + \mathbf{u}_0(x) \qquad x \in \mathbb{R}. \tag{2.1}$$

MAIN THEOREM. Any traveling wave solution $\boldsymbol{\phi}_c$ of (1.1) for c sufficiently large is asymptotically stable to perturbations $\mathbf{u}_0$ if $\mathbf{u}(x, 0)$, given by (2.1), is such that $\mathbf{u}(x, 0) \in S$ and $e^{-c\xi/2}\mathbf{u}_0(\xi) \in (L_p(R))^n$ for some $p \geq 1$.

The meaning of "c sufficiently large" will be clarified within the proof. It is known that sufficiently slow speed waves are unstable in many reaction-diffusion systems [15].

Proof. Let $\mathbf{u}(x, t) = \boldsymbol{\phi}_c(x + ct) + \mathbf{U}(x, t)$; then $\mathbf{U}$ satisfies

$$\begin{aligned} \mathbf{U}_t = D\mathbf{U}_{xx} + \mathbf{f}(\mathbf{U} + \boldsymbol{\phi}_c) - \mathbf{f}(\boldsymbol{\phi}_c) \quad &\text{in} \quad \mathbb{R} \times \mathbb{R}^+, \\ \mathbf{U}(x, 0) = \mathbf{u}(x) \quad &\text{in} \quad \mathbb{R}. \end{aligned} \tag{2.2}$$

Consider a function $\mathbf{P}(x, t)$ which satisfies

$$\mathbf{P}_t = D\mathbf{P}_{xx} + \mathbf{f}(\mathbf{P} + \boldsymbol{\phi}_c) - \mathbf{f}(\boldsymbol{\phi}_c) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+,$$
$$\mathbf{P}(x, 0) = \max\{\mathbf{0}, \mathbf{u}_0(x)\} \quad \text{in} \quad \mathbb{R}.$$

That is, the initial condition is $P_i(x, 0) = \max\{0, u_{0_i}(x)\}$, $i = 1, \ldots, n$. Define $\mathbf{F}(\mathbf{P}; x, t) \equiv \mathbf{f}(\mathbf{P} + \boldsymbol{\phi}_c) - \mathbf{f}(\boldsymbol{\phi}_c)$. Then by previous assumption, $\partial F_i/\partial P_j = \partial f_i/\partial P_j \geq 0$ for $i \neq j$. By the comparison theorem

$$\mathbf{P} \geq \mathbf{0}, \qquad \mathbf{P} \geq \mathbf{U} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+. \tag{2.3}$$

Similarly, consider a function $\mathbf{Q}(x, t)$ which satisfies

$$\mathbf{Q}_t = D\mathbf{Q}_{xx} + \mathbf{F}(\mathbf{Q}; x, t) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+, \qquad \mathbf{Q}(x, 0) = \min\{\mathbf{0}, \mathbf{u}_0(x)\} \quad \text{in} \quad \mathbb{R}.$$

Then by the comparison theorem

$$\mathbf{Q} \leq \mathbf{0}, \qquad \mathbf{Q} \leq \mathbf{U} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+. \tag{2.4}$$

Because of the smoothness of $\mathbf{f}$, and hence $\mathbf{F}$, the existence of S and the fact that $\mathbf{F}(\mathbf{0}; x, t) = \mathbf{0}$, then there is a constant matrix M such that for $\mathbf{P}$ giving (2.3) and $\mathbf{Q}$ giving (2.4), we have

$$\mathbf{F}(\mathbf{P}) \leq M\mathbf{P}, \qquad \mathbf{F}(\mathbf{Q}) \geq M\mathbf{Q}. \tag{2.5}$$

This follows in effect from just expanding $\mathbf{F}$ in a Taylor series and choosing the elements of M appropriately to bound the coefficients, since $\mathbf{P} \geq 0 \geq \mathbf{Q}$. Consider

$$\mathbf{R}_t = D\mathbf{R}_{xx} + M\mathbf{R} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+,$$
$$\mathbf{R}(x, 0) = |\mathbf{u}_0(x)| \quad \text{in} \quad \mathbb{R}. \tag{2.6}$$

(That is, $R_i(x, 0) = |u_{0i}(x)|$, $i = 1, 2, \ldots, n$.) Since $\mathbf{R}_* = -\mathbf{R}$ satisfies the same differential equation (2.6) and $\mathbf{R}_*(x, 0) = -|\mathbf{u}_0(x)|$, we have from (2.3)–(2.5) that

$$|\mathbf{U}| \leq \mathbf{R} \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+.$$

Transform to the traveling coordinate frame $\mathbf{R}(x, t) = \mathbf{r}(\xi, t)$; then $\mathbf{r}$ satisfies

$$\mathbf{r}_t = D\mathbf{r}_{\xi\xi} - c\mathbf{r}_\xi + M\mathbf{r}$$
$$\mathbf{r}(\xi, 0) = |\mathbf{u}_0(\xi)|.$$

Define $\mathbf{z}(\xi, t) \equiv e^{-c\xi/2}\mathbf{r}(\xi, t)$; then $\mathbf{z}$ satisfies

$$\mathbf{z}_t = \mathbf{z}_{\xi\xi} - M_*\mathbf{z}$$
$$\mathbf{z}(\xi, 0) = e^{-c\xi/2}|\mathbf{u}_0(\xi)| \tag{2.7}$$

where

$$M_* \equiv \frac{c^2}{4} I - M, \qquad I \text{ is the } n \times n \text{ identity matrix.}$$

By the parametrix method (see Friedman [9]), there exists a fundamental solution matrix $\Gamma(\xi - \eta, t)$ such that

$$\lim_{t \downarrow \infty} \int_{\mathbb{R}} \Gamma(\xi - \eta, t)\mathbf{z}(\eta, 0)\, d\eta = \mathbf{z}(\xi, 0),$$

and

$$\mathbf{z}(\xi, t) = \int_{\mathbb{R}} \Gamma(\xi - \eta, t)\mathbf{z}(\eta, 0)\, d\eta$$

satisfies (2.7). If M_* has nonnegative eigenvalues, then for some positive constant d and some matrix K of positive constants, $\|\Gamma(\xi - \eta, t)\| \le Kt^{-1/2} \exp\{-d(\xi - \eta)^2/t\}$. Thus, for $\mathbf{z}(\cdot, 0) \in L_p(\mathbb{R})$, for some $p \ge 1$, we have

$$|\mathbf{z}(\xi, t)| \le Ct^{-1/2(1-1/q)}\|\mathbf{z}(\cdot, 0)\|_p$$

by applying Holder's inequality, where $1/q + 1/p = 1$. Therefore,

$$|e^{-c\xi/2}\mathbf{U}(\xi, t)| \le Ct^{-1/2(1-1/q)}\left\{\int_{\mathbb{R}} e^{-pc\xi/2}|u_0(\xi)|^p\, d\xi\right\}^{1/p}$$

and the theorem follows.

Since $M = (m_{ij})$ is constructed to be a nonnegative matrix, then if $c^2/4 \ge m_{ii}$, $i = 1, \ldots, n$, M_* is an M-matrix [2]. If M_* is nonsingular, then its eigenvalues lie in the right half plane. This is the situation in the examples below.

3. Examples. The considerations of the last section were motivated by examples of systems which are known to display traveling wave solutions. We present some examples in this section which are covered under the previously stated restrictions.

The first example is a system representing a scaled version of a model proposed by Field and Noyes [6] to study the propagation of chemical wave fronts in the well-known Belousov-Zhabotinskii chemical reaction. The scaling is due to Murray ([22]; see also Troy [32]). The Belousov reaction is a chemical oscillator consisting of cesium ion catalyzed oxidation by bromate ion. After scaling the equations the scaled versions of bromous acid and bromide ion concentrations, u and v respectively, are

$$u_t = u_{xx} + u(1 - u - rv), \qquad v_t = v_{xx} - buv. \tag{3.1}$$

Murray [22] studied (3.1) with the following boundary conditions

$$u(-\infty, t) = v(+\infty, t) = 0, \qquad t \in \mathbb{R}^+$$
$$u(+\infty, t) = v(-\infty, t) = 1.$$

He showed that if traveling-wave solutions exist for (3.1), then their speed c must satisfy

$$C(b, r) \le c \le 2,$$

where $C(b, r) \equiv [(r^2 + 2b/3)^{1/2} - r](2[2r + b])^{-1/2}$. Troy [32] proved that if $b - 1 > 0$ is sufficiently small, then there exist values c^*, $r^* > 0$ such that (3.1) has a traveling-wave solution with speed $c = c^*$ when $r = r^*$. Then Klaasen and Troy [19] determined the global stability of $(u, v) \equiv (1, 0)$ and investigated the asymptotic behavior of $(u(x + ct, t), v(x + ct, t))$ as $t \to +\infty$ for both large and small values of $c > 0$. Also, in their paper, they established that

$$S \equiv \{(u, v)\,|\,0 \le u \le 1, 0 \le v \le \gamma\} \qquad \gamma \in (0, 1/r)$$

is an invariant region for (3.1). Before we can take advantage of the result in Sec. 2, we must make a transformation. Let $w = \gamma - v$; then substituting in (3.1) we obtain

$$
\begin{aligned}
u_t &= u_{xx} + u(1 - u + rw - r\gamma) = u_{xx} + f_1(u, w), \\
w_t &= w_{xx} + bu(\gamma - w) = w_{xx} + f_2(u, w).
\end{aligned} \tag{3.2}
$$

Certainly since (3.1) has a traveling-wave solution, (3.2) does also, and S is an invariant region for (3.2). Moreover, $\partial f_1/\partial w = ru \geq 0$ and $\partial f_2/\partial u = b(\gamma - w) \geq 0$, so that (3.2) satisfies the comparison theorem hypotheses. Let (ϕ_c, ψ_c) be any traveling wave solution for (3.2) and write $u(x, 0) = \phi_c(x) + u_0(x)$, $w(x, 0) = \psi_c(x) + w_0(x)$, $x \in \mathbb{R}$. Assume

$$(u(x, 0), w(x, 0)) \in S. \tag{3.3}$$

Then we have

COROLLARY. The traveling-wave solution (ϕ_c, ψ_c) of (3.2) with $c = 2$ is asymptotically stable to perturbations (u_0, w_0) which satisfy (3.3), $w + \psi_c \geq \gamma$, and $e^{-c\xi/2}(u_0(\xi), w_0(\xi)) \in L_p \times L_p$ for some $p \geq 1$.

This result follows from the main theorem. The condition $w + \psi_c \leq \gamma$ comes from the observation that, in the notation of Sec. 2, $\partial F_2/\partial u_1 = b(\gamma - u_2 - \psi_c)$, which we want to be nonnegative. From the proof of the main theorem, one can show in this case that M can be chosen to be

$$M = \begin{pmatrix} 1 & r \\ b\gamma & 0 \end{pmatrix}.$$

From the proof of the main theorem, the condition on M_* to guarantee the stability requires $c^2/4 \geq 1$.

A second example is a class of systems proposed by Fife [7, 8] as a general model for studying wave phenomena, threshold behavior, etc. The model can be considered a generalized FitzHugh-Nagumo system because of the qualitative structure imposed on the nonlinearities. The FitzHugh-Nagumo system is a specific prototype model for studying conduction phenomena in nerves (see Rinzel [27]). Fife's model has the form

$$u_t = \varepsilon u_{xx} + f(u, v), \qquad v_t = v_{xx} + g(u, v) \tag{3.4}$$

where f, g have the structure given in Fig. 1. We can assume that f and g are C^1 in their arguments and from the figure we can also assume that f and g satisfy

$$f_v > 0, \qquad g_u > 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^+. \tag{3.5}$$

It follows directly from arguments in Weinberger [33] that the system (3.4) possesses an invariant region S. This is graphically depicted in Fig. 1 and we write for some A, $B > 0$,

$$S = \{(u, v) \mid 0 \leq u \leq A, 0 \leq v \leq B\}.$$

Fife [7] argued from an asymptotic approach that one can show that (3.4) can possess traveling-wave solutions. One can make his arguments more rigorous by transforming the system (3.4) via Fife's "inner" wave frame variable and using a geometric approach

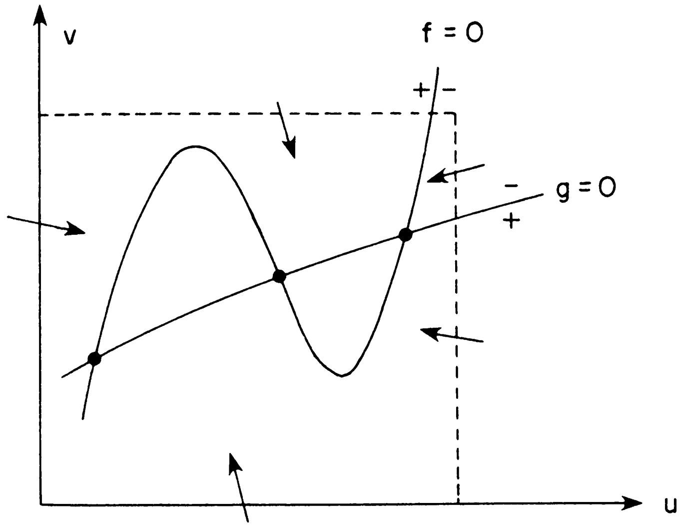

FIG. 1.

of Conley (Bell, unpublished; see Kurland [21]). These approaches all require that $\varepsilon > 0$ be sufficiently small. Thus the system satisfies the constraints of Sec. 2 and we have

COROLLARY. Any traveling-wave solution (ϕ_c, ψ_c) of (3.4) with $c^2/4 \geq \max_S\{f_u, g_v\}$ is asymptotically stable to perturbations $(u_0(x), v_0(x))$ which satisfy $(u(x, 0), v(x, 0)) = (\phi_c(x) + u_0(x), \psi_c(x) + v_0(x)) \in S$ and such that $e^{-c\xi/2}(u_0(\xi), v_0(\xi)) \in (L_p(\mathbb{R}))^2$ for some $p \geq 1$.

A third example is a substrate inhibition model studied by Thomas [31] and Kernevez et al. [18]. It consists of uricase enzyme immobilized on a membrane in which uric acid and oxygen are catalyzed by the enzyme, thus diffusing and reacting to produce allantöin and other products. If w and u are scaled concentrations of uric acid and oxygen, then a dimensionless form of the model may be written as

$$u_t = u_{xx} + \{u_0 - u - \rho w\phi(u)\}$$
$$w_t = \beta w_{xx} + \{\gamma(w_0 - w) - \rho w\phi(u)\}$$
$$\phi(u) = u/(1 + u + ku^2). \tag{3.6}$$

Inhibition models of this type are fairly common (see Seelig [30]) and have been considered as a possible mechanism for pigment distribution in plants [24] and coat color markings in animals [23]. There is strong evidence that such models have traveling-wave solutions [3], but a rigorous proof of this has not been established. In a forthcoming paper we will establish the existence of traveling-wave solutions. The phase plane for (3.6) is shown in Fig. 2 and, by arguments similar to those of the first two examples, it can be shown that (3.6) also has a bounded invariant set S. Thus if we make the transformation $w = V - v$, (3.6) becomes

$$u_t = u_{xx} + \{u_0 - u - \rho(V - v)\phi(u)\} = u_{xx} + f_1(u, v),$$
$$v_t = \beta v_{xx} - \{\gamma(w_0 - V + v) - \rho(V - v)\phi(u)\} = \beta v_{xx} + f_2(u, v). \tag{3.7}$$

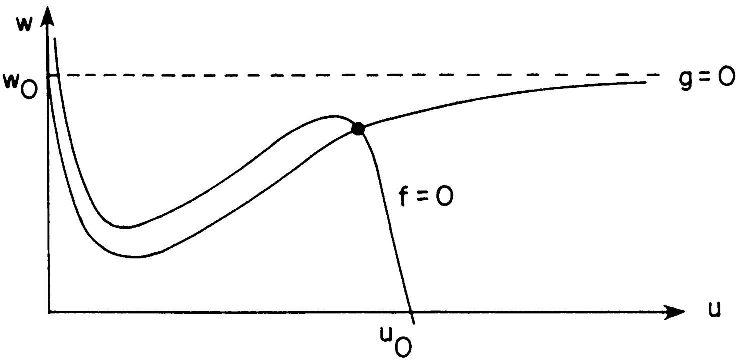

FIG. 2.

Then

$$\partial f_1/\partial v = \rho\phi(u) \geq 0,$$
$$\partial f_2/\partial u = \rho(V - v)(d\phi/du) \geq 0 \quad \text{if} \quad v \leq V \quad \text{and} \quad k \leq u^{-2}.$$

We can choose V sufficiently large because of the existence of a bounded invariant rectangle S; that is, for $(u, v) \in S$, there are constants V_1, U_1 such that $u \leq U_1$ and $v \leq V_1$, so let $V \geq V_1$. Let $R > 0$ be defined by $\min_{0<u}\{d\phi/du\} = -R$. Then M for this example has diagonal elements $\alpha \equiv \rho RV - 1 > 0$ and 0. Thus we have

COROLLARY. Any traveling-wave solution (ϕ_c, ψ_c) of (3.7) with $c^2 \geq 4\alpha$ is asymptotically stable to perturbations $(u_0(x), v_0(x))$ which satisfy $(u(x, 0), v(x, 0)) = (\phi_c(x) + u_0(x), \psi_c(x) + v_0(x)) \in S$ and such that $e^{-c\xi/2}(u_0(\xi), v_0(\xi)) \in (L_p(\mathbb{R}))^2$ for some $p \geq 1$, if $k \leq U_1^{-2}$.

REFERENCES

[1] J. Bell, *Modelling parallel, unmyelinated axons: pulse trapping and ephaptic transmission*, SIAM J. Appl. Math., to appear

[2] A. Berman and R. Plemmons, *Nonnegative matrices in the mathematical sciences*, Academic Press, 1979

[3] N. F. Britton, and J. D. Murray, *Threshold, wave and cell-cell avalanche behavior in a class of substrate inhibition oscillators*, J. Theor. Biol. **77**, 317–332 (1979)

[4] K. N. Chueh, C. C. Conley and J. A. Smoller, *Positively invariant regions for systems of nonlinear diffusion equations*, Ind. U. Math. J. **26**, 373–392 (1977)

[5] C. Conley and J. Smoller, *On the structure of magnetohydrodynamic shock waves*, MRC Report No. 1336, 1973

[6] R. J. Field and R. M. Noyes, *Limit cycle oscillations in a model of a real chemical reaction*, J. Chem. Phys. **60**, 1877–1884 (1974)

[7] P. C. Fife, *Singular perturbation and wave front techniques in reaction-diffusion problems*, In *Asymptotic methods and singular perturbations*, SIAM-AMS Publ. **10**, 23–50 (1976)

[8] P. C. Fife, *Mathematical aspects of reacting and diffusing systems*, Lecture Notes in Biomathematics, **28**, Springer-Verlag, 1979

[9] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, 1964

[10] I. M. Gelfand, *Some problems in the theory of quasilinear equations*, Amer. Math. Soc. Transl. **29**, 295–381 (1963)

[11] A. L. Hodgkin and A. F. Huxley, *A quantitative description of membrane current and its application to conduction and excitation in nerve*, J. Physiol. **117**, 500–544 (19)

[12] F. Hoppensteadt, *Mathematical theories of populations: demographic genetics and epidemics*, Regional Conf. Ser. in Appl. Math., SIAM, Vol. 20, 1975

[13] L. N. Howard and N. Kopell, *Wave trains, shock fronts, and transition layers in reaction-diffusion equations*, pp. 1–12, in *Mathematical aspects of chemical and biochemical problems and quantum chemistry*, Proc. SIAM-AMS Symp. 8, 1974

[14] A. Jeffrey, and T. Kakutani *Stability of the Burgers shock wave and the Korteweg-deVries solution*, Ind. Univ. Math. J. **20**, 463–469 (1970)

[15] D. A. Larson, *On the existence and stability of bifurcated solitary wave solutions to nonlinear diffusion equations*, J. Math. Anal. Appl. preprint, to appear.

[16] Y. I. Kanel', *The behavior of solutions of the Cauchy problem with the time tends to infinity, in the case of quasilinear equations arising in the theory of combustion*, DAN SSSR, **132**, 268–271 (1960)

[17] Y. I. Kanel', *Some problems involving burning-theory equations*, DAN SSSR **136**, 277–280 (1961)

[18] J.-P. Kernevez, G. Joly, M. C. Dubau, B. Bunow and D. Thomas, *Hysteresis oscillations, and pattern formation in realistic immobilized enzyme systems*, J. Math. Biol. **7**, 41–56 (1979)

[19] G. A. Klaasen, and W. C. Troy, *The asymptotic behavior of solutions of a system of reaction-diffusion equations which models the Belovsov-Zhabotinskii chemical reaction*, Preprint, 1979

[20] A. Kolmogoroff, I. Petrovsky, and N. Piscounoff, *Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Bull. Univ. Moshu, Ser. Internat., Sec. A, **1**, 1–25 (1937)

[21] H. Kurland, Dissertation, Univ. of Wisconsin, Madison, 1978

[22] J. D. Murray, *Lectures on nonlinear-differential equation models in biology*, Clarendon Press, 1977

[23] J. D. Murray, *Animal coat markings: a general pattern formation mechanism*, preprint

[24] J. D. Murray, N. F. Britton and G. Joly, *A model for hydranth regeneration in Tubularia*, preprint, 1979

[25] H. J. K. Moet, *A note on the asymptotic behavior of solutions of the KPP equation*, SIAM J. Math. Anal. **10**, 728–732 (1979)

[26] L. A. Peletier, *Asymptotic stability of traveling waves*, in *Instability of continuous systems*, IUTAM Symposium, 1969

[27] J. Rinzel, *Integration and propagation of neuroelectric signals*, in *MAA studies in mathematical biology* (ed. Simon Levine), 1978

[28] D. H. Sattinger, *On the stability of waves of nonlinear parabolic systems*, Adv. in Math. **22**, 312–355 (1976)

[29] D. H. Sattinger, *Weighted norms for the stability of traveling waves*. J. Diff. Eqs. **25**, 130–144 (1977)

[30] F. F. Seelig, *Chemical oscillations by substrate inhibition: a parametrically universal oscillator type in homogeneous catalysis by metal complex formation*, Z. Naturforsch. **31a**, 1168–1172 (1976)

[31] D. Thomas, *Artificial enzyme membranes, transport, memory, and oscillatory phenomena*, in *Proc. int. symp. on analysis and control of immobilized enzyme systems* (ed. D. Thomas and J.-P. Kernevez), 1976

[32] W. C. Troy, *The existence of traveling wave front solutions of a model of the Belousov-Zhabotinskii chemical reaction*, J. Diff. Eqs. (to appear)

[33] H. F. Weinberger, *Invariant sets for weakly coupled parabolic and elliptic systems*, Rend. Mat. **8**, 295–310 (1975)