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A note on the star order in Hilbert spaces

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We study the star order on the algebra $L(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. We present a new interpretation of this order which allows to generalize to this setting many known results for matrices: functional calculus, semi-lattice properties, shorted operators and orthogonal decompositions. We also show several properties for general Hilbert spaces regarding the star order and its relationship with the functional calculus and the polar decomposition, which were unknown even in the finite-dimensional setting. We also study the existence of strong limits of star-monotone sequences and nets.

Keywords: star order; projections; functional calculus; polar decomposition; semi-lattice structure; shorted operators

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1. Introduction

Given two $n \times n$ complex matrices A and B, Hestenes introduced in [9] the concept of $\star$-orthogonality, defined by the equations $A^*B = 0$ and $AB^* = 0$, where A^* (resp. B^*) denotes the transpose and component-wise conjugate of A (resp. B). In the same paper he defined and discussed the relation defined by $A \sim B$ if

$$A^*A = B^*A \quad \text{and} \quad AA^* = AB^*.$$

Later on, Drazin proved in [5] that this relation is in fact a partial ordering on the set of square matrices, and even more generally in semigroups with involution. This order, that we shall denote $\overset{\star}{\leq}$, is nowadays called the star order (or $\star$-order). Although this order can be generalized to a much more general setting, it has been studied specially in the space of complex matrices.

In this article, we study the $\star$-order on the algebra $L(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. Since many of the usual techniques used in finite-dimensional spaces (as pseudoinverses or singular value decompositions) are no longer available

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for general Hilbert spaces, we introduce new techniques which allow us to show that almost all the known properties which hold for matrices can be generalized to operators acting on a Hilbert space $\mathcal{H}$, and to obtain simpler proofs. Indeed, several results of the articles [1–3,6–8,10] concerning the $\star$-order are contained in this note if we consider finite-dimensional spaces. On the other hand, we show several properties for general Hilbert spaces which were unknown even in the finite-dimensional setting, particularly those results concerning the polar decomposition and those regarding the relationship between the $\star$-order and the functional calculus. We also study some questions that only have sense in the infinite-dimensional setting, such as the existence of strong limits for $\star$-monotone sequences and nets.

This article is organized as follows. Section 2 starts with the relationship between the $\star$-order and certain sets of projections. Roughly speaking, given two operators A, $B \in L(\mathcal{H})$, the relation $A \overset{\star}{\leq} B$ says that A is 'a piece' of B. Concretely, it can be proved that $A \overset{\star}{\leq} B$ if and only if $A = P_{\overline{R(A)}} B = BP_{\overline{R(A^*)}}$, where $P_{\mathcal{S}}$ denotes the orthogonal projection onto the closed subspace $\mathcal{S}$. Since the relation $A \overset{\star}{\leq} B$ means that both equalities must hold, it is not true that for every projection P the inequality $PB \overset{\star}{\leq} B$ holds, even if $R(P) \subseteq \overline{R(B)}$. The main aim in the first part of Section 2 is to give a characterization of those projections P, such that there is an operator $A \in L(\mathcal{H})$ so that $A \overset{\star}{\leq} B$ and $P = P_{\overline{R(A)}}$ (and by symmetry also the set of those Q are such that $Q = P_{\overline{R(A^*)}}$). We prove that the mentioned set consists of those projections P that satisfy

$$R(P) \subseteq \overline{R(B)} \quad \text{and} \quad P \cdot BB^* = BB^* \cdot P. \tag{1}$$

Moreover, we show that if P_1 and P_2 satisfy these properties, then $P_1 B \overset{\star}{\leq} P_2 B \iff P_1 \leq P_2$ (Theorem 2.7). These facts can be viewed as a reformulation of the definition of the $\star$-order, and their proofs are quite simple. However, these criteria can be applied to obtain very short proofs of several results throughout this article, and they are particularly useful to work in the context of general Hilbert spaces.

Then, we study what functions preserve the $\star$-order when they are applied to operators using some convenient functional calculi. These results are motivated by the work of Baksalary et al. [1], where these authors prove that the polynomial functions of the form $p(x) = x^{2^k}$ (with $k \in \mathbb{N}$) preserve the $\star$-order, provided that some technical requirement on the ranges of the matrices involved holds. Under a slightly less restrictive additional hypothesis, we prove that any function that can be applied to the operators considered preserves the $\star$-order. We conclude Section 2 by studying the relationship between the $\star$-order and the polar decomposition. One of the main tools used in the finite-dimensional setting is the singular value decomposition, but it is not available for general operators on a Hilbert space. We prove that the polar decomposition behaves very well with respect to the $\star$-order and it may be a natural substitute.

Section 3 is devoted to the study of lattice properties of $L(\mathcal{H})$ endowed with the relation $\overset{\star}{\leq}$. These properties have been studied by Hartwig and Drazin [7] in the finite-dimensional setting. A similar approach can be pursued using the results proved in Section 2 regarding the $\star$-order and the polar decomposition. Nevertheless, we prefer to follow a different way that leads directly to the results. Given $B \in L(\mathcal{H})$, we prove that there exists an order preserving (in both directions) bijection between the sets

(i) $\mathcal{L}_B := \{A \in L(\mathcal{H}) : A \overset{\star}{\leq} B\}$ with the $\star$-order;
(ii) $\mathcal{P}_B := \{P = P^2 = P^* \in L(\mathcal{H}) : R(P) \subseteq \overline{R(B)}$ and $PBB^* = BB^*P\}$ with the usual order (or with the $\star$-order because both orders coincide on this set).

This immediately implies the existence of a minimum between two operators $A, B \in L(\mathcal{H})$, which is denoted by $A \wedge^\star B$. Several properties of this minimum are analysed. Section 3 concludes with the study of some limit theorems. In the finite-dimensional setting, it is not difficult to see that any star monotone sequence is constant from some n_0 onwards. However, in the infinite-dimensional setting, this is no longer true, and a natural question is whether or not a bounded star monotone sequence converges. In the last subsection of Section 3, we answer this question positively and we also study the behaviour of the minimum with respect to monotone sequences. We prove that it behaves well with respect to monotone decreasing sequences, but we exhibit a counterexample which shows that the results proved for monotone decreasing sequences are not valid for monotone increasing ones. In some ways, this is one of the obstructions to pursue the generalization to the infinite-dimensional setting based on the finite-dimensional case and (star) monotone increasing sequences consisting of operators with finite-dimensional ranges (see also Remark 3.7).

In Section 4, we study the so-called star-shorted operator. This notion was introduced for matrices by Mitra in [10]. Recall that given a matrix A and two subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathbb{C}^n$, the star-shorted operator, denoted by $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$, is defined as the star maximum of the set of matrices

$$\overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T}) = \left\{D \overset{\star}{\leq} A,\ \ R(D) \subseteq \mathcal{T} \text{ and } R(D^*) \subseteq \mathcal{S}\right\}.$$

We prove that this maximum also exists in the infinite-dimensional setting, where the subspaces are asked to be closed, and we characterize this maximum as the minimum $A \wedge^\star (P_\mathcal{T} A P_\mathcal{S})$. We also prove that the modulus of $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$ can be characterized as a star-shorted operator of $|A|$ with respect to some suitable subspaces.

Finally, Section 5 is devoted to study the relationship between the $\star$-order and the notion of $\star$-orthogonality. We show that if an operator A admits a suitable $\star$-orthogonal decomposition, then the minima and the shorted operators of A can also be decomposed in terms of the elements of that decomposition.

2. The star order

2.1. *Notations*

Given a Hilbert space $\mathcal{H}$, $L(\mathcal{H})$ denotes the algebra of bounded linear operators on $\mathcal{H}$, $L_{sa}(\mathcal{H})$ the real vector space of self-adjoint operators and $L(\mathcal{H})^+$ the cone of positive operators. For an operator $A \in L(\mathcal{H})$, $R(A)$ denotes the range or image of A, ker A the nullspace of A, $\sigma(A)$ the spectrum of A, A^* the adjoint of A, $|A| = (A^*A)^{1/2}$ the modulus of A and $\|A\|$ the usual norm of A.

The word *projection* is used exclusively for orthogonal projections. We denote by $\mathcal{P}(\mathcal{H}) = \{P \in L(\mathcal{H}) : P = P^* = P^2\}$, the set of all projections in $L(\mathcal{H})$. Throughout this article, $\mathcal{S} \sqsubseteq \mathcal{H}$ means that $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ and $P_{\mathcal{S}} \in \mathcal{P}(\mathcal{H})$ denotes the unique projection onto $\mathcal{S}$. For every $A \in L(\mathcal{H})$, we denote by

$$P_A = P_{\overline{R(A)}} \in \mathcal{P}(\mathcal{H}) \quad \text{and} \quad Q_A = P_{A^*} = P_{\overline{R(A^*)}} = I - P_{\ker A} \in \mathcal{P}(\mathcal{H}) \ .$$

Definition 2.1 Given $A, B \in L(\mathcal{H})$, we say that A is lower or equal than B with respect to the $\star$-order, which is denote by $A \overset{\star}{\leq} B$, if

(1) $A^*A = A^*B = B^*A$;
(2) $AA^* = BA^* = AB^*$.

Remark 2.2 Given $A, B \in L(\mathcal{H})$, it is easy to see that $A \overset{\star}{\leq} B \iff A^* \overset{\star}{\leq} B^*$. Also

$$A \overset{\star}{\leq} B \Longrightarrow AA^* \overset{\star}{\leq} BB^*, \quad A^*A \overset{\star}{\leq} B^*B \quad \text{and} \quad B - A \overset{\star}{\leq} B.$$

2.2. *Star order and projections*

In this section, we describe the relationship between the $\star$-order and some subsets of the Grassmann manifold, viewed as the set of projections. We begin with the following two well-known characterizations of the equalities that define the $\star$-order. Since these characterizations will be very important in the sequel, and for the sake of completeness, we include a short proof valid in our general setting.

PROPOSITION 2.3 *Let* $A, B \in L(\mathcal{H})$. *Then*

$$BA^* = AA^* \iff A = BQ_A \iff A = BQ \text{ for some } Q \in \mathcal{P}(\mathcal{H}). \tag{2}$$

Similarly, it holds that

$$B^*A = A^*A \iff A = P_A B \iff A = PB \text{ for some } P \in \mathcal{P}(\mathcal{H}). \tag{3}$$

Proof Indeed, given $x \in \overline{R(A^*)}$, there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$ contained in $R(A^*)$ such that $u_n \xrightarrow[n\to\infty]{\|\cdot\|} x$. If we assume that $BA^* = AA^*$, then $Bu = Au$ for every $u \in R(A^*)$. Therefore

$$Bx = \lim_{n\to\infty} Bu_n = \lim_{n\to\infty} Au_n = Ax.$$

On the other hand, both $Ax = BP_{A^*}x = 0$ for every $x \in R(A^*)^{\perp} = \ker A$.

Suppose now that $A = BQ$ for some $Q \in \mathcal{P}(\mathcal{H})$. Then

$$BA^* = B(BQ)^* = BQB^* = (BQ)(QB^*) = AA^*.$$

The proof of (3) is almost the same. It can also be obtained from (2). ∎

COROLLARY 2.4 *Let* $A, B \in L(\mathcal{H})$ *such that* $A \overset{\star}{\leq} B$. *Then* $R(A) \subseteq R(B)$ *and* $R(A^*) \subseteq R(B^*)$.

COROLLARY 2.5 *Let* $A, B \in L(\mathcal{H})$ *such that* $A \overset{\star}{\leq} B$ *and* $B^2 = B$. *Then* $A^2 = A$.

Proof Indeed, $A^2 = P_A BBQ_A = P_A BQ_A = P_A A = A$. ∎

The statement of Proposition 2.3 implies the following alternative description of the $\star$-order: Given A, $B \in L(\mathcal{H})$, then

$$A \overset{\star}{\leq} B \iff A = P_A B = BQ_A. \tag{4}$$

This description suggests that the relation $A \overset{\star}{\leq} B$ means that A is 'a piece' of B. Nevertheless, Equation (4) does not say exactly which pieces of B (of the type $A = PB$ for $P \in \mathcal{P}(\mathcal{H})$) are $\star$-smaller than B. In other words, the above results do not characterize which projections P and Q satisfy that $P = P_A$ or $Q = Q_A$ for some $A \overset{\star}{\leq} B$. This characterization is the main goal of this subsection and the first step in that direction is the next lemma which gives a one-sided description (in terms of the action of projections) of the relation $A \overset{\star}{\leq} B$.

Lemma 2.6 *Let $B \in L(\mathcal{H})$.*

(1) *If $P \in \mathcal{P}(\mathcal{H})$ and $R(P) \subseteq \overline{R(B)}$, then $PB \overset{\star}{\leq} B \iff PBB^* = BB^*P$.*
(2) *If $Q \in \mathcal{P}(\mathcal{H})$ and $R(Q) \subseteq \overline{R(B^*)}$, then $BQ \overset{\star}{\leq} B \iff QB^*B = B^*BQ$.*

Proof Denote by $A = PB$. If $A \overset{\star}{\leq} B$, then $PBB^* = AB^* = BA^* = BB^*P$. Conversely, the identity $A = PB$ implies that $B^*A = A^*A$, by Proposition 2.3. On the other hand, since P commutes with BB^*, it holds that

$$BA^* = BB^*P = PBB^*P = AA^*.$$

The proof of the second statement follows *mutatis mutandis*. ■

Theorem 2.7 *Let $A, B, C \in L(\mathcal{H})$. Then it holds that*

$$\begin{aligned} A \overset{\star}{\leq} B &\iff A = P_A B, \quad P_A \leq P_B \quad \text{and} \quad P_A \cdot BB^* = BB^* \cdot P_A \\ &\iff A = BQ_A, \quad Q_A \leq Q_B \quad \text{and} \quad Q_A \cdot B^*B = B^*B \cdot Q_A. \end{aligned} \tag{5}$$

Moreover, if both $A \overset{\star}{\leq} B$ and $C \overset{\star}{\leq} B$, then

$$A \overset{\star}{\leq} C \Leftrightarrow P_A \leq P_C \Leftrightarrow Q_A \leq Q_C \Leftrightarrow R(A) \subseteq R(C) \Leftrightarrow \ker C \subseteq \ker A. \tag{6}$$

Proof Both implications $\Rightarrow$ of (5) follow from Proposition 2.3, Corollary 2.4 and Lemma 2.6. The reverse implications follow from Lemma 2.6.

Assume that $A \overset{\star}{\leq} B$ and $C \overset{\star}{\leq} B$. Then we have that $A = P_A B$ and $C = P_C B$. If $A \overset{\star}{\leq} C$, then Corollary 2.4 shows that all the other conditions of Equation (6) hold. Conversely, suppose that $P_A \leq P_C$. Then $P_A C = P_A P_C B = P_A B = A$, and

$$P_A CC^* = P_A P_C BB^* = P_A BB^* = BB^* P_A = BB^* P_C P_A = CC^* P_A.$$

Therefore, by Lemma 2.6, we can conclude that $A = P_A C \overset{\star}{\leq} C$. The proof of the other case $Q_A \leq Q_C \Longrightarrow A \overset{\star}{\leq} C$ is almost the same. ■

The main advantage of Theorem 2.7 over Proposition 2.3 or Equation (4) is that each equivalence involves only one projection. The price we had to pay for this simplification does not seem to be very high, at least in the applications of this result that we shall consider, because the commutation relation is not too complicated to check in those applications. On the other side, the set of projections of a 'commutant' subalgebra of $L(\mathcal{H})$ has excellent properties. The following results are direct consequences of Theorem 2.7.

Corollary 2.8 *Let $V \in L(\mathcal{H})$ be a partial isometry. Denote by $P = P_V$. Then*

$$W \overset{\star}{\leq} V \iff W = QV \text{ for some } Q \in \mathcal{P}(\mathcal{H}) \text{ such that } Q \leq P.$$

Every such W is also a partial isometry and all of them are $\star$-ordered by the inclusion of their final (resp. initial) spaces.

Corollary 2.9 *Given* $A \in L(\mathcal{H})$ *and* $B \in L(\mathcal{H})^+$, *if* $A \overset{\star}{\leq} B$, *then also* $A \in \mathcal{L}(\mathcal{H})^+$.

Proof It is easy to see that the commutant of $BB^* = B^2$ coincides with $\{B\}'$. ■

A more detailed analysis of the relation between the $\star$-order and the polar decomposition will be done in Section 2.4. The above corollary is no longer true if we replace positive by self-adjoint or normal (for a 2×2 example, see [1]).

2.3. *Star order and functional calculi*

In this subsection, we study the problem of finding out those functions that are monotone with respect to the $\star$-order when they are applied using one of the following functional calculi: the Riesz functional calculus for holomorphic maps and the continuous functional calculus for normal operators [4]. The key remark to obtain these results is the following proposition, which is a direct consequence of Equation (4).

Proposition 2.10 *Let* $A, B \in L(\mathcal{H})$ *such that* $P_A = P_{A^*} = Q_A$. *Then, in terms of the orthogonal decomposition* $\mathcal{H} = R(P_A) \oplus \ker P_A$, *we have that*

$$A \overset{\star}{\leq} B \iff A = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} A_{11} & 0 \\ 0 & B_{22} \end{bmatrix}. \tag{7}$$

In such a case $AB = BA$ *and* $\sigma(A) \subseteq \sigma(B) \cup \{0\}$.

Proof Let $P = P_A = P_{A^*} = Q_A$. By Equation (4), $A \overset{\star}{\leq} B \iff A = BP = PB$, which is a reformulation of Equation (7). ■

The announced results involving the $\star$-order and the functional calculi follows directly from of the above 2×2 decomposition. Note that the type of functions considered in each proposition depends only on the class of operator considered and the functional calculus defined on them, there is no other restriction. It should be mentioned that, besides their applications, these propositions may be interesting by themselves because they provide two different generalizations of Theorem 4.1 of [1], where a similar result is proved for functions of the form $f(t) = t^{2^k}$ for $k \in \mathbb{N}$.

Proposition 2.11 *Let* $A, B \in L(\mathcal{H})$ *such that* $P_A = P_{A^*}$. *Let* f *be a complex analytic function defined in a neighbourhood of* $\{0\} \cup \sigma(A) \cup \sigma(B)$ *such that* $f(0) = 0$. *Then*

$$A \overset{\star}{\leq} B \Longrightarrow f(A) \overset{\star}{\leq} f(B).$$

Moreover, if f *is also injective, then* $A \overset{\star}{\leq} B \iff f(A) \overset{\star}{\leq} f(B)$.

Proof Using Proposition 2.10, and taking 2×2 matrices with respect to the decomposition $\mathcal{H} = R(P_A) \oplus \ker P_A$, we have that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \overset{\star}{\leq} \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix} = B \Longrightarrow f(A) = \begin{bmatrix} f(A_1) & 0 \\ 0 & 0 \end{bmatrix} \overset{\star}{\leq} \begin{bmatrix} f(A_1) & 0 \\ 0 & f(B_1) \end{bmatrix} = f(B),$$

because $f(0) = 0$ and $f(A_1 \oplus B_1) = f(A_1) \oplus f(B_1)$. ■

With almost the same proof, we have the following result regarding the continuous functional calculus for normal operators:

PROPOSITION 2.12 *Let A and B be normal operators. Then, for every continuous function* $f\colon \{0\}\cup\sigma(A)\cup\sigma(B)\to\mathbb{C}$ *satisfying that* $f(0)=0$, *it holds that*

$$A\overset{\star}{\leq}B\Longrightarrow f(A)\overset{\star}{\leq}f(B).$$

Moreover, if f is also injective, then $A\overset{\star}{\leq}B\iff f(A)\overset{\star}{\leq}f(B)$.

The condition $P_A=P_{A^*}$ of Proposition 2.11 may seem too restrictive, but as it can be seen in an example given in [1], without this condition the above results are false even for $f(x)=x^2$. Now we give an useful application of the previous results.

COROLLARY 2.13 *Let* $A,B\in L(\mathcal{H})$ *such that* $A\overset{\star}{\leq}B$. *Then*

$$|A|\overset{\star}{\leq}|B| \quad \text{and} \quad |A^*|\overset{\star}{\leq}|B^*|.$$

Proof By Remark 2.2 we know that $A^*A\overset{\star}{\leq}B^*B$ and $AA^*\overset{\star}{\leq}BB^*$. Then, using Proposition 2.12 with the function $f(t)=t^{1/2}$, we obtain the desired result. ■

2.4. *Star order and the polar decomposition*

One of the main tools used to study the ⋆-order in finite-dimensional spaces is the singular value decomposition, which, in general, is no longer available in the infinite-dimensional setting. This subsection is devoted to the study of the relationship between the ⋆-order and the polar decomposition, which could be seen as a natural substitute of the singular value decomposition in arbitrary Hilbert spaces.

Notation Given $B\in L(\mathcal{H})$, we denote by $U_B\in L(\mathcal{H})$ the unique partial isometry such that $B=U_B|B|$ and $\ker U_B=R(|B|)^\perp=\ker|B|=\ker B$. We shall write that $B=U_B|B|$ is the polar decomposition of B. Observe that also $B=|B^*|U_B$.

Recall that, for every $B\in L(\mathcal{H})$, it holds that $R(|B|)=R(B^*)$ (without closures). Also $\overline{R(B^*)}=\ker B^\perp=\ker U_B{}^\perp=R(U_B^*)$. Therefore, in our notations,

$$P_{|B|}=Q_{|B|}=P_{B^*}=Q_B=Q_{U_B}. \tag{8}$$

LEMMA 2.14 *Let* $A,B\in L(\mathcal{H})$ *such that* $A\overset{\star}{\leq}B$. *If* $B=U|B|$ *is a polar decomposition of* B, *then* $A=U|A|$. *In particular,* $A=U_B|A|$.

Proof By Corollary 2.13, we have that $|A|\overset{\star}{\leq}|B|$. Then $|A|=|B|Q_{|A|}$ and, by Equation (8), it holds that $U|A|=U|B|Q_{|A|}=BQ_A=A$. ■

THEOREM 2.15 *Let* $A,B\in L(\mathcal{H})$. *Then, the following statements are equivalent*

(1) $A\overset{\star}{\leq}B$;
(2) $|A|\overset{\star}{\leq}|B|$ *and* $U_A\overset{\star}{\leq}U_B$.

Proof $1\Rightarrow 2$: By Corollary 2.13 we already know that $|A|\overset{\star}{\leq}|B|$. On the other hand, by Lemma 2.14, $U_A|A|=U_B|A|$. So U_A and U_B coincide on $R(|A|)$, and by continuity, on $\overline{R(|A|)} = R(Q_A) = R(Q_{U_A})$. Therefore, $U_A = U_B P_{|A|} = U_B Q_A = U_B \times Q_{U_A}$. Also

$$|A| \stackrel{\star}{\leq} |B| \Longrightarrow R(U_A^*) = R(Q_A) \subseteq R(Q_B) = R(U_B^*) \Longrightarrow Q_{U_A} \leq Q_{U_B}.$$

So, by Corollary 2.8, we have that $U_A \stackrel{\star}{\leq} U_B$.
$2 \Rightarrow 1$: Denote by $Q = Q_A = Q_{|A|} = P_{|A|} = Q_{U_A}$. By Theorem 2.7, we have that

$$|A| \stackrel{\star}{\leq} |B| \Longrightarrow |A| = |B|Q, \quad Q \leq Q_{|B|} \quad \text{and} \quad QB^*B = Q|B|^2 = |B|^2Q = B^*BQ.$$

Then we get that $Q|B| = |B|Q = |A|$. Similarly, $U_A \stackrel{\star}{\leq} U_B \Longrightarrow U_A = U_B Q$. Hence we have that $BQ = U_B|B|Q = U_BQ|A| = U_A|A| = A$, $Q \leq Q_{|B|} = Q_B$ and $QB^*B = B^*BQ$. Therefore, by Theorem 2.7, $A \stackrel{\star}{\leq} B$. ■

Remark 2.16 The same arguments used to prove $1 \Rightarrow 2$ imply that, if $B = U|B|$ is a polar decomposition of B, then $U_B \stackrel{\star}{\leq} U$. Hence, in Theorem 2.15, the partial isometry U_B can be changed by any partial isometry U that can be used in the polar decomposition of B.

3. Semi-lattice properties

This section is devoted to study the lower semi-lattice properties of $(L(\mathcal{H}), \stackrel{\star}{\leq})$. The key result to pursue these studies is Theorem 3.1.

3.1. *Semi-lattice properties of $(L(\mathcal{H}), \stackrel{\star}{\leq})$*

This study has already been done in the finite-dimensional setting by Hartwig and Drazin in [7]. It is not difficult to see that, for example, the invertible operators are maximal elements with respect to the $\star$-order. So, it is enough to take two different invertible operators, say G_1 and G_2, to see that $L(\mathcal{H})$ cannot be a lattice endowed with $\stackrel{\star}{\leq}$. However, Hartwig and Drazin proved in [7] that the set of matrices endowed with the $\star$-order is a lower semi-lattice, i.e. for every pair of matrices A and B, there exists the $\star$-maximum of the set

$$\{C \in L(\mathcal{H}) : C \stackrel{\star}{\leq} A \quad \text{and} \quad C \stackrel{\star}{\leq} B\}.$$

They first prove the result for pairs of partial isometries and then reduce the general case to this particular case by using a smart trick based on the singular value decomposition. A similar approach can be done by replacing the singular value decomposition by the polar decomposition and using some of the results that have been proved in the previous section. However we prefer to pursue a direct approach based on the next useful reformulation of Theorem 2.7 that contains all we need to recover Hartwig–Drazin's result in our setting.

THEOREM 3.1 *Let $B \in L(\mathcal{H})$. Then, there is an order preserving (in both directions) bijection between the following ordered sets:*

(1) $\mathcal{L}_B := \{A \in L(\mathcal{H}) : A \stackrel{\star}{\leq} B\}$ *with the $\star$-order;*
(2) $\mathcal{P}_B := \{P \in \mathcal{P}(\mathcal{H}) : R(P) \subseteq \overline{R(B)}$ *and* $PBB^* = BB^*P\}$ *with the usual order (or with the $\star$-order because on this set both orders coincide).*

This bijection is given by $\mathcal{L}_B \ni A \mapsto P_A$ and $\mathcal{P}_B \ni P \mapsto PB$.

It is easy to see that $\mathcal{P}_B$ is a lattice with respect to the usual order. Actually, $\mathcal{P}_B$ is the set of all projections of the von Neumann algebra $\mathcal{M}_B = P_B\{BB^*\}' P_B \subseteq L(R(P_B))$, where $\{BB^*\}'$ is the commutant of BB^* in $L(\mathcal{H})$.

Proposition 3.2 *For every $B \in L(\mathcal{H})$, the set $\mathcal{L}_B$ is a lattice. Moreover, the ordered set $(L(\mathcal{H}), \overset{\star}{\leq})$ is a lower semi-lattice, i.e. for every pair $A, B \in L(\mathcal{H})$,*

$$\mathcal{L}(A,B) := \mathcal{L}_A \cap \mathcal{L}_B = \{C \in L(\mathcal{H}) : C \overset{\star}{\leq} A \quad \text{and} \quad C \overset{\star}{\leq} B\} \tag{9}$$

has a $\star$-maximum, called the $\star$-minimum of A and B and denoted by $A \wedge^\star B$.

Proof The fact that $\mathcal{L}_B$ is a lattice follows from Theorem 3.1. Consider the set

$$\mathcal{P}(A,B) = \left\{ Q \in \mathcal{P}(\mathcal{H}) \cap \{A^*A, B^*B\}' : R(Q) \subseteq \overline{R(A)} \cap \overline{R(B)} \cap \ker(A^* - B^*) \right\}.$$

It is clear that $\mathcal{P}(A, B) \subseteq \mathcal{P}_A \cap \mathcal{P}_B$ and that it is also a lattice. Given $Q \in \mathcal{P}(A,B)$, the condition

$$R(Q) \subseteq \ker(A^* - B^*) \iff A^*Q = B^*Q \iff QA = QB,$$

so that $QA = QB \in \mathcal{L}(A,B)$. On the other hand, any $C \in \mathcal{L}(A,B)$ satisfies the condition $C = P_C A = P_C B$, so that $P_C \in \mathcal{P}(A,B)$. Therefore

$$A \wedge^\star B = \max \mathcal{L}(A,B) = PA = PB, \quad \text{where } P = P_{A \wedge^\star B} = \max \mathcal{P}(A,B). \qquad \blacksquare$$

Next, we state some properties of $\star$-minima whose proofs are straightforward.

Proposition 3.3 *Let $A, B, C \in L(\mathcal{H})$. Then*

(1) *$A \wedge^\star B = B \wedge^\star A$ and $(A \wedge^\star B)^* = A^* \wedge^\star B^*$;*
(2) *$(A \wedge^\star B) \wedge^\star C = A \wedge^\star (B \wedge^\star C)$;*
(3) *$(A \wedge^\star B)(A \wedge^\star B)^* \overset{\star}{\leq} AA^* \wedge^\star BB^*$ and $|A \wedge^\star B| \overset{\star}{\leq} |A| \wedge^\star |B|$;*
(4) *If A or B are positive then $A \wedge^\star B$ is also positive.*

Observe that both inequalities of item 3 can be strict. Indeed, take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $A \wedge^\star B = 0$, but $0 \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = A^*A = |A| \overset{\star}{\leq} B = B^*B = |B|$.

3.2. *Some limit theorems*

In finite-dimensional spaces, it is not difficult to see, by using simple arguments of dimension, that any sequence of operators which is non-decreasing (resp. non-increasing) with respect to the $\star$-order is constant from some n onwards. In the infinite-dimensional setting the situation is different, and a natural question is whether or not a star-monotone sequence converge. The following theorems provide a positive answer to this question. Recall that a sequence $\{A_n\}_{n\in\mathbb{N}}$ in $L(\mathcal{H})$ converges strongly to $A \in L(\mathcal{H})$, which is denoted by $A_n \xrightarrow[n\to\infty]{\text{S.O.T.}} A$, if $\|A_n x - Ax\| \underset{n\to\infty}{\longrightarrow} 0$ for every $x \in \mathcal{H}$. It is well known that every sequence (or net) in $L_{sa}(\mathcal{H})$ which is bounded and monotone must converge strongly.

Proposition 3.4 *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence in $L(\mathcal{H})$ which is $\star$-non-increasing. Then, there exists $A \in L(\mathcal{H})$ such that $A_n \xrightarrow[n\to\infty]{\text{S.O.T.}} A$, and $A \overset{\star}{\leq} A_n$ for every $n \in \mathbb{N}$. Moreover, if $B \in L(\mathcal{H})$ satisfies that $B \overset{\star}{\leq} A_n$ for every $n \in \mathbb{N}$, then $B \overset{\star}{\leq} A$.*

Proof Let $P_n = P_{A_n}$. Note that, by Theorem 2.7, for every $n \in \mathbb{N}$ it holds that

$$P_n \geq P_{n+1} \quad \text{and} \quad P_n(A_1A_1^*) = (A_1A_1^*)P_n.$$

Let $P = \inf\{P_n : n \in \mathbb{N}\}$, so that $P_n \xrightarrow[n\to\infty]{\text{S.O.T.}} P$. Then $A_n = P_nA_1 \xrightarrow[n\to\infty]{\text{S.O.T.}} PA_1 =: A$. Clearly, P commutes with $A_1A_1^*$. Hence $A \overset{\star}{\leq} A_1$. Moreover, since $A_n \overset{\star}{\leq} A_1$ and $P \leq P_n$, then Theorem 2.7 assures that $A \overset{\star}{\leq} A_n$. Similarly, if $B \overset{\star}{\leq} A_n$ for every $n \in \mathbb{N}$, then $P_B \leq P_n$ for every $n \in \mathbb{N}$, and therefore $P_B \leq P$. As before, this implies that $B \overset{\star}{\leq} A$ because both $A, B \overset{\star}{\leq} A_1$. ∎

In a similar fashion, we get the next proposition.

Proposition 3.5 *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence in $L(\mathcal{H})$ which is $\star$-non-decreasing and $\star$-bounded from above for some $B \in L(\mathcal{H})$. Then, there exists $A \in L(\mathcal{H})$ such that $A \overset{\star}{\leq} B$, $A_n \xrightarrow[n\to\infty]{\text{S.O.T.}} A$, and $A_n \overset{\star}{\leq} A$ for every $n \in \mathbb{N}$.*

Remark 3.6 Note that in Proposition 3.4, as well as in Proposition 3.5, the sequences can be replaced by nets and the results remain true with almost the same proof.

Remark 3.7 Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence as in Proposition 3.5 and $A \in L(\mathcal{H})$ be the limit of this sequence. Then, it is not difficult to prove that $|A_n| \xrightarrow[n\to\infty]{\text{S.O.T.}} |A|$ and $|A_n^*| \xrightarrow[n\to\infty]{\text{S.O.T.}} |A^*|$. Observe that, if all the operators A_n have finite-dimensional range, so do the operators $|A_n|$ and $|A_n^*|$. Then, simple computations (involving mainly Proposition 2.10) show that $|A|$ (resp. $|A^*|$) have to be diagonalizable, i.e. there exists an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of $|A|$ (resp. $|A^*|$). This, in particular, implies that there are operators in $L(\mathcal{H})$ that cannot be reached using star non-decreasing sequences of finite-range operators, which is one of the obstructions to pursue the generalization to infinite-dimensional Hilbert spaces based in the results for matrices and monotone sequences.

The next result shows that the $\star$-minimum behaves well with respect to $\star$-monotone decreasing sequences.

Proposition 3.8 *Let $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ be two sequences in $L(\mathcal{H})$ which are $\star$-non-increasing and so that $A_n \xrightarrow[n\to\infty]{\text{S.O.T.}} A$ and $B_n \xrightarrow[n\to\infty]{\text{S.O.T.}} B$, for some operators $A, B \in L(\mathcal{H})$. Then, $A_n \wedge^\star B_n \xrightarrow[n\to\infty]{\text{S.O.T.}} A \wedge^\star B$.*

Proof We shall use Proposition 3.4 several times. Since $A_{n+1} \wedge^\star B_{n+1} \overset{\star}{\leq} A_n \wedge^\star B_n$ for every $n \in \mathbb{N}$, then there exists $L \in L(\mathcal{H})$ such that $A_n \wedge^\star B_n \xrightarrow[n\to\infty]{\text{S.O.T.}} L$. Observe that $L \overset{\star}{\leq} A_n \wedge^\star B_n \overset{\star}{\leq} A_n$ for every $n \in \mathbb{N}$. Then $L \overset{\star}{\leq} A$. Analogously, $L \overset{\star}{\leq} B$. On the other hand, let $C \in L(\mathcal{H})$ such that $C \overset{\star}{\leq} A$ and $C \overset{\star}{\leq} B$. As $C \overset{\star}{\leq} A \wedge^\star B \leq A_n \overset{\star}{\leq} B_n$ for every $n \in \mathbb{N}$, we get that $C \overset{\star}{\leq} L$, which completes the proof. ∎

As the next example shows that a similar result for $\star$-non-decreasing sequences is not true.

Example 3.9 Let $\{\mathcal{S}_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{T}_n\}_{n\in\mathbb{N}}$ be two increasing sequences of closed subspaces of $\mathcal{H}$ such that $P_{\mathcal{S}_n} \xrightarrow[n\to\infty]{\text{S.O.T.}} I$ and $P_{\mathcal{T}_n} \xrightarrow[n\to\infty]{\text{S.O.T.}} I$ but $\mathcal{S}_n \cap \mathcal{T}_n = \{0\}$ for each $n \in \mathbb{N}$. Then both sequences $\{P_{\mathcal{S}_n}\}_{n\in\mathbb{N}}$ and $\{P_{\mathcal{T}_n}\}_{n\in\mathbb{N}}$ are $\star$-increasing. However, $P_{\mathcal{S}_n} \wedge^\star P_{\mathcal{T}_n} = 0$ for every $n \in \mathbb{N}$, and $I \wedge^\star I = I \neq 0$.

4. The star shorted

In finite-dimensional spaces, the study of shorted operators related with the $\star$-order was carried out by Mitra [10]. The key tool used by Mitra was the singular value decomposition. In the infinite-dimensional setting, this approach is only available for compact operators. So, in order to generalize Mitra's results to any operator on an arbitrary Hilbert space, we need to develop a new approach. The main goal of this section is to characterize the star-shorted operator as a $\star$-minimum of two operators.

THEOREM 4.1 *Let $A \in L(\mathcal{H})$ and $\mathcal{S}, \mathcal{T} \sqsubseteq \mathcal{H}$. Then, the set of operators*

$$\overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T}) = \left\{D \in L(\mathcal{H}) : D \overset{\star}{\leq} A,\ R(D) \subseteq \mathcal{T} \quad \text{and} \quad R(D^*) \subseteq \mathcal{S}\right\}$$

has a $\star$-maximum given by $S = A \wedge^\star (P_\mathcal{T} A P_\mathcal{S})$.

The next technical lemma is part of the proof of Theorem 4.1, but we write it separately because we think that it could be interesting by itself.

LEMMA 4.2 *Let $A, B \in L(\mathcal{H})$ such that $A \overset{\star}{\leq} B$ and $\mathcal{S}, \mathcal{T} \sqsubseteq \mathcal{H}$ are two subspaces such that $R(A) \subseteq \mathcal{T}$ and $R(A^*) \subseteq \mathcal{S}$. Then $A \overset{\star}{\leq} P_\mathcal{T} B P_\mathcal{S}$.*

Proof Note that $P_A(P_\mathcal{T} B P_\mathcal{S}) = P_A B P_\mathcal{S} = A P_\mathcal{S} = A$. Similarly, it holds that $(P_\mathcal{T} B P_\mathcal{S}) Q_A = A$. Then, using Proposition 2.3, we conclude that $A \overset{\star}{\leq} P_\mathcal{T} B P_\mathcal{S}$. ■

Proof of Theorem 4.1 Let $S = A \wedge^\star (P_\mathcal{T} A P_\mathcal{S})$. By Lemma 4.2, if $D \in \overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T})$ then $D \overset{\star}{\leq} P_\mathcal{T} A P_\mathcal{S}$ and $D \overset{\star}{\leq} A$. Thus, $D \overset{\star}{\leq} S$. Now, it is enough to prove that $S \in \overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T})$. Since $S \overset{\star}{\leq} P_\mathcal{T} A P_\mathcal{S}$, then

$$R(S) \subseteq R(P_\mathcal{T} A P_\mathcal{S}) \subseteq \mathcal{T} \quad \text{and} \quad R(S^*) \subseteq R(P_\mathcal{S} A^* P_\mathcal{T}) \subseteq \mathcal{S}.$$

On the other hand, $S \overset{\star}{\leq} A$. So, $S \in \overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T})$. ■

Definition 4.3 Given $A \in L(\mathcal{H})$ and $\mathcal{S}, \mathcal{T} \sqsubseteq \mathcal{H}$, the maximum of the set $\overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T})$, whose existence is guaranteed by the above theorem, is called the $\star$-shorted operator of A with respect to the subspaces $\mathcal{S}$ and $\mathcal{T}$, and it is denoted by $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$. If $\mathcal{S} = \mathcal{T}$, we abbreviate $\overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T}) = \overset{\star}{\mathcal{M}}(A, \mathcal{S})$ and $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T}) = \overset{\star}{\Sigma}(A, \mathcal{S})$.

The next proposition summarizes some properties of the star-shorted operator which are direct consequences of the definition and the properties of the $\star$-order.

PROPOSITION 4.4 *Let $A, B \in L(\mathcal{H})$ and $\mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V} \sqsubseteq \mathcal{H}$. Then*

(1) $\overset{\star}{\Sigma}(\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T}), \mathcal{S}, \mathcal{T}) = \overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$.
(2) $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T}) = \overset{\star}{\Sigma}(A, \mathcal{S} \cap \overline{R(A^*)}, \mathcal{T} \cap \overline{R(A)})$.
(3) $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})^* = \overset{\star}{\Sigma}(A^*, \mathcal{T}, \mathcal{S})$.
(4) *If A is selfadjoint, then $\overset{\star}{\Sigma}(A, \mathcal{S})$ is selfadjoint.*
(5) *If A is positive then $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$ is positive, even if $\mathcal{S} \neq \mathcal{T}$.*
(6) If $A \overset{\star}{\leq} B$, $\mathcal{S} \subseteq \mathcal{U}$ *and* $\mathcal{T} \subseteq \mathcal{V}$, *then* $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T}) \overset{\star}{\leq} \overset{\star}{\Sigma}(A, \mathcal{U}, \mathcal{V})$. ■

The characterization of the $\star$-shorted operator as a maximum allows to prove the following result in a fairly standard way.

PROPOSITION 4.5 *Let $A, B \in L(\mathcal{H})$ and $\mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V} \sqsubseteq \mathcal{H}$. Then*

$$\overset{\star}{\Sigma}(\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T}), \mathcal{U}, \mathcal{V}) = \overset{\star}{\Sigma}(A, \mathcal{S} \cap \mathcal{U}, \mathcal{T} \cap \mathcal{V}).$$

Proof Let $\mathcal{M}_1 = \overset{\star}{\mathcal{M}}(\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T}), \mathcal{U}, \mathcal{V})$ and $\mathcal{M}_2 = \overset{\star}{\mathcal{M}}(A, \mathcal{S} \cap \mathcal{U}, \mathcal{T} \cap \mathcal{V})$. It is enough to show that $\mathcal{M}_1 = \mathcal{M}_2$. If $D \in \mathcal{M}_1$, then $D \overset{\star}{\leq} \overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$, so $R(D) \subseteq \mathcal{S}$ and $R(D^*) \subseteq \mathcal{T}$. Thus $R(D) \subseteq \mathcal{S} \cap \mathcal{T}$ and $R(D^*) \subseteq \mathcal{T} \cap \mathcal{V}$. But, $D \overset{\star}{\leq} \overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T}) \overset{\star}{\leq} A$, therefore $D \in \mathcal{M}_2$.

On the other hand, if $D \in \mathcal{M}_2$, then $D \overset{\star}{\leq} A$, $R(D) \subseteq \mathcal{S}$ and $R(D^*) \subseteq \mathcal{T}$, therefore $D \overset{\star}{\leq} \overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$. Moreover, $R(D) \subseteq \mathcal{U}$ and $R(D^*) \subseteq \mathcal{V}$, therefore $D \in \mathcal{M}_1$. ∎

Corollary 4.6 *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}, \mathcal{T} \sqsubseteq \mathcal{H}$. Then $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T}) = \overset{\star}{\Sigma}(A, \mathcal{S} \cap \mathcal{T})$.*

Proof By Proposition 4.5, as $B := \overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T}) \geq 0$ we get that

$$B = \overset{\star}{\Sigma}(B, \mathcal{S}, \mathcal{T}) = \overset{\star}{\Sigma}(B^*, \mathcal{S}, \mathcal{T}) = \overset{\star}{\Sigma}(\overset{\star}{\Sigma}(A, \mathcal{T}, \mathcal{S}), \mathcal{S}, \mathcal{T}) = \overset{\star}{\Sigma}(A, \mathcal{S} \cap \mathcal{T}, \mathcal{S} \cap \mathcal{T}) ,$$

which completes the proof. ∎

Proposition 4.7 *Let $A \in L(\mathcal{H})$ and $\mathcal{S}, \mathcal{T} \sqsubseteq \mathcal{H}$ such that $\mathcal{S} \subseteq R(Q_A)$ and $\mathcal{T} \subseteq R(P_A)$. Then, if $A = U|A| = |A^*|U$ is the polar decomposition of A, it holds that*

$$|\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})| = \overset{\star}{\Sigma}(|A|, \mathcal{S}, U^*(\mathcal{T})) \quad \textit{and} \quad |\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})^*| = \overset{\star}{\Sigma}(|A^*|, U(\mathcal{S}), \mathcal{T}). \tag{10}$$

Proof By Corollary 4.6, $\overset{\star}{\Sigma}(|A|, \mathcal{S}, U^*(\mathcal{T})) = \overset{\star}{\Sigma}(A, \mathcal{S} \cap U^*(\mathcal{T}))$. Also, Lemma 2.14 assures that $U^* \overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T}) = |\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})|$. Then, to prove the first identity of (10) it is enough to prove that the map $\Gamma : \overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T}) \to \overset{\star}{\mathcal{M}}(|A|, \mathcal{S} \cap U^*(\mathcal{T}))$, given by $\Gamma(B) = U^*B = |B|$, is an $\star$-order-preserving bijection. Let $B \in \overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T})$. Then $U^*B = |B| \overset{\star}{\leq} |A|$,

$$R(U^*B) = U^*(R(B)) \subseteq U^*(\mathcal{T}) \quad \text{and} \quad R(U^*B) = R(|B|) \subseteq \mathcal{S}.$$

Hence, $\Gamma(B) \in \overset{\star}{\mathcal{M}}(|A|, \mathcal{S} \cap U^*(\mathcal{T}))$. By Corollary 2.13, it is order preserving and it is injective because $U\Gamma(B) = B$ by Lemma 2.14. On the other hand, to prove that Γ is onto, let $C \in \overset{\star}{\mathcal{M}}(|A|, \mathcal{S} \cap U^*(\mathcal{T}))$ and define $B = UC$. Observe that, since $\mathcal{T} \subseteq R(P_A)$, then

$$R(B) = U(R(C)) \subseteq UU^*(\mathcal{T}) = P_A(\mathcal{T}) = \mathcal{T}.$$

Also $R(B^*) \subseteq R(C) \subseteq \mathcal{S}$. Using that $R(C) \subseteq R(|A|)$ and $R(U^*U) = \overline{R(|A|)}$, then $U^*B = C$. Also $BB^* = UCCU^* = UC|A|U^* = BA^*$ and

$$B^*B = CC = C|A| = CU^*U|A| = B^*A ,$$

which shows that $B \overset{\star}{\leq} A$, and therefore $B \in \overset{\star}{\mathcal{M}}(A, \mathcal{S}, \mathcal{T})$. Since $\Gamma(B) = U^*B = C$, then Γ is onto. The other equality of (10) follows in a similar way. ∎

Remark 4.8 Note that, by item 2 of Proposition 4.4, the conditions $\mathcal{S} \subseteq \overline{R(A^*)}$ and $\mathcal{T} \subseteq \overline{R(A)}$ asked in Proposition 4.7 are not too severe.

5. Star orthogonal decompositions

As we have already mentioned, the definition of the $\star$-order has been motivated by the notion of $\star$-orthogonality introduced by Hestenes in [9]. So, it is natural to expect that this relation plays an important role in the different aspects of the $\star$-order. In this section, we shall study how the $\star$-orthogonality is related with the $\star$-minima and the $\star$-shorted operators. We show that if an operator admits a 'suitable decomposition', the $\star$-minima as well as the $\star$-shorted operator can be computed in terms of that decomposition. In the next definition we say what we understand by a suitable decomposition.

Definition 5.1 Given $A \in L(\mathcal{H})$, we shall say that the family of operators $\{A_i\}_{i \in I}$, each $A_i \in L(\mathcal{H})$, is a star orthogonal decomposition ($\star$-OD) of A if

(1) $A_i^* A_k = 0 = A_i A_k^*$ for every $i \neq k$.
(2) The spectra $\sigma(A_i\, A_i^*|_{R(P_{A_i})})$ are pairwise disjoint.
(3) $A = \sum_{i \in I} A_i$, where in the case of decompositions with infinitely many operators, the series converges in the strong operator topology.

Example 5.2 A typical example, which actually motivates Definition 5.1, is the singular value decomposition of compact operators. Given a compact operator $A \in L(\mathcal{H})$, its singular value decomposition can be written as

$$A = \sum_{i=1}^{\infty} s_i U_i, \tag{11}$$

where the numbers s_i are the singular values of A, and the operators U_i are partial isometries of finite rank. In this case, $\{s_i\ U_i\}_{i \in \mathbb{N}}$ is a $\star$-OD of A. Decompositions like (11) are sometimes called Penrose's decompositions [7].

Lemma 5.3 *Let $A \in L(\mathcal{H})$ and let $\{A_i\}_{i \in I}$ be a $\star$-OD of A. Then*

(1) $AA^* = \sum_{i \in I} A_i A_i^*$ *and* $A^* A = \sum_{i \in I} A_i^* A_i$.
(2) *The projections P_{A_i} are pairwise orthogonal, and $P_A = \sum_{i \in I} P_{A_i}$.*
(3) *For every $i \in I$, it holds that $P_{A_i} \in \mathcal{P}_A$, so that $A_i = P_{A_i} A \overset{\star}{\leq} A$.*

Proof Straightforward. ∎

Proposition 5.4 *Let $A, B \in L(\mathcal{H})$ such that $A \overset{\star}{\leq} B$, and let $\{B_i\}_{i \in I}$ be a $\star$-OD of B. Then the sequence $\{A \wedge^\star B_i\}_{i \in I}$ is a $\star$-OD of A.*

Proof By Theorem 2.7, P_A commutes with BB^*. So, by Lemma 5.3 and the second condition in Definition 5.1, P_A commutes with each $B_i B_i^*$ and each P_{B_i}. Denote by $P_i = P_A P_{B_i} \in \mathcal{P}_{B_i}$ and $A_i = P_i B_i \overset{\star}{\leq} B_i$ (by Theorem 2.7). In particular,

$$\sigma\Big(A_i\, A_i^*\Big|_{R(P_{A_i})}\Big) \subseteq \sigma\Big(B_i\, B_i^*\Big|_{R(P_{B_i})}\Big) \Longrightarrow \sigma\Big(A_i\, A_i^*\Big|_{R(P_{A_i})}\Big) \text{ are disjoint.}$$

On the other hand, straightforward computations also show that $A_i^* A_k = 0 = A_i A_k^*$ for every $i \neq k$. Finally, as $P_A \leq P_B = \sum_{i \in I} P_{B_i}$, then $P_A = \sum_{i \in I} P_i$ and $A = P_A B = \sum_{i \in I} A_i$.

Therefore, $\{A_i\}_{i \in I}$ is a $\star$-OD of A. Thus, it is enough to prove that each $A_i = A \wedge^\star B_i$. Fix $i \in I$, and observe that $P_{A \wedge^\star B_i} \leq P_A$ and $P_{A \wedge^\star B_i} \leq P_{B_i}$. So $P_{A \wedge^\star B_i} \leq P_i$. Since $A \wedge^\star B_i \overset{\star}{\leq} B_i$ and also $A_i \overset{\star}{\leq} B_i$, Theorem 2.7 assures that $A \wedge^\star B_i \overset{\star}{\leq} A_i$. The other inequality follows by Lemma 5.3 and the fact that $A_i \overset{\star}{\leq} B_i$. ∎

Corollary 5.5 *Let A, $B \in L(\mathcal{H})$ and let $\{B_i\}_{i \in I}$ be a $\star$-OD of B. Then the sequence $\{A \wedge^\star B_i\}_{i \in I}$ is a $\star$-OD of $A \wedge^\star B$.*

Proof It is a direct consequence of Proposition 5.4 and the fact that

$$(A \wedge^\star B) \wedge^\star B_i = A \wedge^\star (B \wedge^\star B_i) = A \wedge^\star B_i. \quad ∎$$

Lemma 5.6 *Let $A \in L(\mathcal{H})$ and $\{A_i\}_{i\in I}$ be a $\star$-OD of A. For every $i \in I$, let $\{A_{ij}\}_{j\in J_i}$ be a $\star$-OD of A_i. Then $\{A_{ij} : i \in I\ ,\ j \in J_i\}$ is a $\star$-OD of A.*

Proof Straightforward. ■

As a consequence of Lemma 5.6 and Corollary 5.5, we obtain the following result.

Proposition 5.7 *Let $A, B \in L(\mathcal{H})$. Suppose that $\{A_i\}_{i\in I}$ is a $\star$-OD of A and $\{B_j\}_{j\in J}$ is a $\star$-OD of B. Then $\{A_i \wedge^\star B_j\}_{(i,j)\in I\times J}$ is a $\star$-OD of $A \wedge^\star B$. In particular,*

$$A \wedge^\star B = \sum_{(i,j)\,\in\, I\times J} A_i \wedge^\star B_j.$$

Finally, we state the relationship between $\star$-OD and shorted operators.

Proposition 5.8 *Let $A \in L(\mathcal{H})$, $\mathcal{S}, \mathcal{T} \sqsubseteq \mathcal{H}$ and suppose that $\{A_i\}_{i\in I}$ is a $\star$-OD of A. Then, the sequence $\{\overset{\star}{\Sigma}(A_i, \mathcal{S}, \mathcal{T})\}_{i\,\in\, I}$ is a $\star$-OD of $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$.*

Proof Let $B_i = \overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T}) \wedge^\star A_i$. Then, by Proposition 5.4, $\{B_i\}_{i\in I}$ is a $\star$-OD of $\overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$. Therefore, it is enough to prove that $B_i = \overset{\star}{\Sigma}(A_i\,, \mathcal{S}, \mathcal{T})$ for each $i \in I$. So, fix $i \in I$.

On the one hand, Lemma 5.3 assures that $A_i \overset{\star}{\leq} A$, and hence $\overset{\star}{\Sigma}(A_i\,, \mathcal{S}, \mathcal{T}) \overset{\star}{\leq} \overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$ for every $i \in I$. On the other hand, by definition, $\overset{\star}{\Sigma}(A_i\,, \mathcal{S}, \mathcal{T}) \leq A_i$. Therefore each $\overset{\star}{\Sigma}(A_i\,, \mathcal{S}, \mathcal{T}) \overset{\star}{\leq} B_i$. Conversely, by its definition $B_i \overset{\star}{\leq} A_i$. Moreover, as $B_i \overset{\star}{\leq} \overset{\star}{\Sigma}(A, \mathcal{S}, \mathcal{T})$, $R(B_i) \subseteq \mathcal{T}$ and $R(B_i^*) \subseteq \mathcal{S}$. So, $B_i \overset{\star}{\leq} \overset{\star}{\Sigma}(A_i\,, \mathcal{S}, \mathcal{T})$, which concludes the proof. ■

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