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A Note on the Tangent Numbers and Polynomials

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Abstract

In this paper we introduce the tangent numbers T_n and polynomials $T_n(x)$. Some interesting results and relationships are obtained.

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1 Introduction

Recently, many mathematicians have studied in the area of the Bernoulli numbers, Euler numbers, Genocchi numbers, and tangent numbers(see [1-7]). Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$.

The Bernoulli numbers B_n are defined by the generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, (|t| < \pi),$$

where we use the technique method notation by replacing B^n by $B_n (n \geq 0)$ symbolically. We consider the Bernoulli polynomials $B_n(x)$ as follows:

$$\left(\frac{t}{e^t - 1}\right) e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Note that $B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}$.

We introduce the Genocchi polynomials $G_n(x)$ as follows:

$$\left(\frac{2t}{e^t+1}\right) e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}.$$

In the special case, $x = 0$, $G_n(0) = G_n$ are called the n-th Genocchi numbers. As well known definition, the Euler polynomials are defined by

$$\left(\frac{2}{e^t+1}\right) e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},$$

with the usual convention of replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0$, $E_n(0) = E_n$ are called the n-th Euler numbers. Alternatively we may define the Euler numbers by

$$\sec(t) = \sum_{n=0}^{\infty}(-1)^n E_{2n}\frac{t^{2n}}{(2n)!}.$$

They are closely related to the tangent numbers T_n(cf. [3]), which are defined by

$$T_0 = 1, \quad \tan(t) = \sum_{n=0}^{\infty}(-1)^{n+1}T_{2n+1}\frac{t^{2n+1}}{(2n+1)!}, \quad T_{2n} = 0, (n \in \mathbb{N}).$$

Numerous properties of tangent number are known. More studies and results in this subject we may see references [2], [3], [7]. About extensions for the tangent numbers can be found in [7].

Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{0 \le x < p^N} g(x)(-1)^x, \quad (\text{see}[4]). \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0), \ (\text{see } [4\text{-}6]). \tag{1.2}$$

From (1.1), we obtain

$$\int_{\mathbb{Z}_p} g(x+n)d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) + 2\sum_{l=0}^{n-1}(-1)^{n-1-l}g(l). \quad (1.3)$$

Our aim in this paper is to define tangent polynomials $T_n(x)$. We investigate some properties which are related to tangent numbers T_n and polynomials $T_n(x)$. We also derive the existence of a specific interpolation function which interpolate tangent numbers T_n and polynomials $T_n(x)$ at negative integers.

2 Tangent numbers and polynomials

Our primary goal of this section is to define tangent numbers T_n and polynomials $T_n(x)$. We also find generating functions of tangent numbers T_n and polynomials $T_n(x)$ and investigate their properties.

In (1.2), if we take $g(x) = e^{2xt}$, then we easily see that

$$I_{-1}(e^{2xt}) = \int_{\mathbb{Z}_p} e^{2xt}d\mu_{-1}(x) = \frac{2}{e^{2t}+1}.$$

Let us define the tangent numbers T_n and polynomials $T_n(x)$ as follows:

$$I_{-1}(e^{2yt}) = \int_{\mathbb{Z}_p} e^{2yt}d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_n\frac{t^n}{n!}, \quad (2.1)$$

$$I_{-1}(e^{(2y+x)t}) = \int_{\mathbb{Z}_p} e^{(x+2y)t}d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_n(x)\frac{t^n}{n!}. \quad (2.2)$$

By (2.1) and (2.2), we obtain the following Witt's formula.

Theorem 2.1 *For $n \in \mathbb{Z}_+$, we have*

$$\int_{\mathbb{Z}_p} (2x)^n d\mu_{-1}(x) = T_n,$$
$$\int_{\mathbb{Z}_p} (x+2y)^n d\mu_{-1}(y) = T_n(x).$$

By using p-adic integral on $\mathbb{Z}_p$, we obtain,

$$\int_{\mathbb{Z}_p} e^{2xt}d\mu_{-1}(x) = 2\sum_{m=0}^{\infty}(-1)^m e^{2mt}. \quad (2.3)$$

Thus tangent numbers T_n are defined by means of the generating function

$$F(t)=\sum_{n=0}^{\infty}T_n\frac{t^n}{n!}=2\sum_{m=0}^{\infty}(-1)^m e^{2mt}. \tag{2.4}$$

Using similar method as above, by using p-adic integral on $\mathbb{Z}_p$, we have

$$\sum_{n=0}^{\infty}T_n(x)\frac{t^n}{n!}=\left(\frac{2}{e^{2t}+1}\right)e^{xt}. \tag{2.5}$$

By using (2.2) and (2.5), we obtain

$$F(t,x)=\sum_{n=0}^{\infty}T_n(x)\frac{t^n}{n!}=2\sum_{m=0}^{\infty}(-1)^m e^{(2m+x)t}. \tag{2.6}$$

By Theorem 2.1, we easily obtain that

$$\begin{aligned} T_n(x)&=\int_{\mathbb{Z}_p}(x+2y)^n d\mu_{-1}(y)\\ &=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}T_k\\ &=(x+T)^n\\ &=2\sum_{m=0}^{\infty}(-1)^m(x+2m)^n. \end{aligned} \tag{2.7}$$

The following elementary properties of tangent polynomials $T_n(x)$ are readily derived from (2.1) and (2.2). We, therefore, choose to omit the details involved. More studies and results in this subject we may see references [4]-[6].

Theorem 2.2 *For any positive integer n, we have*

$$T_n(x)=(-1)^nT_n(2-x).$$

Theorem 2.3 *For any positive integer m(=odd), we have*

$$T_n(x)=m^n\sum_{i=0}^{m-1}(-1)^iT_n\left(\frac{2i+x}{m}\right),\quad n\in\mathbb{Z}_+.$$

By (1.3), (2.1), and (2.2), we easily see that

$$2^{m+1}\sum_{l=0}^{n-1}(-1)^{n-1-l}l^m=T_m(2n)+(-1)^{n-1}T_m.$$

Hence, we have the following theorem.

Theorem 2.4 *Let $m \in \mathbb{Z}_+$. If $n \equiv 0 \pmod 2$, then*

$$T_m(2n) - T_m = 2^{m+1} \sum_{l=0}^{n-1} (-1)^{l+1} l^m.$$

If $n \equiv 1 \pmod 2$, then

$$T_m(2n) + T_m = 2^{m+1} \sum_{l=0}^{n-1} (-1)^{l} l^m.$$

From (1.3), we note that

$$\begin{aligned} 2 &= \int_{\mathbb{Z}_p} e^{(2x+2)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} e^{2xt} d\mu_{-1}(x) \\ &= \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} (2x+2)^n d\mu_{-2}(x) + \int_{\mathbb{Z}_p} (2x)^n d\mu_{-1}(x) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(T_n(2) + T_n \right) \frac{t^n}{n!}. \end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 2.5 *For $n \in \mathbb{Z}_+$, we have*

$$T_n(2) + T_n = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

By (2.7) and Theorem 2.5, we have the following corollary.

Corollary 2.6 *For $n \in \mathbb{Z}_+$, we have*

$$(T+2)^n + T_n = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}$$

with the usual convention of replacing T^n by T_n.

Theorem 2.7 *For $n \in \mathbb{Z}_+$, we have*

$$T_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} T_k(x) y^{n-k}.$$

By Theorem 2.1, we easily get

$$T_n(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \int_{\mathbb{Z}_p} (2y)^l d\mu_{-1}(y) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} T_l.$$

Therefore, we obtain the following theorem.

Theorem 2.8 *For $n \in \mathbb{Z}_+$, we have*

$$T_n(x) = \sum_{l=0}^{n} \binom{n}{l} T_l x^{n-l}.$$

The tangent polynomials $T_n(x)$ can be determined explicitly. A few of them are

$T_0(x) = 1, \quad T_1(x) = x - 1, \quad T_2(x) = x^2 - 2x, \quad T_3(x) = x^3 - 3x^2 + 2,$
$T_4(x) = x^4 - 4x^3 + 8x, \quad T_5(x) = x^5 - 5x^4 + 20x^2 - 16,$
$T_6(x) = x^6 - 6x^5 + 40x^3 - 96x, \quad T_7(x) = x^7 - 7x^6 + 70x^4 - 336x^2 + 272,$
$T_8(x) = x^8 - 8x^7 + 112x^5 - 896x^3 + 2176x,$
$T_9(x) = x^9 - 9x^8 + 168x^6 - 2016x^4 + 9792x^2 - 7936$

3 The analogue of the Euler zeta function

The Riemann zeta function is defined by

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots ,$$

where s is a real number bigger than 1. Euler found that the Riemann zeta-function for even non-negative integer arguments can be expressed in terms of Bernoulli numbers - the relation is

$$(-1)^{n-1} \frac{B_{2n}}{(2n)!} = \frac{2\zeta(2n)}{(2\pi)^{2n}}.$$

For $s \in \mathbb{C}$ and $\mathrm{Re}(s) > 0$, the Euler zeta function and the Hurwitz–type Euler zeta function are defined by

$$\zeta_E(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \text{ and } \zeta_E(s,x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s},$$

respectively. Notice that the Euler zeta functions can be analytically continued to the whole complex plane, and these zeta functions have the values of the Euler numbers or the Euler polynomials at negative integers. In this section, by using tangent numbers and polynomials, we give the definition for the tangent zeta function and Hurwitz-type tangent zeta functions. These functions interpolate the tangent numbers and tangent polynomials, respectively. From (2.4), we note that

$$\begin{aligned} \frac{d^k}{dt^k} F(t)\Big|_{t=0} &= 2\sum_{m=0}^{\infty} (-1)^m 2^k m^k \\ &= T_k, (k \in \mathbb{N}). \end{aligned} \tag{3.1}$$

By using the above equation, we are now ready to define tangent zeta functions.

Definition 3.1 *Let $s \in \mathbb{C}$ with $Re(s) > 1$.*

$$\zeta_T(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)^s}. \tag{3.2}$$

Note that $\zeta_T(s)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta_T(s)$ and T_k is given by the following theorem.

Theorem 3.2 *For $k \in \mathbb{N}$, we have*

$$\zeta_T(-k) = T_k. \tag{3.3}$$

Observe that $\zeta_T(s)$ function interpolates T_k numbers at non-negative integers. By using (2.7), we note that

$$\begin{aligned} \frac{d^k}{dt^k} F(t,x)\Big|_{t=0} &= 2\sum_{m=0}^{\infty} (-1)^m (x+2m)^k \\ &= T_k(x), (k \in \mathbb{N}), \end{aligned}$$

and

$$\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}\right)\Bigg|_{t=0} = T_k(x), \text{ for } k \in \mathbb{N}. \tag{3.4}$$

By (3.2) and (3.4), we are now ready to define the Hurwitz-type tangent zeta functions.

Definition 3.3 *Let $s \in \mathbb{C}$ with $Re(s) > 1$.*

$$\zeta_T(s,x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+x)^s}. \tag{3.5}$$

Note that $\zeta_T(s,x)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta_T(s,x)$ and $T_k(x)$ is given by the following theorem.

Theorem 3.4 *For $k \in \mathbb{N}$, we have*

$$\zeta_T(-k,x) = T_k(x).$$

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