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A note on the uniqueness of weak solutions to a class of cross-diffusion systems

Xiuqing Chen and Ansgar Jüngel

Abstract. The uniqueness of bounded weak solutions to strongly coupled parabolic equations in a bounded domain with no-flux boundary conditions is shown. The equations include cross-diffusion and drift terms and are coupled self-consistently to the Poisson equation. The model class contains special cases of the Maxwell–Stefan equations for gas mixtures, generalized Shigesada–Kawasaki–Teramoto equations for population dynamics, and volume-filling models for ion transport. The uniqueness proof is based on a combination of the H^{-1} technique and the entropy method of Gajewski.

1. Introduction

Several techniques have been developed for the analysis of nonlinear parabolic systems, including sufficient conditions for the global existence of weak or strong solutions [3,18,22,29]. However, the proof of uniqueness of weak solutions is generally much more delicate, in particular for strongly coupled systems. In this paper, we prove the uniqueness of bounded weak solutions to a class of cross-diffusion systems. The proof is based on a combination of the H^{-1} technique and the method of Gajewski [14], where a certain semimetric measures the distance between two solutions. It is shown that the semimetric is related to relative entropies.

1.1. Model equations

The equations describe the evolution of the concentrations u_i,

$$\partial_t u_i = \operatorname{div} \sum_{j=1}^{n} \big(A_{ij}(u)\nabla u_j + B_{ij}(u)\nabla\phi\big), \quad i = 1, \ldots, n, \tag{1}$$

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in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d \geq 1$), where $u = (u_1, \ldots, u_n)$ and ϕ is a potential solving the Poisson equation

$$- \Delta\phi = \overline{u} - f(x) \quad \text{in } \Omega, \tag{2}$$

where $\overline{u} = \sum_{i=1}^n a_i u_i$ for some constants $a_i \geq 0$, and $f(x)$ is a given background concentration. We complement the equations by no-flux boundary and initial conditions,

$$\sum_{j=1}^n A_{ij}(u)\nabla u_j \cdot \nu = \nabla\phi \cdot \nu = 0 \text{ on } \partial\Omega, \quad u(0) = u^0 \text{ in } \Omega, \quad i = 1, \ldots, n. \tag{3}$$

For consistency, the initial datum has to satisfy the condition

$$\int_\Omega \sum_{i=1}^n a_i u_i^0 \mathrm{d}x = \int_\Omega f(x)\mathrm{d}x.$$

The diffusion coefficients A_{ij} and drift coefficients B_{ij} are defined by

$$A_{ij}(u) = p(\overline{u})\delta_{ij} + a_j u_i q(\overline{u}), \quad B_{ij}(u) = r(\overline{u})u_i\delta_{ij}, \quad i, j = 1, \ldots, n, \tag{4}$$

for some functions p, q, and r and numbers $a_j \geq 0$. Our main assumption is that these functions do not depend on the species number i. Then $\overline{u}$ satisfies a nonlinear drift–diffusion equation (see (12)), and this property allows us to initiate the uniqueness proof. We do not know how to relax this assumption in the context of weak solutions.

The diffusion matrix $A(u) = (A_{ij}(u))$ is *not* assumed to be positive definite, and it may degenerate. The existence theory developed in [17] is based on the assumption that there exists a transformation of variables such that the transformed diffusion matrix becomes positive semidefinite, allowing for some degeneracy; see [18] for details.

Under some conditions, model (1) and (4) can be derived formally from a master equation for a continuous-time, discrete-space random walk in the macroscopic limit [26,32] or from a fluid dynamical model in the inertia approximation [18, Section 4.2]. The variables u_i may describe the density of the ith population species or the ith component of a gas mixture with electrically charged components. In the former case, ϕ models the environmental potential; in the latter case, it denotes the electric potential. Because of these applications, it is reasonable to assume that $u_i \geq 0$ in Ω, $t > 0$.

For special choices of A_{ij} and B_{ij}, including condition (4), the existence of global bounded weak solutions can be shown. We give some examples and references in Sect. 4. In this paper, we are only concerned with the uniqueness of weak solutions.

1.2. State of the art

Before stating and explaining our assumptions and the main result, let us review some techniques to show the uniqueness of (weak) solutions to nonlinear parabolic equations. We focus on generalized solutions since uniqueness of strong solutions is usually proved by standard L^2 estimations.

One important technique is based on the use of the test function $\mathrm{sign}_+(u^{(1)} - u^{(2)})$, where $u^{(1)}$ and $u^{(2)}$ are two solutions and sign_+ is the positive sign function ($\mathrm{sign}_+(s) = 1$ for $s > 0$ and $\mathrm{sign}_+(s) = 0$ else). The use of this test function can be justified by employing the technique of doubling the variables, first developed by Kružkov for hyperbolic equations [21] and later extended by Carrillo to scalar parabolic equations [8] and by Blanchard and Porretta to allow for renormalized solutions [6]. We refer to the review [4] for an extensive bibliography. All these results hold for scalar equations only.

Nonlinear semigroup methods provide powerful abstract tools for proving the uniqueness of (mild of integral) solutions; see, e.g., [5]. However, this approach seems to be generally not accessible to cross-diffusion systems.

One of the first uniqueness theorems for diffusion systems was shown by Alt and Luckhaus [2] under the assumptions that the time derivative of u_i is integrable and the elliptic operator is linear. The first hypothesis was relaxed to finite-energy solutions by Otto [27], and the ellipticity condition was generalized by Agueh using methods from optimal transport [1], but in both cases for scalar equations only.

Another powerful approach is the dual method which consists in choosing a test function which satisfies an appropriate dual problem [10]. This includes the H^{-1} method, where a test function of an *elliptic* dual problem is chosen. In some sense, the uniqueness problem is reduced to an existence problem of the dual problem [23]. The dual method allows one to treat diffusion systems that are, to some extent, weakly coupled; see, e.g., [10,16,25]. Based on a dual method, Pham and Temam [28] proved recently a uniqueness result for a strongly coupled population system assuming a strictly positive definite diffusion matrix.

The uniqueness of (weak) solutions may be also proven by using an entropy method. One idea is to differentiate the relative entropy $H(u^{(1)}|u^{(2)})$, where $u^{(1)}$ and $u^{(2)}$ are two solutions emanating from the same initial data, with respect to time and to show that $\frac{\mathrm{d}}{\mathrm{d}t}H(u^{(1)}|u^{(2)}) \le CH(u^{(1)}|u^{(2)})$ for some constant $C > 0$, which implies from Gronwall's lemma that $H(u^{(1)}|u^{(2)}) = 0$ and hence $u^{(1)} = u^{(2)}$. This approach has been used to show the weak–strong uniqueness for compressible Navier–Stokes equations [11,12] and reaction–diffusion systems (with diagonal diffusion matrix) [13]. A second idea, due to Gajewski [14], is to time differentiate the semimetric

$$d(u^{(1)}, u^{(2)}) = H(u^{(1)}) + H(u^{(2)}) - 2H\left(\frac{u^{(1)} + u^{(2)}}{2}\right) \tag{5}$$

for convex entropies H and to show that $\frac{\mathrm{d}}{\mathrm{d}t}d(u^{(1)}, u^{(2)}) \le 0$, implying again that $u^{(1)} = u^{(2)}$. The technique has been applied to nonlinear drift–diffusion equations for semiconductors [14] and later to cross-diffusion systems [20,32]. Compared to other methods, it has the advantage that only weak solutions are needed [18, Chapter

4.7]. The Gajewski method is related to the approach of using relative entropies; see Remark 4.

1.3. Assumptions and main result

Our approach is to combine the H^{-1} technique and the method of Gajewski and to generalize the results from [20,32]. The novelty is the inclusion of the potential term and the general structure of $A_{ij}(u)$. With hypothesis (4), equations (1) can be formulated as

$$\partial_t u_i = \operatorname{div}\big(p(\overline{u})\nabla u_i + q(\overline{u})u_i\nabla\overline{u} + r(\overline{u})u_i\nabla\phi\big), \quad i = 1, \ldots, n. \tag{6}$$

This can be interpreted as a drift–diffusion equation with field term $q(\overline{u})\nabla\overline{u}+r(\overline{u})\nabla\phi$. Since $\overline{u}$ depends on all u_i, this is still a cross-diffusion system. However, the drift–diffusion structure is essential in the uniqueness proof. Our main result is as follows.

THEOREM 1 (Uniqueness of weak solutions). *Let (u, ϕ) with $u = (u_1, \ldots, u_n)$ be a weak solution to (1)–(3) such that $\overline{u}(x, t) \in [0, L]$ for $x \in \Omega$, $t \in (0, T)$ and some $L > 0$. Let $u^0 \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$. We assume that there exists $M > 0$ such that for all $s \in [0, L]$,*

$$p(s) \geq 0, \quad p(s) + q(s)s \geq 0, \tag{7}$$

$$r(s)s \in C^1([0, L]), \quad \left(\frac{d}{ds}(r(s)s)\right)^2 \leq M\big(p(s) + q(s)s\big). \tag{8}$$

Then (u, ϕ) is unique in the class of solutions satisfying $\int_\Omega \phi dx = 0$, $\nabla\phi \in L^\infty(0, T; L^\infty(\Omega))$, and

$$u_i \in L^2(0, T; H^1(\Omega)), \quad \partial_t u_i \in L^2(0, T; H^1(\Omega)'), \quad i = 1, \ldots, n.$$

In the case $r \equiv 0$, the boundedness of $\overline{u}$ is not needed, provided that

$$\sqrt{p(\overline{u})}\nabla u_i,\ \sqrt{|q(\overline{u})|}\nabla u_i \in L^2(\Omega \times (0, T)). \tag{9}$$

REMARK 2.
1. The regularity assumption on the potential can be relaxed to $\nabla\phi \in L^\infty(0, T; L^\alpha(\Omega))$ for $\alpha > d$ if $p(s) + q(s)s = \text{const.} > 0$; see Remark 3.
2. If $\partial\Omega \in C^{1,1}$ and $f \in L^\alpha(\Omega)$ with $\alpha > d$, the regularity $\nabla\phi \in L^\infty(0, T; L^\infty(\Omega))$ is a consequence of elliptic theory. Indeed, since $\overline{u}$ is bounded, $\overline{u} - f \in L^\infty(0, T; L^\alpha(\Omega))$, which implies, by Sobolev embedding, that $\phi \in L^\infty(0, T; W^{2,\alpha}(\Omega)) \hookrightarrow L^\infty(0, T; W^{1,\infty}(\Omega))$. □

The idea of the proof is first to show the uniqueness of $(\overline{u}, \phi)$. Indeed, multiplying (1) by a_i and summing over $i = 1, \ldots, n$ leads to a nonlinear drift–diffusion equation for $\overline{u}$,

$$\partial_t\overline{u} = \operatorname{div}\big(\nabla Q(\overline{u}) + R(\overline{u})\nabla\phi\big), \tag{10}$$

coupled with Poisson equation (2), where

$$R(s) = r(s)s, \quad Q(s) = \int_0^s (p(\tau) + q(\tau)\tau)d\tau. \tag{11}$$

Since the diffusion operator in (10) may degenerate, it is natural to apply the H^{-1} technique. Indeed, given two solutions $(u^{(1)}, \phi^{(1)})$, $(u^{(2)}, \phi^{(2)})$ with the same initial data, we use the test function $\phi := \phi^{(1)} - \phi^{(2)}$ in (12), which solves the dual problem $-\Delta\phi = u^{(1)} - u^{(2)}$ in Ω, $\nabla\phi \cdot \nu = 0$ on $\partial\Omega$. Then, using conditions (7)–(8), it can be shown that $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi\|^2_{L^2(\Omega)} \leq C\|\nabla\phi\|^2_{L^2(\Omega)}$, which implies that $u^{(1)} = u^{(2)}$ and $\phi^{(1)} = \phi^{(2)}$. In this step, we need the regularity $\nabla\phi^{(2)} \in L^\infty$.

The second step is to prove the uniqueness of (6). For this, we employ the method of Gajewski [14], based on an estimation of the semimetric

$$d(u, v) = \sum_{i=1}^n \int_\Omega \left(h(u_i) + h(v_i) - 2h\left(\frac{u_i + v_i}{2}\right)\right)\mathrm{d}x,$$

where $h(s) = s(\log s - 1) + 1$ is called an entropy. Let $u^{(1)} = (u_1^{(1)}, \ldots, u_n^{(1)})$, $u^{(2)} = (u_1^{(2)}, \ldots, u_n^{(2)})$ be two weak solutions to (6). A formal computation shows that $\frac{\mathrm{d}}{\mathrm{d}t}d(u^{(1)}, u^{(2)}) \leq 0$ and hence $d(u^{(1)}, u^{(2)}) = 0$. The convexity of h implies that $u^{(1)} = u^{(2)}$. In order to make this argument rigorous, we need to regularize the entropy, since terms with $\log u_i$ may be not defined on sets where $u_i = 0$. We discuss in Remark 4 the applicability of the Gajewski method.

The paper is organized as follows. Theorem 1 is proved in Sect. 2. Section 3 is concerned with some comments on the techniques and the proof. Some examples satisfying conditions (7)–(8) are detailed in Sect. 4.

2. Proof of Theorem 1

Step 1. Uniqueness for $(\overline{u}, \phi)$. We multiply (1) by a_i and sum over $i = 1, \ldots, n$:

$$\begin{aligned}
\partial_t \overline{u} &= \sum_{i,j=1}^n \operatorname{div}\left(\delta_{ij} p(\overline{u}) a_i \nabla u_j + q(\overline{u}) a_i u_i \nabla(a_j u_j) + \delta_{ij} r(\overline{u}) a_i u_i \nabla\phi\right) \\
&= \operatorname{div}\left(p(\overline{u})\nabla\overline{u} + q(\overline{u})\overline{u}\nabla\overline{u} + r(\overline{u})\overline{u}\nabla\phi\right) \\
&= \operatorname{div}\left(\nabla Q(\overline{u}) + R(\overline{u})\nabla\phi\right),
\end{aligned} \tag{12}$$

where Q and R are defined in (11). Clearly, it holds

$$\nabla Q(\overline{u}) \cdot \nu = \sum_{i=1}^n a_i \sum_{j=1}^n A_{ij} \nabla u_j \cdot \nu = 0 \quad \text{on } \partial\Omega.$$

In view of condition (7), the function Q is nondecreasing. We use the H^{-1} method to prove that from (2), (12) possesses at most one solution. Let $(\overline{u}^{(1)}, \phi^{(1)})$ and $(\overline{u}^{(2)}, \phi^{(2)})$

be two solutions to (2), (12), subject to no-flux boundary conditions and the same initial condition (3). We set $w = \overline{u}^{(1)} - \overline{u}^{(2)}$ and $\psi = \phi^{(1)} - \phi^{(2)}$. It holds that $\int_\Omega w \mathrm{d}x = 0$, and ψ solves

$$-\Delta\psi = w \quad \text{in } \Omega, \quad \nabla\psi\cdot\nu = 0 \quad \text{on } \partial\Omega, \quad \int_\Omega \psi \mathrm{d}x = 0.$$

Since $w \in L^2(\Omega\times(0,T))$ and $\partial_t w \in L^2(0,T;H^1(\Omega)')$, we have $\psi \in L^2(0,T;H^2(\Omega))$ and $\partial_t\Delta\psi \in L^2(0,T;H^1(\Omega)')$. By applying a standard mollification procedure, we can prove that $t\mapsto \|\nabla\psi(t)\|^2_{L^2(\Omega)}$ is continuous on $[0,T]$ (possibly after redefinition on a set of measure zero) and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\psi(t)\|^2_{L^2(\Omega)} = -\langle\partial_t\Delta\psi(t),\psi(t)\rangle = \langle\partial_t w(t),\psi(t)\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the dual product between $H^1(\Omega)'$ and $H^1(\Omega)$. Observe that at time $t=0$, $-\Delta\psi(0)=0$ and hence, $\psi(0)=0$. Using ψ as a test function in the difference of the weak formulations of (12) for $\overline{u}^{(1)}$ and $\overline{u}^{(2)}$, respectively, it follows that

$$\begin{aligned}
\frac{1}{2}\|\nabla\psi(t)\|^2_{L^2(\Omega)} &= -\int_0^t\int_\Omega \nabla\big(Q(\overline{u}^{(1)}) - Q(\overline{u}^{(2)})\big)\cdot\nabla\psi\mathrm{d}x\mathrm{d}s \\
&\quad -\int_0^t\int_\Omega \big(R(\overline{u}^{(1)})\nabla\phi^{(1)} - R(\overline{u}^{(2)})\nabla\phi^{(2)}\big)\cdot\nabla\psi\mathrm{d}x\mathrm{d}s \\
&= -\int_0^t\int_\Omega \big(Q(\overline{u}^{(1)}) - Q(\overline{u}^{(2)})\big)(\overline{u}^{(1)} - \overline{u}^{(2)})\mathrm{d}x\mathrm{d}s \\
&\quad -\int_0^t\int_\Omega R(\overline{u}^{(1)})|\nabla\psi|^2\mathrm{d}x\mathrm{d}s \\
&\quad -\int_0^t\int_\Omega \big(R(\overline{u}^{(1)}) - R(\overline{u}^{(2)})\big)\nabla\phi^{(2)}\cdot\nabla\psi\mathrm{d}x\mathrm{d}s. \qquad (13)
\end{aligned}$$

The second integral on the right-hand side is estimated as

$$\begin{aligned}
\left|\int_0^t\int_\Omega R(\overline{u}^{(1)})|\nabla\psi|^2\mathrm{d}x\mathrm{d}s\right| &\le \|R(\overline{u}^{(1)})\|_{L^\infty(0,T;L^\infty(\Omega))}\|\nabla\psi\|^2_{L^2(Q_T)} \\
&\le C_1\|\nabla\psi\|^2_{L^2(Q_T)}
\end{aligned}$$

for some constant $C_1>0$, where $Q_T=\Omega\times(0,T)$. This is the only place where the boundedness of $\overline{u}^{(1)}$ is needed.

To estimate the last integral in (13), we use the assumption $\nabla\phi^{(2)}\in L^\infty(0,T;L^\infty(\Omega))$ and Young's inequality with $\varepsilon>0$:

$$\begin{aligned}
&\left|\int_0^t\int_\Omega \big(R(\overline{u}^{(1)}) - R(\overline{u}^{(2)})\big)\nabla\phi^{(2)}\cdot\nabla\psi\mathrm{d}x\mathrm{d}s\right| \\
&\quad \le C_2\int_0^t\int_\Omega \big|R(\overline{u}^{(1)}) - R(\overline{u}^{(2)})\big||\nabla\psi|\mathrm{d}x\mathrm{d}s
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^t \int_\Omega \big((Q(\overline{u}^{(1)}) - Q(\overline{u}^{(2)}))\overline{u} + \varepsilon\big)\mathrm{d}x\mathrm{d}s \\
&\quad + \frac{C_2^2}{4}\int_0^t \int_\Omega \frac{(R(\overline{u}^{(1)}) - R(\overline{u}^{(2)}))^2}{(Q(\overline{u}^{(1)}) - Q(\overline{u}^{(2)}))\overline{u} + \varepsilon}|\nabla\psi|^2\mathrm{d}x\mathrm{d}s. \qquad (14)
\end{aligned}$$

We claim that the quotient is bounded. Indeed, by assumption (8), $(R')^2 \leq MQ'$ on $[0, L]$ and hence, by Hölder's inequality,

$$\begin{aligned}
\left(\int_0^1 R'(\theta\overline{u}^{(1)} + (1-\theta)\overline{u}^{(2)})d\theta\right)^2 &\leq \int_0^1 \big(R'(\theta\overline{u}^{(1)} + (1-\theta)\overline{u}^{(2)})\big)^2\mathrm{d}x \\
&\leq C_3\int_0^1 Q'(\theta\overline{u}^{(1)} + (1-\theta)\overline{u}^{(2)})d\theta.
\end{aligned}$$

This shows that

$$\frac{(R(\overline{u}^{(1)}) - R(\overline{u}^{(2)}))^2}{(Q(\overline{u}^{(1)}) - Q(\overline{u}^{(2)}))\overline{u} + \varepsilon} = \frac{\left(\int_0^1 R'(\theta\overline{u}^{(1)} + (1-\theta)\overline{u}^{(2)})d\theta\right)^2\overline{u}^2}{\int_0^1 Q'(\theta\overline{u}^{(1)} + (1-\theta)\overline{u}^{(2)})d\theta\overline{u}^2 + \varepsilon} \leq C_3.$$

Then (14) becomes

$$\begin{aligned}
&\left|\int_0^t \int_\Omega \big(R(\overline{u}^{(1)}) - R(\overline{u}^{(2)})\big)\nabla\phi^{(2)}\cdot\nabla\psi\mathrm{d}x\mathrm{d}s\right| \\
&\quad \leq \int_0^t \int_\Omega \big((Q(\overline{u}^{(1)}) - Q(\overline{u}^{(2)}))\overline{u} + \varepsilon\big)\mathrm{d}x\mathrm{d}s \\
&\qquad + \frac{1}{4}C_2^2C_3\int_0^t \int_\Omega |\nabla\psi|^2\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

In the limit $\varepsilon \to 0$, we obtain

$$\begin{aligned}
\left|\int_0^t \int_\Omega \big(R(\overline{u}^{(1)}) - R(\overline{u}^{(2)})\big)\nabla\phi^{(2)}\cdot\nabla\psi\mathrm{d}x\mathrm{d}s\right| &\leq \int_0^t \int_\Omega (Q(\overline{u}^{(1)}) - Q(\overline{u}^{(2)}))\overline{u}\mathrm{d}x\mathrm{d}s \\
&\quad + \frac{1}{4}C_2^2C_3\|\nabla\psi\|^2_{L^2(Q_T)}.
\end{aligned}$$

The first integral on the right-hand side is absorbed by the first integral on the right-hand side of (13), and we end up with

$$\|\nabla\psi(t)\|^2_{L^2(\Omega)} \leq C_4\int_0^t \|\nabla\psi\|^2_{L^2(\Omega)}\mathrm{d}s,$$

where $C_4 = 2C_1 + C_2^2C_3/2$. Finally, by Gronwall's lemma, it follows that $\nabla\psi(t) = 0$ in Ω and $w(t) = -\Delta\psi(t) = 0$. Since $\int_\Omega \psi(t)\mathrm{d}x = 0$, we also have $\psi(t) = 0$ for $t \in (0, T)$. This shows that from (2), (12) is uniquely solvable.

Step 2. Uniqueness for u_i. Let $u^{(1)} = \left(u_1^{(1)}, \ldots, u_n^{(1)}\right)$ and $u^{(2)} = \left(u_1^{(2)}, \ldots, u_n^{(2)}\right)$ be two weak solutions to (1). In this step, the solutions are not required to be bounded. We set $\overline{u}^{(1)} = \sum_{i=1}^n a_i u_i^{(1)}$, $\overline{u}^{(2)} = \sum_{i=1}^n a_i u_i^{(2)}$. Step 1 shows that $\overline{u} := \overline{u}^{(1)} = \overline{u}^{(2)}$ and the corresponding potential ϕ is unique. Then $u^{(1)}$ and $u^{(2)}$ solve, respectively,

$$\partial_t u_i^{(j)} = \operatorname{div}\left(p(\overline{u})\nabla u_i^{(j)} + u_i^{(j)} F\right), \quad j = 1, 2, \tag{15}$$

with corresponding no-flux and initial conditions, where $F = q(\overline{u})\nabla\overline{u} + r(\overline{u})\nabla\phi \in L^2(Q_T)$. Let $0 < \varepsilon < 1$. We introduce, as in [32], the regularized entropy

$$h_\varepsilon(s) = (s + \varepsilon)\big(\log(s + \varepsilon) - 1\big) + 1, \quad s \geq 0,$$

and the semimetric

$$d_\varepsilon(u, v) = \sum_{i=1}^n \int_\Omega \left(h_\varepsilon(u_i) + h_\varepsilon(v_i) - 2h_\varepsilon\left(\frac{u_i + v_i}{2}\right)\right)\mathrm{d}x$$

for appropriate functions $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$. Since h_ε is convex, we have $d_\varepsilon(u, v) \geq 0$.

We recall the following result. Let $0 \leq w \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^1(\Omega)')$. Then $t \mapsto \int_\Omega h_\varepsilon(w(t))\mathrm{d}x$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega h_\varepsilon(w(t))\mathrm{d}x = \langle \partial_t w, \log(w + \varepsilon)\rangle.$$

Therefore, we can differentiate $t \mapsto d_\varepsilon(u^{(1)}(t), u^{(2)}(t))$, yielding

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} d_\varepsilon(u^{(1)}, u^{(2)}) &= \sum_{i=1}^n \Bigg(\langle \partial_t u_i^{(1)}, \log(u_i^{(1)} + \varepsilon)\rangle + \langle \partial_t u_i^{(2)}, \log(u_i^{(2)} + \varepsilon)\rangle \\
&\quad - \left\langle \partial_t\big(u_i^{(1)} + u_i^{(2)}\big), \log\left(\frac{u_i^{(1)} + u_i^{(2)}}{2} + \varepsilon\right)\right\rangle\Bigg) \\
&= -\sum_{i=1}^n \int_\Omega \big(p(\overline{u})\nabla u_i^{(1)} + u_i^{(1)} F\big) \cdot \frac{\nabla u_i^{(1)}}{u_i^{(1)} + \varepsilon}\mathrm{d}x \\
&\quad - \sum_{i=1}^n \int_\Omega \big(p(\overline{u})\nabla u_i^{(2)} + u_i^{(2)} F\big) \cdot \frac{\nabla u_i^{(2)}}{u_i^{(2)} + \varepsilon}\mathrm{d}x \\
&\quad + \sum_{i=1}^n \int_\Omega \big(p(\overline{u})\nabla\big(u_i^{(1)} + u_i^{(2)}\big) + \big(u_i^{(1)} + u_i^{(2)}\big)F\big)\frac{\nabla\big(u_i^{(1)} + u_i^{(2)}\big)}{u_i^{(1)} + u_i^{(2)} + 2\varepsilon}\mathrm{d}x.
\end{aligned}$$

Rearranging the terms, we end up with

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}d_\varepsilon(u^{(1)},u^{(2)}) &= -\sum_{i=1}^n\int_\Omega p(\overline{u})\Big(\frac{|\nabla u_i^{(1)}|^2}{u_i^{(1)}+\varepsilon}+\frac{|\nabla u_i^{(2)}|^2}{u_i^{(2)}+\varepsilon}-\frac{|\nabla\big(u_i^{(1)}+u_i^{(2)}\big)|^2}{u_i^{(1)}+u_i^{(2)}+2\varepsilon}\Big)\mathrm{d}x\\ &\quad-\sum_{i=1}^n\int_\Omega F\cdot\nabla u_i^{(1)}\Big(\frac{u_i^{(1)}}{u_i^{(1)}+\varepsilon}-\frac{u_i^{(1)}+u_i^{(2)}}{u_i^{(1)}+u_i^{(2)}+2\varepsilon}\Big)\mathrm{d}x\\ &\quad-\sum_{i=1}^n\int_\Omega F\cdot\nabla u_i^{(2)}\Big(\frac{u_i^{(2)}}{u_i^{(2)}+\varepsilon}-\frac{u_i^{(1)}+u_i^{(2)}}{u_i^{(1)}+u_i^{(2)}+2\varepsilon}\Big)\mathrm{d}x.\end{aligned}$$

Since for suitable functions u, v,

$$\frac{|\nabla u|^2}{u+\varepsilon}+\frac{|\nabla v|^2}{v+\varepsilon}-\frac{|\nabla(u+v)|^2}{u+v+2\varepsilon}=\frac{1}{u+v+2\varepsilon}\bigg|\sqrt{\frac{v+\varepsilon}{u+\varepsilon}}\nabla u-\sqrt{\frac{u+\varepsilon}{v+\varepsilon}}\nabla v\bigg|^2,$$

the first term is nonpositive. Then, integrating in time and observing that $d_\varepsilon(u^{(1)}(0), u^{(2)}(0))=0$, it follows that

$$\begin{aligned}d_\varepsilon(u^{(1)}(t),u^{(2)}(t)) &\le -\sum_{i=1}^n\int_0^t\int_\Omega F\cdot\nabla u_i^{(1)}\Big(\frac{u_i^{(1)}}{u_i^{(1)}+\varepsilon}-\frac{u_i^{(1)}+u_i^{(2)}}{u_i^{(1)}+u_i^{(2)}+2\varepsilon}\Big)\mathrm{d}x\mathrm{d}s\\ &\quad-\sum_{i=1}^n\int_0^t\int_\Omega F\cdot\nabla u_i^{(2)}\Big(\frac{u_i^{(2)}}{u_i^{(2)}+\varepsilon}-\frac{u_i^{(1)}+u_i^{(2)}}{u_i^{(1)}+u_i^{(2)}+2\varepsilon}\Big)\mathrm{d}x\mathrm{d}s.\end{aligned}\tag{16}$$

Expanding $h_\varepsilon\big(u_i^{(1)}\big)$ and $h_\varepsilon\big(u_i^{(2)}\big)$ at $\big(u_i^{(1)}+u_i^{(2)}\big)/2$ up to second order and summing the resulting expressions, we find that, for some $\theta_i^{(k)}\in(0,1)$ ($k=1,2$) and $\xi_i^{(k)}=\theta_i^{(k)}u_i^{(k)}+\big(1-\theta_i^{(k)}\big)\big(u_i^{(1)}+u_i^{(2)}\big)/2$,

$$\begin{aligned}d_\varepsilon(u^{(1)},u^{(2)}) &= \frac{1}{8}\sum_{i=1}^n\int_\Omega\big(h''_\varepsilon(\xi_i^{(1)})+h''_\varepsilon(\xi_i^{(2)})\big)\big(u_i^{(1)}-u_i^{(2)}\big)^2\mathrm{d}x\\ &\ge\frac{1}{4}\sum_{i=1}^n\int_\Omega\frac{\big(u_i^{(1)}-u_i^{(2)}\big)^2}{\max\{u_i^{(1)},u_i^{(2)}\}+\varepsilon}\mathrm{d}x\ge\frac{1}{4}\sum_{i=1}^n\int_\Omega\frac{\big(u_i^{(1)}-u_i^{(2)}\big)^2}{\max\{u_i^{(1)},u_i^{(2)}\}+1}\mathrm{d}x.\end{aligned}$$

Since $F\cdot\nabla u_i^j\in L^1(Q_T)$ for $j=1,2$, we may apply the dominated convergence theorem giving, as $\varepsilon\to0$,

$$\int_0^t\int_\Omega F\cdot\nabla u_i^j\Big(\frac{u_i^j}{u_i^j+\varepsilon}-\frac{u_i^{(1)}+u_i^{(2)}}{u_i^{(1)}+u_i^{(2)}+2\varepsilon}\Big)\mathrm{d}x\mathrm{d}s\to0,\quad j=1,2.$$

Therefore, (16) becomes

$$0 \le \int_\Omega \frac{\big(u_i^{(1)} - u_i^{(2)}\big)(t)^2}{\max\{u_i^{(1)}(t), u_i^{(2)}(t)\} + 1} \mathrm{d}x = 0,$$

and thus, $u_i^{(1)}(t) = u_i^{(2)}(t) = 0$ for $t \in (0, T)$ since $u_i^{(j)}(t)$ is finite a.e. in Ω.

If $r \equiv 0$ and $\overline{u}$ is not bounded, then we need integrability (9) to make the computations rigorous. This concludes the proof of Theorem 1.

3. Remarks

We give two comments on the regularity of the drift term and on the relation of Gajewski's semimetric to relative entropies.

REMARK 3 *(Lower regularity of $\nabla\phi$).* We claim that the regularity on ϕ can be relaxed to $\nabla\phi \in L^\infty(0, T; L^\alpha(\Omega))$ with $\alpha > d$ if $p(s) + q(s)s = D = \text{const.} > 0$. For simplicity, we assume that $D = 1$. In this case, we do not need to apply the H^{-1} method and can use standard L^2 estimates. Let $(u^{(1)}, \phi^{(1)})$ and $(u^{(2)}, \phi^{(2)})$ be two solutions to (2), (12) with the same boundary and initial conditions. Taking $u^{(1)} - u^{(2)}$ as a test function in (12), we find that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (\overline{u}^{(1)} - \overline{u}^{(2)})^2(t)\mathrm{d}x + \int_0^t \int_\Omega |\nabla(\overline{u}^{(1)} - \overline{u}^{(2)})|^2 \mathrm{d}x\mathrm{d}s \\
&\quad = -\int_0^t \int_\Omega (R(\overline{u}^{(1)}) - R(\overline{u}^{(2)}))\nabla\phi^{(1)} \cdot \nabla(\overline{u}^{(1)} - \overline{u}^{(2)})\mathrm{d}x\mathrm{d}s \\
&\qquad - \int_0^t \int_\Omega R(\overline{u}^{(2)})\nabla(\phi^{(1)} - \phi^{(2)}) \cdot \nabla(\overline{u}^{(1)} - \overline{u}^{(2)})\mathrm{d}x\mathrm{d}s \\
&\quad =: I_1 + I_2.
\end{aligned} \tag{17}$$

By the boundedness of $u_i^{(2)}$ and the elliptic estimate for the Poisson equation, the second integral is estimated as

$$\begin{aligned}
I_2 &\le C_5 \|\nabla(\phi^{(1)} - \phi^{(2)})\|_{L^2(Q_t)} \|\nabla(\overline{u}^{(1)} - \overline{u}^{(2)})\|_{L^2(Q_t)} \\
&\le \frac{1}{4}\|\nabla(\overline{u}^{(1)} - \overline{u}^{(2)})\|_{L^2(Q_t)}^2 + C_6 \|\overline{u}^{(1)} - \overline{u}^{(2)}\|_{L^2(Q_t)}^2,
\end{aligned}$$

where $Q_t = \Omega \times (0, t)$ and $C_5 > 0$ depends on the L^∞ norm of $\overline{u}^{(2)}$. For the first integral I_1, we employ the Lipschitz continuity of R, the Cauchy–Schwarz inequality, the Hölder inequality, the Gagliardo–Nirenberg inequality with $\theta = d/2 - d/\beta \in (0, 1)$, and eventually the Young inequality with parameter θ:

$$\begin{aligned}
I_1 &\le \frac{1}{4}\|\nabla(\overline{u}^{(1)} - \overline{u}^{(2)})\|_{L^2(Q_t)}^2 + C_7 \int_0^t \|\overline{u}^{(1)} - \overline{u}^{(2)}\|_{L^\beta(\Omega)}^2 \|\nabla\phi^{(1)}\|_{L^\alpha(\Omega)}^2 \mathrm{d}s \\
&\le \frac{1}{4}\|\nabla(\overline{u}^{(1)} - \overline{u}^{(2)})\|_{L^2(Q_t)}^2 + C_8\big(\|\nabla\phi^{(1)}\|_{L^\infty(0,T;L^\alpha(\Omega))}\big)
\end{aligned}$$

$$\times \int_0^t \big(\|\nabla(\overline{u}^{(1)} - \overline{u}^{(2)})\|_{L^2(\Omega)}^{2\theta} \|\overline{u}^{(1)} - \overline{u}^{(2)}\|_{L^2(\Omega)}^{2(1-\theta)} + \|\overline{u}^{(1)} - \overline{u}^{(2)}\|_{L^2(\Omega)}^2\big)\mathrm{d}s$$
$$\le \frac{1}{2}\|\nabla(\overline{u}^{(1)} - \overline{u}^{(2)})\|_{L^2(Q_t)}^2 + C_9\big(\|\nabla\phi^{(1)}\|_{L^\infty(0,T;L^\alpha(\Omega))}\big)\int_0^t \|\overline{u}^{(1)} - \overline{u}^{(2)}\|_{L^2(\Omega)}^2\mathrm{d}s.$$

Therefore, (13) becomes

$$\|(\overline{u}^{(1)} - \overline{u}^{(2)})(t)\|_{L^2(\Omega)}^2 \le C_{10}\int_0^t \|\overline{u}^{(1)} - \overline{u}^{(2)}\|_{L^2(\Omega)}^2\mathrm{d}s,$$

and Gronwall's lemma shows that $(\overline{u}^{(1)} - \overline{u}^{(2)})(t) = 0$ in Ω, $t > 0$. □

REMARK 4 *(Comparison of Gajewski's semimetric and relative entropies).* In the second step of the proof of Theorem 1, we may work with another semimetric, based on the relative entropy

$$H(u|v) = H(u) - H(v) - H'(v)\cdot(u - v),$$

as done in [13], where $H(u) = \sum_{i=1}^n \int_\Omega h(u_i)\mathrm{d}x$ with $h(u_i) = u_i(\log u_i - 1) + 1$. Setting $h(u) = (h(u_1), \ldots, h(u_n))$ by a slight abuse of notation, we see that $h : \mathbb{R}^n \to \mathbb{R}^n$ is a convex function. Instead of the expression from [13], we use its symmetrized version to obtain a semimetric:

$$d_0(u, v) = H(u|v) + H(v|u) = \int_\Omega (h'(u) - h'(v))\cdot(u - v)\mathrm{d}x. \tag{18}$$

Semimetrics (5) and (18) are strongly related although they are different. First, both expressions behave like $|u-v|^2$ for "small" $|u-v|$, since a Taylor expansion shows that both semimetrics can be estimated from below by, up to a factor, $(u-v)^\top h''(\xi)(u-v)$, where $h''(\xi)$ is the Hessian of h at some point $\xi \in \mathbb{R}^n$. Second, when differentiating $d_0(u^{(1)}, u^{(2)})$ with respect to time and inserting (1), the drift terms cancel, as they do when differentiating $d(u^{(1)}, u^{(2)})$. A formal computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}d_0(u^{(1)}, u^{(2)}) = -\sum_{i=1}^n \int_\Omega p(\overline{u})(u_i + v_i)\left|\nabla\log\frac{u_i}{v_i}\right|^2\mathrm{d}x \le 0,$$

implying that $u^{(1)} = u^{(2)}$. In order to make this argument rigorous, we need to work as in Sect. 2 with a regularization (replacing $u_i^{(j)}$ by $u_i^{(j)} + \varepsilon$).

In fact, the previous argument can be generalized to the following family of semimetrics. Let $d_1(u, v) = \int_\Omega g(u, v)\mathrm{d}x$ for some smooth symmetric convex function g, and let $u^{(1)}$ and $u^{(2)}$ be two solutions to the scalar equation

$$\partial_t u = \mathrm{div}(a(x)\nabla u + uF(x)), \tag{19}$$

which resembles (15), with no-flux boundary conditions and the same initial condition. We assume that $a(x) \ge 0$ and $F(x) \in \mathbb{R}^n$. Set

$$g_{11} = \frac{\partial^2 g}{\partial u^2}(u^{(1)}, u^{(2)}), \quad g_{12} = \frac{\partial^2 g}{\partial u\partial v}(u^{(1)}, u^{(2)}), \quad g_{22} = \frac{\partial^2 g}{\partial v^2}(u^{(1)}, u^{(2)}).$$

Then, formally,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&d_1(u^{(1)}, u^{(2)})\\ &= -\int_\Omega a(x)\big(g_{11}|\nabla u^{(1)}|^2 + 2g_{12}\nabla u^{(1)}\cdot\nabla u^{(2)} + g_{22}|\nabla u^{(2)}|^2\big)\mathrm{d}x\\ &\quad -\int_\Omega F(x)\cdot\big((u^{(1)}g_{11} + u^{(2)}g_{12})\nabla u + (u^{(1)}g_{12} + u^{(2)}g_{22})\nabla u^{(2)}\big)\mathrm{d}x.\end{aligned}$$

Since g is convex, the first integral is nonnegative. If we assume that

$$u\frac{\partial^2 g}{\partial u^2} + v\frac{\partial^2 g}{\partial u\partial v} = 0 \quad \text{for all } u, v, \tag{20}$$

then the second integral vanishes (using the symmetry of g) and consequently, $\frac{\mathrm{d}}{\mathrm{d}t}d_1(u^{(1)}, u^{(2)}) \le 0$, which implies that $u^{(1)} = u^{(2)}$. The integrands of semimetrics (5) and (18) satisfy condition (20). This argument shows that the linearity in the diffusion term of (19) is essential for the entropy method. □

4. Examples

Theorem 1 can be applied to some cross-diffusion systems arising in applications.

4.1. Maxwell–Stefan equations

The first example is the Maxwell–Stefan equations [24,31]

$$\partial_t u_i + \operatorname{div} J_i = 0, \quad \nabla u_i = -\sum_{j=1,\, j\neq i}^{n+1} d_{ij}(u_j J_i - u_i J_j), \quad i = 1, \ldots, n+1, \tag{21}$$

where J_i are the fluxes and d_{ij} the diffusion coefficients. For a formal derivation, see [18, Section 4.2]. We assume that the sum of all concentrations is constant, $\sum_{i=1}^{n+1} u_i = 1$, which implies that $\sum_{i=1}^{n+1} J_i = 0$. In contrast to (1), the fluxes are not a linear combination of the gradients ∇u_i, and we need to invert the flux–gradient relations. However, because of $\sum_{i=1}^{n+1} J_i = 0$, the relations cannot be directly inverted. One idea is to remove the variable $u_{n+1} = 1 - \sum_{i=1}^{n} u_i$, ending up with n equations, formulated as $\nabla u' = -A_0 J'$ [19], where $u' = (u_1, \ldots, u_n)$, $J' = (J_1, \ldots, J_n)$, and $A_0 = (A^0_{ij}) \in \mathbb{R}^{n\times n}$ with

$$\begin{aligned}A^0_{ij} &= -(d_{ij} - d_{i,n+1})u_i, \quad i \neq j,\ i, j = 1, \ldots, n,\\ A^0_{ii} &= \sum_{j=1,\, j\neq i}^{n} (d_{ij} - d_{i,n+1})u_j + d_{i,n+1}, \quad i = 1, \ldots, n.\end{aligned}$$

is invertible. The existence of global bounded weak solutions was shown in [19].

COROLLARY 5 (Maxwell–Stefan model). *Let $d_{ij} = D_0$ and $d_{i,n+1} = D$ for $i, j = 1, \ldots, n$. Then from Maxwell–Stefan system (3), (21) has at most one weak solution.*

Proof. By assumption, we have

$$A^0_{ij} = \delta_{ij}\Bigg(D + (D_0 - D)\sum_{k=1,\, k\neq i}^{n} u_k\Bigg) - (1-\delta_{ij})(D_0 - D)u_i.$$

A computation shows that the inverse $A(u) = A_0^{-1}$ is given by

$$A_{ij}(u) = \frac{\delta_{ij} D + (D_0 - D)u_i}{D^2 + D(D_0 - D)\sum_{k=1}^n u_i}.$$

This expression is of form (4) with $a_i = 1$ and

$$p(s) = \frac{D}{D^2 + D(D_0 - D)s}, \quad q(s) = \frac{D_0 - D}{D^2 + D(D_0 - D)s}.$$

The assumptions of Theorem 1 are satisfied since $r(s) = 0$ and

$$p(s) \geq \frac{1}{\max\{D_0, D\}} > 0, \quad p(s) + q(s)s = \frac{1}{D} > 0.$$

This concludes the proof. □

4.2. Shigesada–Kawasaki–Teramoto equations

The second example is Shigesada–Kawasaki–Teramoto system (1) arising in population dynamics [30] with coefficients

$$A_{ij}(u) = \delta_{ij}\Bigg(a_{i0} + \sum_{j=1}^{n} a_{ij}u_j\Bigg) + a_{ij}u_i, \quad B_{ij}(u) = \delta_{ij}u_i, \quad i, j = 1, \ldots, n, \tag{22}$$

where $a_{ij} > 0$ for $i = 0, \ldots, n$, $j = 1, \ldots, n$. The variables u_i model population densities of interacting species subject to some environmental potential. A formal derivation was given in [18, Section 4.2]. The existence of global weak solutions was proved in [9] (with $B_{ij} = 0$) under the assumption that there exists a vector $(\pi_1, \ldots, \pi_n)$ such that the detailed-balance condition $\pi_i a_{ij} = \pi_j a_{ij}$ for all $i, j = 1, \ldots, n$ holds or the self-diffusion a_{ii} dominates cross-diffusion a_{ij} $(i \neq j)$. Under additional conditions and for $n = 2$, the weak solutions are bounded [20].

COROLLARY 6 (Population dynamics model). *Let $a_{i0} = a_0 > 0$ and $a_{ij} = a_j > 0$ for $i, j = 1, \ldots, n$. Then from (1)–(3), (22) has at most one bounded weak solution.*

Note that under the conditions of the corollary, the detailed-balance condition is satisfied with $\pi_i = a_i$. The corollary follows from Theorem 1 by setting $p(s) = a_0 + s \geq a_0 > 0$, $q(s) = 1$, and $r(s) = 1$.

4.3. A volume-filling model for ion transport

The ion-transport model is defined by

$$A_{ij}(u) = D_i u_i \quad \text{for } i \neq j, \quad A_{ii}(u) = D_i(1-\overline{u}+u_i), \quad B_{ij} = z_i(1-\overline{u})u_i\delta_{ij}, \quad (23)$$

where $\overline{u} = \sum_{i=1}^n u_i$ and $D_i > 0$, $z_i \in \mathbb{R}$ are some constants [7]. The variables u_i represent the ion concentration of the ith species and $u_{n+1} := 1 - \overline{u}$ the solvent concentration. The model can be derived formally from a random-walk lattice model [18,26]. The existence of global bounded weak solutions was shown in [32] without potential and in [15] including the potential term. Formulation (4) is obtained for $D_i = D > 0$ and $z_i = z \in \mathbb{R}$ by setting $a_i = 1$, $p_i(s) = D(1-s)$, $q_i(s) = D$, and $r_i(s) = z(1-s)$. The following result was already proved in [15]. We show here that the model fits in our framework.

COROLLARY 7 (Ion-transport model). *Let $D_i = D > 0$ and $z_i = z \in \mathbb{R}$ for $i = 1, \ldots, n$. Then from (1)–(3), (23) has at most one bounded weak solution with $\nabla\phi \in L^\infty(0,T; L^\alpha(\Omega))$ and $\alpha > d$.*

Proof. Conditions (7)–(8) are satisfied since $p(s) + q(s)s = D > 0$ and $r(s)$ is continuous on $[0,1]$. By Remark 3, the uniqueness result holds for potentials satisfying $\nabla\phi \in L^\infty(0,T; L^\alpha(\Omega))$ with $\alpha > d$. □

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Xiuqing Chen
School of Sciences
Beijing University of Posts and Telecommunications
Beijing 100876
China
E-mail: buptxchen@yahoo.com

Ansgar Jüngel
Institute for Analysis and Scientific Computing
Vienna University of Technology
Wiedner Hauptstrasse 8–10
1040 Wien
Austria
E-mail: juengel@tuwien.ac.at