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A Note on the (Weighted) Bivariate Poisson Distribution

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Abstract. In the recent statistical literature, the univariate Poisson distribution has been generalized by many authors, among them: the univariate weighted Poisson distribution [13], the generalized univariate Poisson distribution [7], the bivariate Poisson distribution according to Holgate [11], the bivariate Poisson distribution according to Lakshminarayana, Pandit and Srinivasa Rao [15], the bivariate Poisson distribution according to Berkhout and Plug [4], the bivariate weighted Poisson distribution according to Elion et *al.* [8] and the generalized bivariate Poisson distribution according to Famoye [9]. In this paper, We highlight the weighted bivariate Poisson distribution and show that it is the synthesis of all the bivariate Poisson distributions which, under certain conditions, converge in distribution towards the bivariate Poisson distribution according to Berkhout and Plug [4] which can be considered like the standard distribution in $\mathbb{N}^2$ as is the univariate Poisson distribution in $\mathbb{N}$.

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1. Introduction

The bivariate Poisson distribution was discussed for the first time by Campbell [6] who considered the limit of the distribution of a two-dimensional contingency table. Practically, at the same period Guldberg [10] obtains the bivariate distribution of independent Poisson distributions as the limit of the distribution of independent binomial distributions. The explicit form of the bivariate Poisson distribution is due a few years later to Aitken [1]. We had to wait Holgate [11] to obtain a bivariate Poisson variable from three independent univariate Poisson variables, i.e. with a non-diagonal variance-covariance matrix. A few years later, Kawamura [12] considered the structure of a bivariate Poisson distribution as the limit of a bivariate Bernoulli distribution and found the results of Holgate. We can refer to Morin [17] for a better edification.

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Several authors have studied bivariate Poisson distributions, in particular Berkhout & Plug [4] and Lakshminarayana et *al.* [15]. Elion et *al.* [8] through the crossing of two weighted Poisson distributions revealed the bivariate weighted Poisson distribution. Batsindila Nganga et *al.* [2] showed that the bivariate Poisson distribution according to Holgate converges in distribution to the bivariate Poisson distribution according to Berkhout & Plug [4].
In this paper, we highlight the important role played by the bivariate Poisson distribution according to Berkhout & Plug [4] which allows to generate all the bivariate Poisson distributions. We show that the bivariate weighted Poisson distribution evidenced by Elion et *al.* [8] is a weighted bivariate Poisson distribution. We highlight the weighted bivariate Poisson distribution and show that it is the synthesis of all the bivariate Poisson distributions which, under certain conditions, converge in distribution towards the bivariate Poisson distribution according to Berkhout & Plug [4] which can be considered like the standard distribution in $\mathbb{N}^2$ as is the univariate Poisson distribution in $\mathbb{N}$.
The rest of this paper is organized as follows. In sections 2 and 3, we respectively recall the notion of univariate weighted Poisson distribution and the notion of generalized Poisson distribution. In section 4, we review the bivariate Poisson distributions according to Berkhout & Plug [4], according to Holgate [11], according to Lakshminarayana et *al.* [15] and the generalized bivariate Poisson distribution according to Famoye [9] then we show that these distributions converge in distribution towards the bivariate Poisson distribution according to Berkhout & Plug [4]. In section 5, we construct the weighted bivariate Poisson distribution and show that under certain conditions, this distribution is equal to the bivariate Poisson distribution according to Berkhout & Plug [4]. The section 6 presents the conclusion of this paper.

2. Univariate weighted Poisson distribution

Suppose the realization y of the random variable Y of mass function $p(y;\delta)$ is recorded with a probability proportional to $\omega(y)$; the record y is the realization of a random variable Y^{ω} called *weighted version* of Y and which has the probability distribution:

$$p_{\omega}(y;\delta) = \mathbb{P}[Y^{\omega} = y] = \frac{\omega(y)}{\mathbb{E}_{\delta}[\omega(Y)]} p(y;\delta), \quad y \in \mathbb{N} := \{0,1,\cdots\}, \ \delta \in \mathbb{R}_{+}^{*} \tag{1}$$

called *weighted distribution* where $\omega(y)$ is called *weight function*, a positive function and $\mathbb{E}_{\delta}[\omega(Y)] = \sum_{y\in\mathbb{N}} \omega(y)\, p(y;\delta)$ *the constant of normalization* which is the mean relative to the distribution of Y depending on δ such that $0 < \mathbb{E}_{\delta}[\omega(Y)] < +\infty$. The function $\omega(y) = \omega(y;\phi)$ can depend on a parameter ϕ which represents the mechanism of saving data. Note that $\omega(y) = \omega(y;\delta,\phi)$ can also depend on the canonical parameter δ. The data of a weight function makes it possible to generate a weighted probability distribution [13]. In this case, we can say that this distribution is generated by the weight function.
In this paper, the distribution $p(y;\delta)$ will be called the basic distribution. When the basic distribution is equal to the univariate Poisson distribution of parameter δ, the Expression (1) is called the univariate weighted Poisson distribution.
The univariate weighted Poisson distribution has the following characteristics [3]:

$$\mathbb{E}_{\delta}(Y^{\omega}) = \delta\left(1 + \frac{d}{d\delta}\ln \mathbb{E}_{\delta}[\omega(Y)]\right)$$

$$\operatorname{var}(Y^{\omega}) = \mathbb{E}_{\delta}(Y) + \delta^{2}\frac{d^{2}}{d\delta^{2}}\ln \mathbb{E}_{\delta}[\omega(Y)].$$

Example 1. *The univariate COM-Poisson distribution [5] with a probability mass function:*

$$\mathbb{P}(Y = y|\lambda,\nu) = \frac{\lambda^{y}}{(y!)^{\nu}}\frac{1}{Z(\lambda,\nu)},\ y = 0.1,\ldots;\lambda > 0,\nu \geq 0,$$

is a univariate weighted Poisson distribution of weight function $\omega(y,\nu) = (y!)^{\nu-1}$ *and constant of normalization:* $\mathbb{E}[\omega(Y,\nu)] = e^{-\lambda}Z(\lambda,\nu)$*, with* $Z(\lambda,\nu) = \sum_{n=0}^{+\infty}\lambda^{n}/(n!)^{\nu}$*.*

3. Generalized Poisson distribution

The generalized Poisson distribution [7] of a random variable Y has the mass function:

$$\mathbb{P}(Y = y;\delta,\alpha) = \begin{cases} \frac{\delta^{y}}{y!}(1+\alpha y)^{y-1}e^{-\delta(1+\alpha y)}, & y \in \mathbb{N} \\ \\ 0 & \text{for } y > m \text{ if } \alpha < 0, \end{cases} \tag{2}$$

with $\max\left(-\delta^{-1},-m^{-1}\right) < \alpha < \delta^{-1}$, where m (≥ 4) is the largest positive integer such as $1+\alpha m > 0$, when $\alpha < 0$. This distribution has the following characteristics [7]:

$$\begin{aligned} \mathbb{E}_{\delta}(Y) &= \delta(1-\alpha\delta)^{-1} \\ \operatorname{var}(Y) &= \delta(1-\alpha\delta)^{-3} \\ \mathbb{E}_{\delta}\left(e^{-Y}\right) &= e^{\delta(s-1)},\ \text{with } \ln(s) - \alpha\delta(s-1) + 1 = 0. \end{aligned}$$

4. Bivariate Poisson distributions

4.1. Bivariate Poisson distribution according to Berkhout and Plug [4]

Let Y_i $(i = 1,2)$ a random variable which follows the univariate Poisson distribution with parameter δ_i $(i = 1,2)$. The vector (Y_1,Y_2) follows the bivariate Poisson distribution according to Berkhout and Plug [4] if its mass function denoted f_{BP} is equal to

$$f_{BP}(y_1,y_2;\delta_1,\delta_2) = \left(\frac{\delta_1^{y_1}}{y_1!}e^{-\delta_1}\right)\left(\frac{\delta_2^{y_2}}{y_2!}e^{-\delta_2}\right),\ y_1 \in \mathbb{N}, y_2 \in \mathbb{N}, \delta_1 \in \mathbb{R}_{+}^{*}, \delta_2 \in \mathbb{R}_{+}^{*}, \tag{3}$$

under the conditions

$$\ln\delta_1 = x'\beta_1 \tag{4}$$

and

$$\ln\delta_2 = x'\beta_2 + \eta y_1, \tag{5}$$

where β_1, β_2 and η are parameters and $x' = \left(x_1, x_2, \ldots, x_p\right)$ the vector of deterministic variables or factors. The Expression (4) results in $\mathbb{P}(Y_1 = y_1; \delta_1) = \left(\delta_1^{y_1}/y_1!\right)e^{-\delta_1}$ is a marginal distribution of Y_1 and the Expression (5) means that $\mathbb{P}(Y_2 = y_2; \delta_2) = \mathbb{P}(Y_2 = y_2/Y_1 = y_1) = \left(\delta_2^{y_2}/y_2!\right)e^{-\delta_2}$ is a conditional probability.

Thus, we have $f_{BP}(y_1, y_2; \delta_1, \delta_2) = \mathbb{P}(Y_1 = y_1; \delta_1)\mathbb{P}(Y_2 = y_2/Y_1 = y_1)$. When $\eta = 0$, then the variables Y_1 and Y_2 are independent. The generalized linear model of Expression (4) has for response variable Y_1 and the model of Expression (5) has for response variable Y_2. The resolution of these models makes it possible to highlight, not only the independence between the variables Y_1 and Y_2 but also the effect of the factor x' on these same variables. The bivariate Poisson distribution according to [4] has the following characteristics [3]:

$$\begin{aligned} \mathbb{E}_{\delta_1}(Y_1) &= \operatorname{var}(Y_1) = \delta_1 & (6)\\ \mathbb{E}_{\delta_2}(Y_2) &= e^{x'\beta_2 + c_2 + \delta_1(e^{\eta}-1)}, & (7) \end{aligned}$$

where c_2 is the intercept of the model (5).

$$\begin{aligned} \operatorname{var}(Y_2) &= \mathbb{E}_{\delta_2}[Y_2] + [\mathbb{E}_{\delta_2}(Y_2)]^2\left(e^{\delta_1(e^{\eta}-1)} - 1\right) & (8)\\ \operatorname{cov}(Y_1, Y_2) &= \delta_1\mathbb{E}_{\delta_2}[Y_2](e^{\eta} - 1). & (9) \end{aligned}$$

The expression (8) shows that the variable Y_2 is overdispersed. The Expression (9) confirms the fact that the variables Y_1 and Y_2 are independent if and only if $\eta = 0$. And the covariance is negative, zero or positive depending on whether η is negative, zero or positive.

4.2. Bivariate Poisson distribution according to Holgate [11]

Let be three univariate random variables V_1, V_2 and U independent of Poisson with respective parameters λ_1, λ_2 and λ_3. With these three variables, we construct two new dependent variables Y_1 and Y_2 such as:

$$Y_j = V_j + U, \text{ where } j = 1, 2. \tag{10}$$

Then the joint distribution of the couple (Y_1, Y_2) is written:

$$\mathbb{P}(Y_1 = y_1, Y_2 = y_2) = e^{-\lambda_1-\lambda_2-\lambda_3}\sum_{\ell=0}^{\min(y_1,y_2)}\frac{\lambda_3^{\ell}}{\ell!}\frac{\lambda_1^{y_1-\ell}}{(y_1-\ell)!}\frac{\lambda_2^{y_2-\ell}}{(y_2-\ell)!};\ y_1, y_2 = 0, 1, 2, \ldots \tag{11}$$

By setting $\delta_1 = \lambda_1 + \lambda_3$ and $\delta_2 = \lambda_2 + \lambda_3$, we have the following result [2]:

$$\mathbb{P}(Y_1 = y_1, Y_2 = y_2) = \left(\frac{\delta_1^{y_1}}{y_1!}e^{-\delta_1}\right)\left(\frac{\delta_2^{y_2}}{y_2!}e^{-\delta_2}\right) \times b(y_1, y_2; \delta_1, \delta_2, \lambda_3) \tag{12}$$

with

$$b(y_1, y_2; \delta_1, \delta_2, \lambda_3) = e^{\lambda_3}\left(1 - \frac{\lambda_3}{\delta_1}\right)^{y_1}\left(1 - \frac{\lambda_3}{\delta_2}\right)^{y_2}\sum_{\ell=0}^{\min(y_1,y_2)}(-y_1)^{[\ell]}(-y_2)^{[\ell]}\frac{z^{\ell}}{\ell!} \tag{13}$$

and $z = \lambda_3 / (\delta_1 - \lambda_3)(\delta_2 - \lambda_3)$, $(-y_1)^{[\ell]} = (-1)^{\ell} y_1! / (y_1 - \ell)!$. We denote the distribution given by the Expression (11) by $f_H(y_1, y_2, \lambda_1, \lambda_2, \lambda_3)$. The pair of variables (Y_1, Y_2) has the following characteristics [11]:

$$\begin{aligned} \mathbb{E}_{\delta_i}(Y_i) &= \operatorname{var}(Y_i) = \delta_i \ , (i = 1, 2) \qquad (14) \\ \operatorname{cov}(Y_1, Y_2) &= \lambda_3. \qquad (15) \end{aligned}$$

The marginal variable Y_i $(i = 1, 2)$ is a univariate Poisson variable with parameter δ_i $(i = 1, 2)$. The variables Y_1 and Y_2 are dependent because their covariance is strictly positive.
By taking

$$\frac{\delta_1^{y_1}}{y_1!} e^{-\delta_1} = \mathbb{P}[Y_1 = y_1],$$

as the marginal distribution of Y_1 and

$$\frac{\delta_2^{y_2}}{y_2!} e^{-\delta_2} = \mathbb{P}[Y_2 = y_2 / Y_1 = y_1],$$

as the conditional distribution of Y_2 when we consider $Y_1 = y_1$, under the constraints (4) and (5), we find:

$$\mathbb{P}[Y_1 = y_1, Y_2 = y_2] = \mathbb{P}[Y_1 = y_1]\,\mathbb{P}[Y_2 = y_2 / Y_1 = y_1], \qquad (16)$$

$$\mathbb{P}[Y_1 = y_1, Y_2 = y_2] = \mathbb{P}[Y_1 = y_1]\,\mathbb{P}[Y_2 = y_2 / Y_1 = y_1] = f_{BP}(y_1, y_2; \delta_1, \delta_2)$$

and

$$f_H(y_1, y_2; \delta_1, \delta_2, \lambda_3) = f_{BP}(y_1, y_2; \delta_1, \delta_2) \times b(y_1, y_2; \delta_1, \delta_2, \lambda_3), \qquad (17)$$

which are the results found by Batsindila Nganga et *al.* [2].

By setting $\lambda_3 = 1/n$ with $n \in \mathbb{N}^*$, Batsindila Nganga et *al.* [2] constructed the family of bivariate Poisson distributions according to Holgate $\{f_{H,n} / n \in \mathbb{N}^*\}$, with $f_{H,n}(y_1, y_2; \delta_1, \delta_2) = f_H(y_1, y_2; \delta_1, \delta_2, 1/n)$. By making n tend to infinity, we have the following results [2]:

$$\lim_{n \longrightarrow +\infty} b(y_1, y_2; \delta_1, \delta_2, 1/n) = 1 \qquad (18)$$

and

$$\lim_{n \longrightarrow +\infty} f_{H,n}(y_1, y_2; \delta_1, \delta_2) = f_{BP}(y_1, y_2; \delta_1, \delta_2). \qquad (19)$$

The bivariate Poisson distribution according to Holgate [11] converges in distribution to the bivariate Poisson distribution according to Berkhout & Plug [4].

4.3. Bivariate Poisson distribution according to Lakshminarayana et *al.* [15]

Lakshminarayana et *al.* [15] defined the bivariate Poisson distribution, which is the joint distribution of the pair of random variables (Y_1, Y_2), as the product of Poisson marginal distributions with a multiplicative factor. The probability mass function of this bivariate Poisson distribution that we denote by f_{LPS} is defined by:

$$f_{LPS}\left(y_1, y_2; \delta_1, \delta_2, \lambda\right) = \left(\frac{\delta_1^{y_1}}{y_1!}e^{-\delta_1}\right)\left(\frac{\delta_2^{y_2}}{y_2!}e^{-\delta_2}\right)\left[1 + \lambda\left(e^{-y_1} - e^{-d\delta_1}\right)\left(e^{-y_2} - e^{-d\delta_2}\right)\right], \quad (20)$$

with $y_1, y_2 \in \mathbb{N}, (\delta_1, \delta_2) \in \left(\mathbb{R}_+^*\right)^2$, $\lambda \in \mathbb{R}_+^*$ et $d = 1 - e^{-1}$.
This distribution has the characteristics (Lakshminarayana et *al.*, 1999):

$$\begin{aligned} \mathbb{E}_{\delta_i}\left(Y_i\right) &= \delta_i, \ (i = 1, 2) \\ \mathrm{cov}\left(Y_1, Y_2\right) &= \delta_1\delta_2 d^2 e^{-c(\delta_1+\delta_2)}. \end{aligned}$$

The marginal variables are Poisson with parameters δ_i $(i = 1, 2)$ et $e^{-d\delta_i} = \mathbb{E}_{\delta_i}\left(e^{Y_i}\right)$ $(i = 1, 2)$. We have the following result.

Proposition 1. *Taking into account Expressions* (3)*,* (4) *and* (5)*, we have the following expression.*

$$f_{LPS}\left(y_1, y_2; \delta_1, \delta_2, \lambda\right) = f_{BP}\left(y_1, y_2; \delta_1, \delta_2\right) \times \psi\left(y_1, y_2; \delta_1, \delta_2, \lambda\right), \quad (21)$$

with $\psi\left(y_1, y_2; \delta_1, \delta_2, \lambda\right) = 1 + \lambda\left(e^{-y_1} - e^{-d\delta_1}\right)\left(e^{-y_2} - e^{-d\delta_2}\right)$.

Proof. The proof is obvious.

Corollary 1. *By setting* $\lambda = \lambda_n, n \in \mathbb{N}$*, such that* $\lim_{n\longrightarrow+\infty} \lambda_n = 0$*, we build a family of the bivariate Poisson distributions according to Lakshminarayana et al. [15],* $\left\{f_{LPS,n}\left(y_1, y_2; \delta_1, \delta_2\right) / n \in \mathbb{N}^*\right\}$ *such that* $f_{LPS,n}\left(y_1, y_2; \delta_1, \delta_2\right) = f_{LPS}\left(y_1, y_2; \delta_1, \delta_2, \lambda_n\right)$*.*

We have $\lim_{n\longrightarrow+\infty} \psi\left(y_1, y_2; \delta_1, \delta_2, \lambda_n\right) = 1$ *and therefore*

$$\lim_{n\longrightarrow+\infty} f_{LPS,n}\left(y_1, y_2; \delta_1, \delta_2\right) = f_{BP}\left(y_1, y_2; \delta_1, \delta_2\right). \quad (22)$$

The bivariate Poisson distribution according to Lakshminarayana et al. [15] converges in distribution to the bivariate Poisson distribution according to Berkhout and Plug [4].

We can therefore notice, through Expression (21), that the bivariate Poisson distribution according to Lakshminarayana et *al.*[15] is the product of the bivariate Poisson distribution according to Berkhout & Plug with a multiplicative factor. The Expression (22) shows that the bivariate Poisson distribution according to Berkhout & Plug is a limit case of the bivariate Poisson distribution according to Lakshminarayana et *al.*[15]

4.4. Bivariate generalized Poisson distribution

Famoye [9] combines the generalized Poisson distribution of Consul & Jain [7] and the bivariate Poison distribution of Lakshminarayana et *al.* [15] to construct the distribution whose probability mass function is

$$\mathbb{P}(Y_1 = y_1, Y_2 = y_2) = \prod_{i=1}^{2}\left[\frac{\delta_i^{y_i}}{y_i!}(1+\alpha_i y_i)^{y_i-1}\, e^{-\delta_i(1+\alpha_i y_i)}\right]\left[1+\lambda\left(e^{-y_1}-c_1\right)\left(e^{-y_2}-c_2\right)\right], \tag{23}$$

with $c_i = \mathbb{E}\left(e^{-Y_i}\right)$, $y_i \in \mathbb{N}$, $\delta_i \in \mathbb{R}_+^*$, $\alpha_i \in \mathbb{R}$, $(i = 1, 2)$.
We will denote the distribution given in Expression (23) by $f_F(y_1, y_2, \delta_1, \delta_2, \alpha_1, \alpha_2, \lambda)$. This distribution has the following characteristics [9]:

$$\begin{aligned}\mathbb{E}_{\delta_i}(Y_i) &= \delta_i(1-\alpha_i\delta_i)^{-1}, \ i = 1,2\\ \operatorname{var}(Y_i) &= \delta_i(1-\alpha_i\delta_i)^{-3}, \ i = 1,2\\ \operatorname{cov}(Y_1, Y_2) &= \lambda(c_{11}-c_1\delta_1)(c_{22}-c_2\delta_2),\end{aligned}$$

with $c_{ii} = \mathbb{E}_{\delta_i}\left(Y_i e^{-Y_i}\right) = \delta_i(1-\alpha_i\theta_i s_i)^{-1} e^{\delta_i(1+\alpha_i)(s_i-1)-1}$ where $\ln(s_i) - \alpha_i\theta_i(s_i-1)+1 = 0 \ (i = 1,2)$.
We have the following result.

Proposition 2. *Under the conditions* (4) *and* (5)*, the Expression* (23) *becomes*

$$f_F(y_1, y_2, \delta_1, \delta_2, \alpha_1, \alpha_2, \lambda) = f_{BP}(y_1, y_2, \delta_1, \delta_2)\, \psi_F(y_1, y_2, \delta_1, \delta_2, \alpha_1, \alpha_2, \lambda), \tag{24}$$

where

$$\psi_F(y_1, y_2, \delta_1, \delta_2, \alpha_1, \alpha_2, \lambda) = \left(\prod_{i=1}^{2}(1+\alpha_i y_i)^{y_i-1} e^{-\alpha_i\delta_i y_i}\right)\left[1+\lambda\left(e^{-y_1}-c_1\right)\left(e^{-y_2}-c_2\right)\right]. \tag{25}$$

Proof. Note that Expression (23) can still be written

$$\begin{aligned}\mathbb{P}(Y_1 = y_1, Y_2 = y_2) &= \left(\prod_{i=1}^{2}\frac{\delta_i^{y_i}}{y_i!}e^{-\delta_i}\right)\left(\prod_{i=1}^{2}(1+\alpha_i y_i)^{y_i-1} e^{-\delta_i\alpha_i y_i}\right)\left[1+\lambda\left(e^{-y_1}-c_1\right)\left(e^{-y_2}-c_2\right)\right]\\ &= \left(\frac{\delta_1^{y_1}}{y_1!}e^{-\delta_1}\right)\left(\frac{\delta_2^{y_2}}{y_2!}e^{-\delta_2}\right)\left(\prod_{i=1}^{2}(1+\alpha_i y_i)^{y_i-1} e^{-\delta_i\alpha_i y_i}\right)\left[1+\lambda\left(e^{-y_1}-c_1\right)\left(e^{-y_2}-c_2\right)\right].\end{aligned}$$

Taking into account Expressions (3), (4) and (5), we get

$$\mathbb{P}(Y_1 = y_1, Y_2 = y_2) = f_{BP}(y_1, y_2; \delta_1, \delta_2)\left(\prod_{i=1}^{2}(1+\alpha_i y_i)^{y_i-1} e^{-\delta_i\alpha_i y_i}\right)\left[1+\lambda\left(e^{-y_1}-c_1\right)\left(e^{-y_2}-c_2\right)\right].$$

By setting

$$\psi_F(y_1, y_2, \delta_1, \delta_2, \alpha_1, \alpha_2, \lambda) = \left(\prod_{i=1}^{2}(1+\alpha_i y_i)^{y_i-1} e^{-\alpha_i\delta_i y_i}\right)\left[1+\lambda\left(e^{-y_1}-c_1\right)\left(e^{-y_2}-c_2\right)\right], \tag{26}$$

it follows that

$$f_F(y_1, y_2, \delta_1, \delta_2, \alpha_1, \alpha_2, \lambda) = f_{BP}(y_1, y_2, \delta_1, \delta_2)\, \psi_F(y_1, y_2, \delta_1, \delta_2, \alpha_1, \alpha_2, \lambda).$$

The proof is finished.

Corollary 2. *In Expression* (26)*, let* $\alpha_i = \alpha_{in}$*,* $n \in \mathbb{N}$ *with* $\lim_{n\longrightarrow+\infty} \alpha_{in} = 0 \ \ (i = 1, 2)$ *and* $\lambda = \lambda_n$*,* $n \in \mathbb{N}$ *with* $\lim_{n\longrightarrow+\infty} \lambda_n = 0$*. We can then build a family of Famoye distributions* $\{f_{F,n}/n \in \mathbb{N}\}$ *such that* $f_{F,n}(y_1, y_2, \delta_1, \delta_2, \alpha_1, \alpha_2,) = f_F(y_1, y_2, \delta_1, \delta_2, \alpha_{1n}, \alpha_{2n}, \lambda_n)$*.*
Like $\lim_{n\longrightarrow+\infty} \psi_F(y_1, y_2, \delta_1, \delta_2, \alpha_{1n}, \alpha_{2n}, \lambda_n) = 1$*, then*

$$\lim_{n\longrightarrow+\infty} f_{F,n}(y_1, y_2, \delta_1, \delta_2, \alpha_1, \alpha_2,) = f_{BP}(y_1, y_2, \delta_1, \delta_2),$$

the distribution of Famoye [9] converges in distribution towards the bivariate Poisson distribution according to Berkhout and Plug. Expression (24) *confirms that the distribution evidenced by Famoye [9] is a bivariate Poisson distribution.*

5. Weighted bivariate Poisson distribution

Definition 1. *Consider* $f_{BP}(y_1, y_2; \delta_1, \delta_2)$ *the basic distribution of the pair of random variables* (Y_1, Y_2)*. We call the weighted bivariate Poisson distribution, the probability mass function defined by:*

$$f^{\omega}(y_1, y_2; \delta_1, \delta_2, \lambda) = \frac{\omega(y_1, y_2; \delta_1, \delta_2, \lambda)}{\mathbb{E}_{\delta_1,\delta_2}[\omega(Y_1, Y_2; \delta_1, \delta_2, \lambda)]} \times f_{BP}(y_1, y_2; \delta_1, \delta_2), \tag{27}$$

where $\omega(y_1, y_2; \delta_1, \delta_2, \lambda)$ *is called the weight function, a positive function, and*

$$\mathbb{E}_{\delta_1,\delta_2}[\omega(Y_1, Y_2; \delta_1, \delta_2, \lambda)] = \sum_{y_1}\sum_{y_2} \omega(y_1, y_2; \delta_1, \delta_2, \lambda)\, f_{BP}(y_1, y_2; \delta_1, \delta_2)$$

the constant of normalization such that $0 < \mathbb{E}_{\delta_1,\delta_2}[\omega(Y_1, Y_2; \delta_1, \delta_2, \lambda)] < +\infty$*.*
Let

$$\psi(y_1, y_2; \delta_1, \delta_2, \lambda) = \frac{\omega(y_1, y_2; \delta_1, \delta_2, \lambda)}{\mathbb{E}_{\delta_1,\delta_2}[\omega(Y_1, Y_2; \delta_1, \delta_2, \lambda)]}, \tag{28}$$

the normalized weight function ([16], [14]). The expression (28) *results in*

$$\omega(y_1, y_2; \delta_1, \delta_2, \lambda) = \psi(y_1, y_2; \delta_1, \delta_2) \times \mathbb{E}_{\delta_1,\delta_2}[\omega(Y_1, Y_2; \delta_1, \delta_2, \lambda)]. \tag{29}$$

From the Expression (29)*, we can deduce that the constant of normalization* $\mathbb{E}_{\delta_1,\delta_2}[\omega(Y_1, Y_2; \delta_1, \delta_2, \lambda)]$ *makes it possible to calculate the weight functions and consequently it also generates the weighted bivariate Poisson distribution.*

Example 2. *suppose that* $\omega(y_1, y_2; \delta_1, \delta_2, \lambda) = \omega_1(y_1)\omega_2(y_2)$ *and* $\mathbb{E}_{\delta_1\delta_2}[\omega(Y_1, Y_2; \delta_1, \delta_2, \lambda)] = \mathbb{E}_{\delta_1}[\omega_1(Y_1)]\mathbb{E}_{\delta_2}[\omega_2(Y_2)]$ *. This last expression does not mean that the random variables* Y_1 *and* Y_2 *are independent. The mass function* $f^{\omega}(y_1, y_2; \delta_1, \delta_2)$ *given in Expression* (27) *is equal to:*

$$f^{\omega}(y_1, y_2; \delta_1, \delta_2) = \frac{\omega_1(y_1)}{\mathbb{E}_{\delta_1}[\omega_1(Y_1)]}\frac{\omega_2(y_2)}{\mathbb{E}_{\delta_2}[\omega_2(Y_2)]} \times f_{BP}(y_1, y_2; \delta_1, \delta_2). \tag{30}$$

The Expression (30) *is the crossing between two univariate weighted Poisson distributions. It is called the bivariate weighted Poisson distribution [8]. Expression* (30) *shows that the bivariate weighted Poisson distribution is a weighted bivariate Poisson distribution. Its characteristics are ([3]):*

$$\begin{aligned}
\mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right] &= e^{x'\beta_2 + c_2 + \delta_1(e^{\eta}-1)}\frac{\mathbb{E}_{e^{\eta}\delta_1}[\omega_1(Y_1)]}{\mathbb{E}_{\delta_1}[\omega_1(Y_1)]} \\
var\left(Y_2^{\omega_2}\right) &= \mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right] + \left[\mathbb{E}_{\delta_2}\left(Y_2^{\omega_2}\right)\right]^2\left(e^{\delta_1(e^{\eta}-1)}\frac{\mathbb{E}_{\delta_1}[\omega_1(Y_1)]\mathbb{E}_{\delta_1 e^{2\eta}}[\omega_1(Y_1)]}{\left(\mathbb{E}_{\delta_1 e^{\eta}}[\omega_1(Y_1)]\right)^2} - 1\right) \\
cov\left(Y_1^{\omega_1}, Y_2^{\omega_2}\right) &= \mathbb{E}_{\delta_2}\left[Y_2^{\omega_2}\right]\left(\delta_1 e^{\eta} + \frac{d}{d\eta}\left(\ln \mathbb{E}_{\delta_1 e^{\eta}}[\omega_1(Y_1)]\right) - \mathbb{E}_{\delta_1}\left[Y_1^{\omega_1}\right]\right).
\end{aligned}$$

Proposition 3. *If the univariate random variables* Y_1 *and* Y_2 *are punctually dual, then the bivariate weighted Poisson distribution given by Expression* (30) *is equal to the bivariate Poisson distribution* $f_{BP}(y_1, y_2; \delta_1, \delta_2)$.

Proof. If Y_1 and Y_2 are punctually dual [13], then $\omega_1(y_1)\omega_2(y_2) = 1$, $\forall (y_1, y_2) \in \mathbb{N}^2$. So $\mathbb{E}_{\delta_1}[\omega_1(Y_1)]\mathbb{E}_{\delta_2}[\omega_2(Y_2)] = 1$, therefore $f^{\omega}(y_1, y_2; \delta_1, \delta_2) = f_{BP}(y_1, y_2; \delta_1, \delta_2)$.

Example 3. *In Expression* (13)*, let*

$$\psi(y_1, y_2; \delta_1, \delta_2) = b(y_1, y_2; \delta_1, \delta_2, \lambda_3) = e^{\lambda_3}\left(1 - \tfrac{\lambda_3}{\delta_1}\right)^{y_1}\left(1 - \tfrac{\lambda_3}{\delta_2}\right)^{y_2}\sum_{\ell=0}^{\min(y_1,y_2)}(-y_1)^{[\ell]}(-y_2)^{[\ell]}\frac{z^{\ell}}{\ell!}.$$

From the Expression (28)*, if we take*

$$\mathbb{E}_{\delta_1,\delta_2}[\omega(Y_1, Y_2; \delta_1, \delta_2, \lambda)] = e^{-\lambda_3},$$

as the constant of normalization, then the weight function is equal to (Cf. Expression (29)*):*

$$\omega(y_1, y_2; \mu_1, \mu_2, \lambda) = \left(1 - \frac{\lambda_3}{\delta_1}\right)^{y_1}\left(1 - \frac{\lambda_3}{\delta_2}\right)^{y_2}\sum_{\ell=0}^{\min(y_1,y_2)}(-y_1)^{[\ell]}(-y_2)^{[\ell]}\frac{z^{\ell}}{\ell!}.$$

We deduce, from Definition 1, that the bivariate Poisson law according to Holgate [11] is a weighted bivariate Poisson distribution.

Example 4. *From Expression* (21)*, we have*

$$\psi\left(y_1,y_2;\mu_1,\mu_2,\lambda\right)=1+\lambda\left(e^{-y_1}-e^{-d\mu_1}\right)\left(e^{-y_2}-e^{-d\mu_2}\right),\ \textit{with } d=1-e^{-1}.$$

If we take

$$\mathbb{E}_{\delta_1,\delta_2}\left[\omega\left(Y_1,Y_2;\delta_1,\delta_2,\lambda\right)\right]=e^{-(\delta_1-\delta_2)\lambda},$$

as the constant of normalization, the weight function is equal to

$$\omega\left(y_1,y_2;\mu_1,\mu_2,\lambda\right)=\left[1+\lambda\left(e^{-y_1}-e^{-d\mu_1}\right)\left(e^{-y_2}-e^{-d\mu_2}\right)\right]e^{(\delta_1-\delta_2)\lambda}.$$

We deduce, from Definition 1, that the bivariate Poisson distribution according to Lakshminarayana and al. [15] is a weighted bivariate Poisson distribution.

Example 5. *Let Y_i $(i=1,2)$ be random variables of COM-Poison [5] with parameters (δ_i,ν_i), $(i=1,2)$. The COM-Poisson distribution is a weighted univariate Poisson distribution (Cf. Expression* (1)*) with a weight function*

$$\omega_i\left(y_i,\delta_i\right)=\left(y_i\right)^{1-\nu_i},\ (i=1,2), \tag{31}$$

and the constant of normalization

$$\mathbb{E}_{\delta_i}\left[\omega\left(Y_i\right)\right]=e^{-\delta_i}Z\left(\delta_i,\nu_i\right), \tag{32}$$

with $Z\left(\delta_i,\nu_i\right)=\sum_{n=0}^{+\infty}\delta_i^n/\left(n!\right)^{\nu_i}$. Taking into account the Expressions (31) *and* (32)*, the distribution given by the Expression* (30)*, called bivariate COM-Poisson distribution [5], is a weighted bivariate Poisson distribution of weight function*

$$\omega\left(y_1,y_2,\delta_1,\delta_2,\nu_1,\nu_2\right)=\left(y_1\right)^{1-\nu_1}\left(y_2\right)^{1-\nu_2}$$

and the constant of normalization

$$\mathbb{E}_{\delta_1,\delta_2}\left[\omega\left(y_1,y_2,\delta_1,\delta_2,\nu_1,\nu_2\right)\right]=Z\left(\delta_1,\nu_1\right)Z\left(\delta_2,\nu_2\right)e^{-\delta_1-\delta_2}.$$

Example 6. *From Expression* (25)*, we have*

$$\psi\left(y_1,y_2,\delta_1,\delta_2,\alpha_1,\alpha_2,\lambda\right)=\left(\prod_{i=1}^{2}\left(1+\alpha_i y_i\right)^{y_i-1}e^{-\alpha_i\delta_i y_i}\right)\left[1+\lambda\left(e^{-y_1}-c_1\right)\left(e^{-y_2}-c_2\right)\right].$$

If we take

$$\mathbb{E}_{\delta_1,\delta_2}\left[\omega\left(Y_1,Y_2;\delta_1,\delta_2,\lambda\right)\right]=1,$$

as the normalization constant and the weight function is equal to

$$\omega\left(y_1,y_2;\delta_1,\delta_2,\lambda\right)=\psi\left(y_1,y_2,\delta_1,\delta_2,\alpha_1,\alpha_2,\lambda\right).$$

The bivariate generalized Poisson distribution according to Famoye [9] is a weighted bivariate Poisson distribution, that is to say a bivariate Poisson distribution.

6. Conclusion

We have reviewed the bivariate Poisson distributions and determined the functional relationships that exist between them. We have highlighted the important role played by the bivariate Poisson distribution according to Berkhout and Plug [4] which allows to generate all the bivariate Poisson distributions. The bivariate weighted Poisson distribution evidenced by Elion et *al*. [8] is a weighted bivariate Poisson distribution. The weighted bivariate Poisson distribution that we have defined is the synthesis of all the bivariate Poisson distributions which, under certain conditions, converge in distribution towards the bivariate Poisson distribution according to Berkhout and Plug [4]. This last distribution can be considered as the standard distribution in $\mathbb{N}^2$ as is the univariate Poisson distribution in $\mathbb{N}$.

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