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A NOTE ON THE YAMADA–EXPONENTIAL SOFTWARE RELIABILITY MODEL

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Abstract: In this paper we study the Hausdorff approximation of the Heaviside step function $h_r(t)$ by deterministic curve models based on Yamada–exponential software reliability model, Yamada–Rayleigh model and Yamada–Weibul model and find an expression for the error of the best approximation. Some comparisons are made.

1. Introduction

The Gompertz and logistic curves are still used in industry, because these curves are well fitted to the cumulative number of faults observed in existing software development processes. Japanese software development companies prefer regression analysis based on deterministic functions such as Gompertz and Gompertz–type curves to estimate the number of residual faults (see, for instance [11]). In the context of reliability engineering, the Gompertz curve is, for example, used to assess the reliability growth phenomenon of hardware products (see, [6]). A residual–based approach for fault detection at rolling

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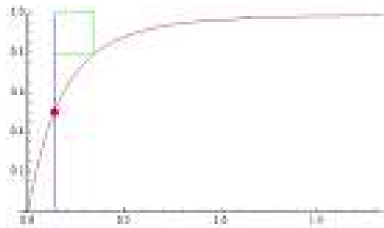

Figure 1: The model (1) with $a = 1$, $r = 2.5$, $\alpha = 2$, $\beta = 1.1$, $t_0 = 0.135664$; H–distance $d = 0.2075655$.

mills based on data–driven soft computing techniques, can be found in [8]. For other results, see [4], [10], [12], [7], [1], [2], [3]. Ohishi, Okamura and Dohi [11] formulate Gompertz software reliability model based on the following *deterministic curve model*: $M(t) = \omega a^{b^t}$, $a, b \in (0, 1)$. Satoh [5] and Satoh and Yamada [9] introduced a discrete Gompertz curve by discretization of the differential equations for the Gompertz curve and applied the discrete Gompertz curve to predict the number of detected software faults. A new class of Gompertz–type software reliability models and some deterministic reliability growth curves for software error detection, also approximation and modeling aspects, can be found in [17], [18]. In this note we study the Hausdorff approximation of the Heaviside step function $h_r(t)$ by some models based on Yamada–exponential software reliability model, Yamada–Rayleigh model and Yamada–Weibul model and find an expression for the error of the best approximation.

2. The Yamada exponential software reliability model

We consider the Yamada–exponential software reliability model:

$$Y(t; a, r, \alpha, \beta) = a\left(1 - e^{-r\alpha(1-e^{-\beta t})}\right). \tag{1}$$

We examine the special case $a = 1$, $t_0 = -\frac{1}{\beta}\ln\left(1 - \frac{\ln 2}{r\alpha}\right)$, i.e. $Y(t_0; 1, r, \alpha, \beta) = \frac{1}{2}$. The H–distance d between the Heaviside step function

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ \frac{1}{2}, & \text{if} \quad t = t_0 \\ 1, & \text{if} \quad t > t_0 \end{cases} \tag{2}$$

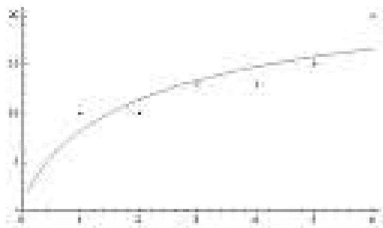

Figure 2: Approximate solution by Yamada–exponential software reliability model (6.26).

and the sigmoid (1) satisfies the relation

$$Y(t_0 + d; 1, r, \alpha, \beta) = 1 - d. \tag{3}$$

The deterministic model (1) for $a = 1$, $r = 2.5$, $\alpha = 2$, $\beta = 1.1$, $t_0 = 0.135664$ is visualized on Fig. 1.

Numerical example. We examine the following data. (The data were reported by C. Ravindranth Pandian and Murali Kumar S. K. [13] and represent the failures observed during system testing for 6 weeks).

Week	*Cumulative failures*
1	10
2	10
3	13
4	13
5	15
6	20

Table 1: Failures in test period and cumulative failures [13]

The fitted model based on the data of Table 1 and the estimated parameters is:

$$Y(t) = 20\left(1 - e^{-1.43126\times 1.95961(1-e^{-0.17438t})}\right).$$

The approximate solution is plotted on Fig. 2.

The following theorem gives upper and lower bounds for d

Theorem 1. The one–sided Hausdorff distance d between h_{t_0} and the curve

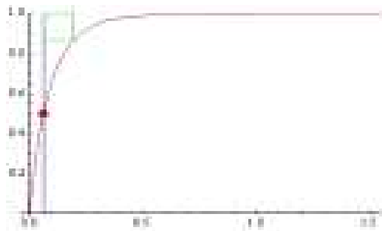

Figure 3: The model (1) with $a = 1$, $r = 5.5$, $\alpha = 1.9$, $\beta = 1.1$, $t_0 = 0.0623928$; H–distance $d = 0.132782$.

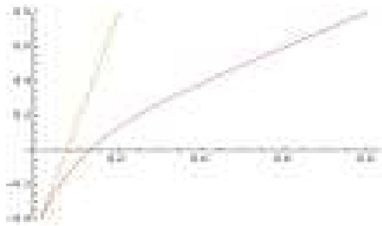

Figure 4: The functions $F(d)$ and $G(d)$ for $a = 1$, $r = 5.5$, $\alpha = 1.9$, $\beta = 1.1$.

(1) the following inequalities hold for:

$$r\alpha\beta - \beta\ln 2 > \frac{e^{1.1}}{1.1} - 2 \approx 0.73106$$

$$d_l = \frac{1}{1.1(2 + r\alpha\beta - \beta\ln 2))} < d < \frac{\ln(1.1(2 + r\alpha\beta - \beta\ln 2))}{1.1(2 + r\alpha\beta - \beta\ln 2))} = d_r. \tag{4}$$

Proof. Let us examine the functions:

$$F(d) = Y(t_0 + d; 1, r, \alpha, \beta) - 1 + d. \tag{5}$$

$$G(d) = -0.5 + 0.5(2 + r\alpha\beta - \beta\ln 2)d. \tag{6}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$. Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 4.). In addition $G'(d) > 0$. Further, for $r\alpha\beta - \beta\ln 2 > \frac{e^{1.1}}{1.1} - 2 \approx 0.73106$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

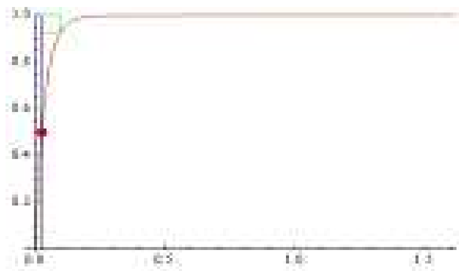

Figure 5: The model (1) with $a = 1$, $r = 10$, $\alpha = 2.2$, $\beta = 1.3$, $t_0 = 0.0246259$; H–distance $d = 0.0728743$.

r	α	β	d_l	d_r	d from (3)
2.5	2	1.1	0.134929	0.270264	0.207565
5.5	1.9	1.1	0.071399	0.188456	0.132782
6.5	1.8	1.1	0.0644401	0.176696	0.123511
8.5	2.1	1.2	0.0402462	0.129301	0.0885994
10	2.2	1.3	0.0306102	0.10672	0.0728743
100	2.5	1.4	0.00258978	0.0154252	0.0110156

Table 2: Bounds for d computed by (4) for various r, α, β.

The deterministic model (1) for $a = 1$, $r = 5.5$, $\alpha = 1.9$, $\beta = 1.1$, $t_0 = 0.0623928$ is visualized on Fig. 3. The deterministic model (1) for $a = 1$, $r = 10$, $\alpha = 2.2$, $\beta = 1.3$, $t_0 = 0.0623928$ is visualized on Fig. 5.

Some computational examples using relations (4) are presented in Table 2. The last column of Table 2 contains the values of d computed by solving the nonlinear equation (3).

3. The logistic–exponential software reliability model

In some cases it is appropriate to use the software reliability growth model with logistic–exponential test–effort function. We will consider the following logistic–exponential cumulative TEF (Testing Effort Function) over time period $(0, t]$ [14]:

$$W(t) = \alpha \frac{\left(e^{\lambda t} - 1\right)^k}{1 + \left(e^{\lambda t} - 1\right)^k}, \tag{7}$$

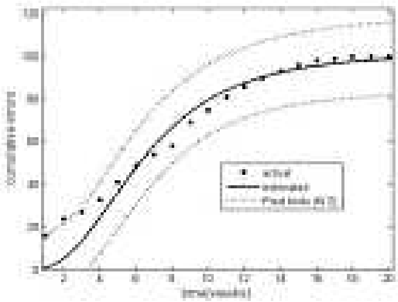

Figure 6: Typical cumulative (TEF) with confidence bounds [15].

where α is the total expenditure; k is a positive shape parameter and λ is a positive scale parameter (see, also [15]). Typical cumulative (TEF) with confidence bounds are ploted on Fig. 6.

We examine the special case $\alpha = 1$, $t_0 = \frac{1}{\lambda}\ln 2$, i.e. $W(t_0) = \frac{1}{2}$. The H–distance d between the Heaviside step function

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0 \\ 1, & \text{if} \quad t > t_0 \end{cases} \tag{8}$$

and the sigmoid (7) satisfies the relation

$$W(t_0 + d) = \frac{\left(e^{\lambda(t_0+d)} - 1\right)^k}{1 + \left(e^{\lambda(t_0+d)} - 1\right)^k} = 1 - d. \tag{9}$$

The following theorem gives upper and lower bounds for d

Theorem 2. The Hausdorff distance d between h_{t_0} and the curve (7) the following inequalities hold for:

$$k\lambda > \frac{2e^{1.05}}{2.1} - 2 \approx 0.721572$$

$$d_l = \frac{1}{2.1\left(1 + \frac{k\lambda}{2}\right)} < d < \frac{\ln\left(2.1\left(1 + \frac{k\lambda}{2}\right)\right)}{2.1\left(1 + \frac{k\lambda}{2}\right)} = d_r. \tag{10}$$

Proof. Here we will only sketch the proof. Let us examine the functions

$$F(d) = W(t_0 + d) - 1 + d. \tag{11}$$

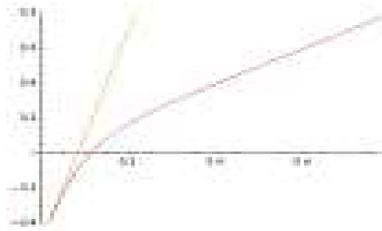

Figure 7: The functions $F(d)$ and $G(d)$ for $\alpha = 1$, $k = 5$, $\lambda = 2$.

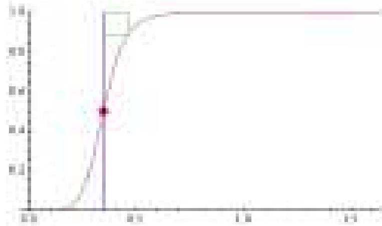

Figure 8: The model (7) with $\alpha = 1$, $k = 5$, $\lambda = 2$, $t_0 = 0.346574$; H–distance $d = 0.113351$.

and

$$G(d) = -\frac{1}{2} + \left(1 + \frac{k\lambda}{2}\right) d. \tag{12}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$. Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 7). In addition $G'(d) > 0$. Further, for $k\lambda > \frac{2e^{1.05}}{2.1} - 2 \approx 0.721572$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

The deterministic model (7) for $\alpha = 1$, $k = 5$, $\lambda = 2$, $t_0 = 0.346574$ is visualized on Fig. 8. The deterministic model (7) for $\alpha = 1$, $k = 10$, $\lambda = 3$, $t_0 = 0.231049$ is visualized on Fig. 9.

Remarks.

Remark 1. Of course, the two–sided bounds (10) can be improved, but this aim in the proposed monograph.

Remark 2. The model (7) can be used with success in debugging theory.

Remark 3. In grnrral software testing effort can be defined as the amount of effort spends during the software testing.

Cumulative testing effort can be described in $(0, t]$ by the following curves:

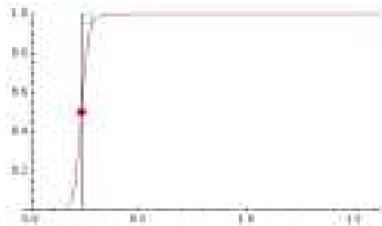

Figure 9: The model (7) with $\alpha = 1$, $k = 10$, $\lambda = 3$, $t_0 = 0.231049$; H–distance $d = 0.05191693$.

Rayleigh curve:

$$W(t) = \alpha\left(1 - e^{-\beta t^2}\right), \tag{13}$$

where β is a scale parameter represents the consumption rate of the testing effort (**Yamada–Rayleigh software reliability model**);

Weibul curve:

$$W(t) = \alpha\left(1 - e^{-\beta t^m}\right), \tag{14}$$

where m is a shape parameter and β is a scale parameter (**Yamada–Weibul software reliability model**).

I. We examine the **Yamada–Rayleigh software reliability model** (13) for the special case $\alpha = 1$, $t_0 = \sqrt{\frac{1}{\beta} ln2}$, i.e. $W(t_0) = \frac{1}{2}$. The H–distance d between the Heaviside step function

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0 \\ 1, & \text{if} \quad t > t_0 \end{cases} \tag{15}$$

and the sigmoid (13) satisfies the relation

$$W(t_0 + d) = 1 - e^{-\beta(t_0+d)^2} = 1 - d. \tag{16}$$

The following theorem gives upper and lower bounds for d

Theorem 3. The Hausdorff distance d between h_{t_0} and the curve (13) the following inequalities hold for:

$$\beta > \frac{1}{\ln 2}\left(\frac{e^{1.05}}{2.1} - 1\right)^2 \approx 0.187791$$

$$d_l = \frac{1}{2.1\left(1+\sqrt{\beta \ln 2}\right)} < d < \frac{\ln\left(2.1\left(1+\sqrt{\beta\ln 2}\right)\right)}{2.1\left(1+\sqrt{\beta \ln 2}\right)} = d_r. \tag{17}$$

Proof. Here we will only sketch the proof. Let us examine the functions

$$F(d) = W(t_0 + d) - 1 + d. \tag{18}$$

and

$$G(d) = -\frac{1}{2} + \left(1 + \sqrt{\beta \ln 2}\right) d. \tag{19}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$. Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$. In addition $G'(d) > 0$. Further, for $\beta > \frac{1}{\ln 2}\left(\frac{e^{1.05}}{2.1} - 1\right)^2 \approx 0.187791$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

II. We examine the **Yamada–Weibul software reliability model** (14) for the special case $\alpha = 1$,

$$t_0 = \left(\frac{1}{\beta}\ln 2\right)^{\frac{1}{m}},$$

i.e. $W(t_0) = \frac{1}{2}$. The H–distance d between the Heaviside step function

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0 \\ 1, & \text{if} \quad t > t_0 \end{cases} \tag{20}$$

and the sigmoid (14) satisfies the relation

$$W(t_0 + d) = 1 - e^{-\beta(t_0+d)^m} = 1 - d. \tag{21}$$

The following theorem gives upper and lower bounds for d

Theorem 4. The Hausdorff distance d between h_{t_0} and the curve (14) the following inequalities hold for:

$$\frac{m}{\left(\frac{1}{\beta}\ln 2\right)^{\frac{1}{m}}} > \frac{2}{\ln 2}\left(\frac{e^{1.05}}{2.1} - 1\right)^2 \approx 0.375582$$

$$d_l = \frac{1}{2.1\left(1 + \frac{1}{2}\frac{m \ln 2}{\left(\frac{1}{\beta}\ln 2\right)^{\frac{1}{m}}}\right)} < d < \frac{\ln\left(2.1\left(1 + \frac{1}{2}\frac{m\ln 2}{\left(\frac{1}{\beta}\ln 2\right)^{\frac{1}{m}}}\right)\right)}{2.1\left(1 + \frac{1}{2}\frac{m\ln 2}{\left(\frac{1}{\beta}\ln 2\right)^{\frac{1}{m}}}\right)} = d_r. \tag{22}$$

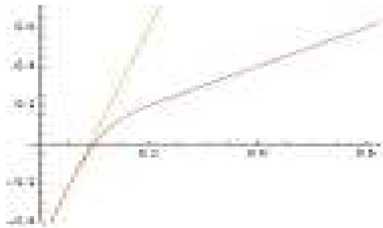

Figure 10: The functions $F(d)$ and $G(d)$ for $\alpha = 1$, $m = 10$, $\beta = 9.9$.

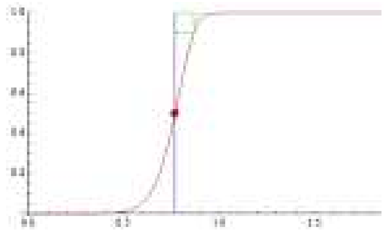

Figure 11: The model (14) for $\alpha = 1$, $m = 10$, $\beta = 9.9$, $t_0 = 0.766512$, H–distance $d = 0.0983845$.

Proof. Here we will only sketch the proof. Let us examine the functions:

$$F(d) = W(t_0 + d) - 1 + d. \tag{23}$$

$$G(d) = -\frac{1}{2} + \left(1 + \frac{1}{2}\frac{m \ln 2}{\left(\frac{1}{\beta}\ln 2\right)^{\frac{1}{m}}}\right) d. \tag{24}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$. Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see, Fig. 10). In addition $G'(d) > 0$. Further, for $\frac{m}{\left(\frac{1}{\beta}\ln 2\right)^{\frac{1}{m}}} > \frac{2}{\ln 2}\left(\frac{e^{1.05}}{2.1} - 1\right)^2 \approx 0.375582$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

The deterministic model (14) for $\alpha = 1$, $m = 10$, $\beta = 9.9$, $t_0 = 0.766512$, H–distance $d = 0.0983845$ is visualized on Fig. 11. The deterministic model (14) for $\alpha = 1$, $m = 30$, $\beta = 8.9$, $t_0 = 0.918434$, H–distance $d = 0.046656$ is visualized on Fig. 12. The deterministic model (14) for $\alpha = 1$, $m = 50$, $\beta = 8.5$, $t_0 = 0.951104$, H–distance $d = 0.0311342$ is visualized on Fig. 13.

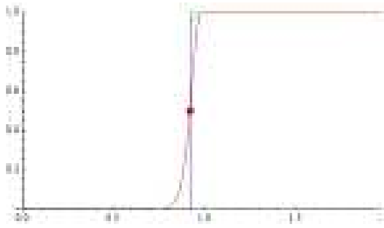

Figure 12: The model (14) for $\alpha = 1$, $m = 30$, $\beta = 8.9$, $t_0 = 0.918434$, H–distance $d = 0.046656$.

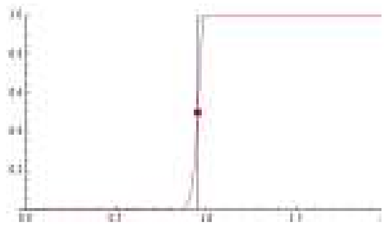

Figure 13: The model (14) for $\alpha = 1$, $m = 50$, $\beta = 8.5$, $t_0 = 0.951104$, H–distance $d = 0.0311342$.

Some computational examples using relations (22) are presented in Table 3. The last column of Table 3 contains the values of d computed by solving the nonlinear equation (21).

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m	β	d_l	d_r	d from (21)
10	9.9	0.0852439	0.21134	0.0983845
30	8.9	0.03865	0.125736	0.046656
50	8.5	0.0247764	0.0916197	0.0311342
56	4.5	0.0226032	0.0856586	0.0286411

Table 3: Bounds for d computed by (22) for various m and β.

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