European Research Studies,
Volume XIII, Issue (4), 2010

A Note on Theory of Productive Efficiency and Stochastic Frontier Models

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Abstract:

Neoclassical economics assume that producers in an economy always operate efficiently, however in real terms, producers are not always fully efficient. This difference may be explained both in terms of efficiency, as well as unforeseen exogenous shocks outside the producer control. This paper aims to analyse the productive efficiency estimation through a stochastic frontier analysis approach. Particularly, this paper attempts to examine systematically the theoretical background of stochastic frontier function estimation, focusing on the analysis of the efficiency function, in order to provide a solid background for productive efficiency estimation.

JEL Classification: C14, C23

Keywords: Stochastic Frontier Analysis, Productive Efficiency Function, Efficiency

1. Introduction

The main core of the modern economic theory is based on the assumption of optimising behaviour, either from a producer or a consumer approach. Economic theory assumes that producers optimise both from a technical and economic perspective:

1. From a technical perspective, producers optimise by not wasting productive resources.
2. From an economic perspective producers optimise by solving allocation problem involving prices.

However, not all producers succeed in solving both types of optimisation problem in all circumstances. Performance at firm or industry level, defined as the ratio of output(s) a production unit produces to the input(s) that a production unit uses, yielding a relative measure of performance applied to factors of production

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[*] *The author should thank Prof. Dr. Richard Harris, Department of Economics, University of Glasgow for all the guidance and contributing remarks. However, any faults are of sole responsibility of the author.*

(Fried et al., 1993, Lovell, 1993), may depend: a) on differences in production technology, b) on differences in the efficiency of the production process, or, c) on differences in the environment where production occurs[2]. However, at a given moment of time, even when technology and production environment are 'essentially the same', firms or industries may exhibit different productivity levels due to differences in their production efficiency (Korres, 2007).

For this reason it is important to have a way of analysing the degree to which producers fail to optimise, the departures from full technical and economic efficiency. Based on this general notion, one of the main analytical approaches to efficiency measurement is the analysis of production frontiers, a tool which has expanded greatly in the last decades.

However, even though the concept of production efficiency is central in production performance, its estimation has been proved to be rather complex, with relevant literature providing a range of different methodologies and approaches (Lovell, 1993), with one of the major approaches to be the 'stochastic frontier analysis'.

This paper examines the main characteristics of stochastic frontier models, especially the alternative model specifications.

2. The Theoretical Background

The stochastic frontier model was originally developed by Aigner, Lovell and Schmidt (1977). Typically, the production or cost model is based on a Cobb – Douglas function:

$$\log y = \beta' x + v - u \tag{1}$$

where y is the observed outcome $\beta' x + v$ is the optimal production frontier (e.g. maximum production output or minimum cost), $\beta' x$ is the deterministic part of the frontier and $v \sim N(0, \sigma_v^2)$ is the stochastic part, respectively. The components of x are generally logs of inputs for a production model or logs of output and input prices for a cost model, or their squares and/or cross products. These two parts constitute the stochastic frontier. The amount by which the observed individual fails to reach the optimum (the frontier) is u, namely inefficiency, where $u=|U|$ and $U \sim N[0, \sigma_u^2]$. The stochastic frontier model becomes:

$$y = \beta' x + v - u \text{ , } u = | U | \tag{2}$$

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[2] *In early economic studies, productivity change was allocated exclusively to shifts in production technology (the magnitude of neutral technical change), eventually roles were also assigned to the biases of technical change and the structure of the technology, namely scale economies (Kumbhakar and Lovell, 2000).*

In the stochastic frontier model, the error term ε is made up of two independent components, *v - u,* where *u* measures technical inefficiency, namely the shortfall of output *y* from its maximal possible value given by the stochastic frontier $[g(x_0, \beta) + v]$. When a model of this form is estimated, the obtained residuals $\hat{\varepsilon} = y - g(x - \hat{\beta})$, may be regarded as estimates of the error term *ε* (Jondrow et al., 1982). The conditional distribution of *u* given *ε, E[u|ε]* is the mean productive efficiency. Under each of the assumed possible distributional forms for the inefficiency term in a model, this mean hat this distribution contains whatever information ε yields about u. The predicted value is $\beta' x$. The residual is computed by the Jondrow et al. (1982) formula:

$$E[u \mid v - u] \text{ or } E[u \mid v + u] \tag{3}$$

or

$$\hat{E}[u \mid \varepsilon] = \frac{\sigma\lambda}{1 + \sigma^2}\left[\frac{\phi(z)}{1 - \Phi(z)} - z\right], \ \varepsilon = v \pm u, \ z = \frac{\varepsilon\lambda}{\sigma} \tag{4}$$

The marginal effects in the model are the coefficients β. Estimation of the model parameters is usually of secondary interest, whereas, estimation and analysis of the inefficiency of individuals in the sample and of the aggregated sample are of greater concern. The results obtained are critically dependent on the model form and the assumptions set. Regarding this, special focus has been given to panel data estimation technique.

3. The Model Specifications

The stochastic frontier model follows Battese and Coelli (1995) and consists of two equations, one to represent the production frontier and a second to measure technical inefficiency:

$$Y_{it} = exp(X_{ijt}\beta + V_{it} - U_{it}) \tag{5}$$

and

$$E_{it} = exp(-U_{it}) = exp(-Z_{it}\delta - \mathrm{W}_{it}) \tag{6}$$

In the first equation, Y_{it} represents output of the ith firm at time *t*. X_{ijt} is a vector of productive inputs and indicator variables for the ith firm at time *t*. The parameter vectors β and δ are estimated together with the variance parameters:

$$\sigma^2 = \sigma_v^2 + \sigma_u^2 \text{ and } \gamma = \sigma_u^2 / \sigma^2 \tag{7}$$

Technical efficiency is measured using the conditional expectation

$$E_{it} = exp(-U_{it})$$

given the composed error term. The first component, (v_{it}), accounts for random events. The second component, u_{it}, is a non-negative random variable which captures unobservable inefficiency effects relative to the stochastic frontier. The random component, *v* is assumed to be independently and identically distributed with $N(0, \sigma_v^2)$. The technical inefficiency component, *u*, is assumed to follow an arbitrary distributional form, in this case a half-normal distribution $N(Z_{it}\delta, \sigma_u^2)^2$. The inefficiency model random component, *w*, is not identically distributed nor is it required to be non-negative (Battese and Coelli, 1995)[3].

Basically, there are two methods of estimation in the literature. In the first, the estimation of the parameters of the production frontier is done conditionally on fixed values of the u_{it}*'s* which leads to the fixed effects model and the within estimator of the frontier coefficients. In the second, the estimation is carried out marginally on the firm specific effects u_{it}*'s* which leads to the random effects model and either the Generalised Least Squares (GLS) estimation of the parameters.

Although the fixed effects models have the advantage of following correlation between the inefficiency term and the independent variables, and of allowing no distributional assumption on efficiency, the results should be interpreted carefully. Simar (1992) has shown that the fixed effects model appears to provide a poor estimation of the intercepts and of the slope coefficients of frontier production functions and consequently unreasonable measures of technical efficiency. In the random effects model, the stochastic nature of the efficiency effects is explicitly taken into account in the estimation process. In the fixed effects model, the coefficients of time – invariant regressors, even though they may vary across firms, cannot be estimated because these time – invariant regressors will be eliminated in the within transformation, as shown in the equation:

$$(y_{it} - \bar{y}_i) = \beta'(x_{it} - \bar{x}_i) + v'_{it} \quad (8)$$

In this case, the firm – specific technical efficiency effects will include the influence of all variables that are time – invariant at the firm level within the sample. This would make technical efficiency comparisons difficult unless the excluded fixed regressors influence all firms in the sample equally.

4. Extended Model

In the literature there are several variants of the previous model allowing for different distributions of the *u* and *v* term (see Kalirajan and Shand, 1999):

[3] *Before Battese and Coelli (1995), Jondrow et al.(1982) provided an initial solution by deriving the conditional distribution of [-u_i|($v_i - u_i$)] which contains all the information ($v_i - u_i$) contains about u_i. This enabled to derive the expected value of this conditional distribution, from which they proposed to estimate the technical efficiency of each producer:* $T\hat{E}_0(x_i, y_i) = \{\exp\{E[-\hat{u}_i \mid (v_i - u_i)]\}\}^{-1} \geq 1$ *which is a function of the MLE parameter estimates. Later, Batesse and Coelli (1988) proposed to estimate the technical efficiency of each producer from:* $T\hat{E}_0(x_i, y_i) = \{E[\exp\{-\hat{u}_i\} \mid (v_i - u_i)]\}^{-1} \geq 1$ *which is slightly different function of the same MLE parameter estimates.*

1. The half – normal model
2. The fixed effects model
3. The random effects model
4. The latent class model

4.1 The Half – Normal Model

The essential form of the stochastic production frontier model [see Aigner et al. (1977)] is:

$$y_{it} = \alpha + x_{it}\beta + v_{it} - u_i \tag{9}$$

where

$$v_i \sim N[0,\sigma_v^2],\ u_i = |U_i|, \text{ and } U_i \sim N[0,\sigma_u^2] \tag{10}$$

As described in Greene (2003), this is the canonical 'half normal' model. A central parameter in the model is the asymmetry parameter, $\lambda = \sigma_u/\sigma_v$; the larger is λ, the greater is the inefficiency component in the data. Parameters are estimated by maximum likelihood, rather than least squares. Estimation of u_i is the central focus of the analysis. With the model estimated in logarithms, u_i would correspond *to 1 - TE$_i$*. Individual specific efficiency is typically estimated with $\exp(-\hat{u}_i)$. Alternatively, $\hat{u}_i$ provides an estimate of proportional inefficiency. With parameter estimates in hand, one can only obtain a direct estimate of $\varepsilon_i = v_i - u_i$. This is translated into an estimate of u_i using Jondrow et al. (1982) formula:

$$E[u_i \mid \varepsilon_i] = \frac{\sigma\lambda}{1+\lambda^2}\left[z_i + \frac{\phi(z_i)}{\Phi(z_i)}\right],\ z_i = \frac{-\varepsilon_i\lambda}{\sigma} \tag{11}$$

where $\sigma = \left(\sigma_u^2 + \sigma_v^2\right)^{1/2}$ and $\varphi(z)$ and $\Phi(z)$ are the density and CDF of the standard normal distribution, respectively.

The narrow assumption of half normality is a viewed as significant drawback in this model. This feature leads to the extension of the model to a truncated normal model by allowing the mean of U_i to be nonzero (Stevenson, 1980). The major shortcoming here is that the strict assumption suppresses individual heterogeneity in inefficiency that is allowed, for example, by the fixed effects formulation. Letting $\boldsymbol{h}_i$ denote a set of variables that measure the group heterogeneity, we write:

$$E[U_i] = \mu_i = h_i'\delta \tag{12}$$

The Jondrow et al. (1982) result is now changed by replacing z_i with $z_i^* = z_i - \mu_i/(\sigma\lambda)$ representing a significant extension of the model.

4.2 The Fixed Effects Model

The fixed effects model is based on the Schmidt and Sickles (1982) formulation:

$$y_{it} = (\alpha - u_i) + x_{it}^{'}\beta + v_{it} = \alpha_i + x_{it}^{'}\beta + v_{it} \tag{13}$$

where

$$\hat{u}_i = \max{}_i(\hat{\alpha}_i) - \hat{\alpha}_i \geq 0 \tag{14}$$

This variation has two important restrictions. First, any time invariant heterogeneity will be pushed into α_i and ultimately into $\hat{u}_i$. Second, the model assumes that inefficiency is time invariant. For short time intervals, this may be a reasonable assumption. But, this is to be questionable. Both of these restrictions can be relaxed by placing country specific constant terms in the stochastic frontier model – we call this a 'true' fixed effects model:

$$y_{it} = \alpha + x_{it}^{'}\beta + v_{it} - u_{it} \tag{15}$$

where u_{it} has the stochastic specifications noted earlier for the stochastic frontier model. The model is still fit by maximum likelihood, not least squares.

4.3 The Random Effects Model

As referred in Greene (2003), the random effects model is obtained by assuming that u_i is time invariant and also uncorrelated with the included variables in the model:

$$y_{it} = \alpha + x_{it}^{'}\beta + v_{it} - u_i \tag{16}$$

In the linear regression case, the parameters are estimated by two step generalized least squares (Green, 2003). The regression based random effects model has a significant drawback: there is no implied estimator of inefficiency in this model, that is, no estimator of technical efficiency TE_i as in the fixed effects case. So, the model would not have been useful in any event.
With this extension, the Jondrow et al. (1982) estimator becomes:

$$E\left[u_i \mid \varepsilon_1, \varepsilon_2, \ldots\ldots, \varepsilon_T h_i\right] = Z_i + \psi\left[\frac{\phi\left(Z_i/\psi\right)}{\Phi\left(Z_i/\psi\right)}\right] \tag{17}$$

where

$$Z_i = \gamma\mu_i - (1-\gamma)\bar{\varepsilon}_i\ ,\ \gamma = 1/\left(1+T\lambda^2\right),\ \psi^2 = \gamma\sigma_u^2\ , \tag{18}$$

$$\bar{\varepsilon}_i = \left(1/T\right)\sum{}_i \varepsilon_{it}$$

The time invariance of the inefficiency component of the random effects model is a potential drawback in the random effects model. Battese and Coelli (1988, 1995) have proposed a modification of the model that allows some systematic variation in the following model: $u_{it} = \eta_t |U_i|$ where $\eta_t = 1 + \eta_1(t-T) + \eta_2(t-T)^2$ and U_i ~ $N[0,\sigma_u^2]$. A random effects counterpart to the true fixed effects model would be a 'true random effects' stochastic frontier model: $y_{it} = (\alpha + w_i) + x_{it}'\beta + v_{it} - u_{it}$. This form of the model overcomes both of the drawbacks noted earlier. As broadly presented in Greene (2003), the preceding has suggested various ways to accommodate both cross country heterogeneity and time variation in inefficiency in the stochastic frontier model. Time variation in inefficiency is achieved by removing restrictions on u_{it} and allowing it to vary unsystematically through time.

4.4 The Latent Class Model

Presented in Greene (2003) is also another model variation, the latent class model:

$$\left(y_{it} \mid class = j\right) = \alpha_j + x_{it}'\beta_j + v_{it}|j - u_i|j \tag{19}$$

and a model for the mixing probabilities:

$$\Pr ob\left[country\ i\ is\ a\ member\ of\ class\ j\right] = F_j\left(h_i, \theta\right),\ 0 \le Fi_j \le 1 \tag{20}$$

Heterogeneity enters this model through the prior mixing probabilities. As before, it can also enter through the distribution of u_{it}. The latent class model is an alternative to the random parameters model described in the preceding section. With a sufficient number of classes, the finite mixture can provide a good approximation to continuous parameter variation. In practical terms, this model is somewhat less flexible than the random parameters model discussed above. Greene (2003) has extended it to the most general variant of the Battese and Coelli formulation of the random effects model. Since this approach is new to the literature, its usefulness as an empirical tool remains to be established.

A number of studies have also attempted to estimate time-varying inefficiency. Cornwell, Schmidt and Sickles (1990) replaced the firm effect by a squared function of time with parameters that vary over time. Kumbhakar (1990) allowed a time-varying inefficiency measure assuming that it was the product of the specific firm inefficiency effect and an exponential function of time. This allows flexibility in inefficiency changes over time, although no empirical applications have been developed using this approach (Coelli, Rao and Battese 1998).

5. Concluding remarks

In summary, in stochastic frontier model it is acknowledged that the estimation of production functions must respect the fact that actual production cannot exceed maximum possible production given input quantities. There is an extensive body of literature on factors influencing productivity growth. In this literature body, it is widely accepted that decision making units are not homogeneous producing units and, they are not therefore, all operating at the same level of efficiency. On the contrary, in the real world some producers are more efficient than others. Empirical analyses have shown that productivity level may considerably differ even if they operate in the same market. While some units operate at the technological frontier and earn high profits, others lag considerably behind.

As a management tool, stochastic frontier analysis focuses on productive efficiency analysing variables under or beyond decision-makers' control. These factors are neither inputs to the production process nor outputs of it but nonetheless exert an influence on producer performance.[4]. In this context, the term environment is used to describe factors that could influence the efficiency of a firm, where such factors are not traditional inputs and are not under the control of management. In other words, some exogenous variables may influence the productive efficiency with which inputs are converted into outputs. In particular, to investigate the determinants of the productive efficiency we distinguish between firm / industry -specific and external factors. External factors are not under direct control of the firm, at least in the short-run, as industry affiliation and firm location. Firm-specific factors, on the other hand, refer to characteristics that can be influenced by the firm in the short-run, as firm size, R&D intensity and degree of outsourcing. These factors may express social aspects, geographical or climatic conditions, as well as regulatory and institutional constraints.

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[4] *In many cases, the distinction between decision-maker controlled and external variables is not always distinct. As in McMillan and Chan (2006), external variables here include purely exogenous variables as well as firm-specific variables representing production methods and output characteristics.*

It becomes apparent that in the area of stochastic frontier models, estimation of the model parameters is usually of secondary interest, whereas, estimation and analysis of the inefficiency are of greater concern. The important task is to relate inefficiency to a number of factors that are likely to be determinants, and measure the extent to which they contribute to the presence of inefficiency.

Stochastic frontier approach has found wide acceptance within the production economics literature, because of their consistency with theory, versatility and relative ease of estimation. Some literature focused on stochastic frontier model with distributional assumptions by which efficiency effects can be separated from stochastic element in the model and for this reason a distributional assumption has to be made. However, these are computationally more complex, there are no priori reasons for choosing one distributional form over the other, and all have advantages and disadvantages. Within this framework, the analysis so far provides a solid background for further development of the model.

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