A Note on Two Methods for Estimating Missing Pairwise Preference Values

Francisco Chiclana, Enrique Herrera-Viedma, and Sergio Alonso

***Abstract*—This note analyzes two methods for calculating missing values of an incomplete reciprocal fuzzy preference relation. The first method by Herrera-Viedma *et al.* appeared in the IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS—PART B: CYBERNETICS [vol. 37, no. 1 (2007) 176–189], while the second one by Fedrizzi and Giove appeared later in the *European Journal of Operational Research* [vol. 183 (2007) 303–313]. The underlying concept driving both methods is the additive consistency property. We show that both methods, although different, are very similar. Both methods derive the same estimated values for the independent-missing-comparison case, while they differ in the dependent-missing-comparison case. However, it is shown that a modification of the first method coincides with the second one. Regarding the total reconstruction of an incomplete preference relation, it is true that the second method performs worse than the first one. When Herrera-Viedma *et al.*'s method is unsuccessful, Fedrizzi–Giove's method is as well. However, in those cases when Fedrizzi–Giove's method cannot guarantee the successful reconstruction of an incomplete preference relation, we have that Herrera-Viedma *et al.*'s method can. These results lead us to claim that both methods should be seen as complementary rather than competitors in their application, and as such, we propose a reconstruction policy of incomplete fuzzy preference relations using both methods. By doing this, the only unsuccessful reconstruction case is when there is a chain of missing pairwise comparisons involving each one of the feasible alternatives at least once.**

***Index Terms*—Consistency, incomplete preference relation, missing values, pairwise comparison, transitivity.**

I. INTRODUCTION

To reach a decision, experts have to express their preferences by means of a set of evaluations over a set of alternatives. Different alternative preference elicitation methods were compared in [18], where it was concluded that pairwise comparison methods are more accurate than nonpairwise methods. Given two alternatives of a finite set of all potentially available ones, denoted as X, an expert either prefers one to the other or is indifferent between them. Obviously, there is another possibility: that of an expert being unable to compare them.

Given three alternatives x_i, x_j, and x_k such that x_i is preferred to x_j and x_j to x_k, the question of whether the "degree or strength of preference" of x_i over x_j exceeds, equals, or is less than the "degree or strength of preference" of x_j over x_k cannot be answered by the classical preference modeling. The implementation of the degree of preference between alternatives may be essential in many situations. Take, for example, the case of three alternatives $\{x, y, z\}$ and two experts. If one of the experts prefers x to y to z and the other prefers z to y to x, then it may be difficult or impossible to decide which alternative is the best.

Manuscript received February 10, 2009. First published June 23, 2009; current version published November 18, 2009. This paper was recommended by Associate Editor Q. Shen.

F. Chiclana is with the Centre for Computational Intelligence, Faculty of Technology, De Montfort University, Leicester LE1 9BH, U.K. (e-mail: chiclana@dmu.ac.uk).

E. Herrera-Viedma is with the Department of Computer Science and Artificial Intelligence, University of Granada, 18071 Granada, Spain (e-mail: viedma@decsai.ugr.es).

S. Alonso is with the Department of Software Engineering, University of Granada, 18071 Granada, Spain (e-mail: zerjioi@ugr.es).

Digital Object Identifier 10.1109/TSMCB.2009.2023923

The introduction of the concept of a fuzzy set as an extension of the classical concept of a set when applied to a binary relation leads to the concept of a fuzzy relation. A fuzzy preference value can be associated with the following two semantics [9]: "the intensity of preference (to what extent x_i is preferred to x_j)" and "the uncertainty about the preference (how sure it is that x_i is preferred to x_j)." The fuzzy interpretation of intensity of preferences was introduced by Bezdek *et al.* [4] via the concept of a reciprocal fuzzy relation and later reinterpreted by Nurmi [19]. The adapted definition of a reciprocal preference relation is the following [6].

Definition 1 (Reciprocal Fuzzy Preference Relation): A reciprocal fuzzy preference relation R on a finite set of alternatives X is a fuzzy relation in $X \times X$ with membership function $\mu_R : X \times X \longrightarrow [0, 1]$, $\mu(x_i, x_j) = r_{ij}$, verifying

$$r_{ij} + r_{ji} = 1, \quad \forall\, i, j \in \{1, \ldots, n\}.$$

When the cardinality of X is small, the reciprocal fuzzy preference relation may conveniently be denoted by the matrix $R = (r_{ij})$. The following interpretation is also usually assumed.

1) $r_{ij} = 1$ indicates the maximum degree of preference for x_i over x_j.
2) $r_{ij} \in]0.5, 1[$ indicates a definite preference for x_i over x_j.
3) $r_{ij} = 0.5$ indicates indifference between x_i and x_j.

Fishburn pointed out that indifference might arise in three different ways [12]:

1) when an expert truly feels that there is no real difference, in a preference sense, between the alternatives;
2) when the expert is uncertain as to his/her preference between the alternatives because "he[/she] might find their comparison difficult and may decline to commit himself[/herself] to a strict preference judgement while not being sure that he[/she] regards [them] equally desirable (or undesirable)";
3) when both alternatives are considered incomparable on a preference basis by the expert.

Therefore, incomparability and indifference are equivalent concepts for Fishburn. However, we believe that when an expert is unable to compare two alternatives, then this situation should not be reflected in the preference relation as an indifference situation but instead with a missing entry for that particular pair of alternatives. In other words, a missing value in a preference relation is not equivalent to a lack of preference of one alternative over another. A missing value might be also the result of the incapacity of experts to quantify the degree of preference of one alternative over another because of "time pressure, lack of knowledge or data, and their limited expertise related to the problem domain" [15], in which case they may decide not to "guess" to maintain the consistency of the values already provided [2]. To model these situations, the following definitions express the concept of an incomplete preference relation.

Definition 2: A function $f : X \longrightarrow Y$ is *partial* when not every element in the set X necessarily maps to an element in the set Y. When every element from the set X maps to one element of the set Y, then we have a *total* function.

Definition 3: A preference relation P on a set of alternatives X with a *partial* membership function is an *incomplete preference relation.*

In [13], Herrera-Viedma *et al.* developed a method for calculating the missing values of an incomplete fuzzy preference relation. This calculation is done by using only the known preference values, therefore assuring that the reconstruction of the incomplete fuzzy preference relation is compatible with the rest of the information provided by that expert. An aim in the design of this method is to maintain or maximize the expert's global consistency. In [11], Fedrizzi and Giove proposed a new and apparently different method for calculating the missing values of an incomplete fuzzy preference relation. This method was based on the resolution of an optimization problem, and therefore, when the estimated missing values exist, it results in a complete fuzzy preference relation with maximum global consistency. Consistency is modeled in these two studies via the additive transitivity property [14], [20].

In this note, in Section II, we briefly describe both reconstruction methods, while in Section III, we analyze them and show that both methods, although different, are very similar. Indeed, Fedrizzi–Giove's method can be derived by introducing a modification to Herrera-Viedma *et al.*'s method. After analyzing the conditions that guarantee the successful application of Fedrizzi–Giove's method, we conclude that there are many cases when the reconstruction of an incomplete preference relation cannot be guaranteed with it, but it can with Herrera-Viedma *et al.*'s method. All this leads us to consider both methods as complementary in their application, and as such, we propose a reconstruction policy of incomplete fuzzy preference relations using both methods. By doing this, the only unsuccessful reconstruction case is when there is a chain of missing pairwise comparisons involving all the feasible alternatives at least once. Finally, conclusions are drawn in Section IV.

II. Missing Pairwise Preference Values

It is quite often the case in empirical studies to discard a whole questionnaire when some data are missing. One example of this practice is reported by Millet [18]. Carmone *et al.* [5] investigate the effect of reduced sets of pairwise comparisons. They compared results obtained for a complete pairwise comparison matrix and an incomplete one derived by eliminating known elements of the complete one. Their result suggests that "random deletion of as much as 50% of the comparisons provides good results without compromising the accuracy." However, because this process relies on the *a priori* knowledge of the complete pairwise comparison matrix, it is therefore inapplicable in real-life applications. When a complete pairwise comparison matrix is not available, Carmone *et al.* suggest the selection of an appropriate methodology to "build" the matrix. A strong argument supporting this type of methodology is given by Ebenbach and Moore [10]: "scenarios with missing values are normally penalized and rated more negatively than the same scenario with a value provided." A system that helps experts to build a complete fuzzy preference relation in decision-making contexts has been developed in [1]. This system reacts to an expert input of preference values by providing him/her with recommendations on the preference values that he/she has not yet expressed.

In group decision making, procedures that correct the lack of knowledge of a particular expert using the information provided by the rest of the experts, together with some aggregation procedures, can be found in [16] and [17]. These approaches have several disadvantages. Among them, we can cite the following.

- These approaches require multiple experts to learn the missing value of a particular one.
- These procedures normally do not take into account the differences between experts' preferences, which could lead to the estimation of a missing value that would not naturally be compatible with the rest of the preference values given by that expert.
- Some of these missing-information-retrieval procedures are interactive, that is, they need experts to collaborate in "real time," an option that is not always possible.

Different approaches to the above ones have been developed by Herrera-Viedma *et al.* [13] and by Fedrizzi and Giove [11]. In these two approaches, the computation of missing values in an expert's incomplete preference relation is done using only the preference values provided by that particular expert. By doing this, it is assured that the reconstruction of the incomplete fuzzy preference relation is compatible with the rest of the information provided by that expert. Furthermore, the main aim in the design of these approaches is to maintain or maximize the expert's global consistency, which is modeled and measured via Tanino's "additive transitivity" property

$$p_{ij} = p_{ik} + p_{kj} - 0.5, \quad \forall i, j, k \in \{1, 2, \dots, n\}. \tag{1}$$

Obviously, additive transitivity implies additive reciprocity, as well as indifference between any alternative and itself. In the next sections, we review these two reconstruction methods.

A. Herrera-Viedma et al.'s Reconstruction Method

Given a reciprocal fuzzy preference relation, (1) can be used to calculate an estimated value of a preference degree using other preference degrees. Indeed, using an intermediate alternative x_k, the following local estimated value of p_{ij} $(i \neq j)$ is obtained:

$$ep_{ij}^k = p_{ik} + p_{kj} - 0.5. \tag{2}$$

The overall estimated value ep_{ij} of p_{ij} is obtained as the average of all possible values ep_{ij}^k, i.e.,

$$ep_{ij} = \sum_{\substack{k=1 \\ k \neq i,j}}^{n} \frac{ep_{ij}^k}{n-2}. \tag{3}$$

It is easy to prove that

$$\left| e^r p_{ij} - e^{r-1} p_{ij} \right| = \left(\frac{2}{n-2} \right)^{r-1} |ep_{ij} - p_{ij}|, \qquad (r > 1) \tag{4}$$

i.e., this process of estimating preference values converges toward perfect consistency [8].

In [13], an iterative procedure was introduced to estimate the missing values of an incomplete fuzzy preference relation based on the values known. To that end, the following sets were introduced:

$$\begin{aligned} A &= \{(i,j) \,|\, i,j \in \{1,\dots,n\} \wedge i \neq j\} \\ MV &= \{(i,j) \in A \,|\, p_{ij} \text{ is unknown}\} \\ EV &= A \setminus MV \\ H_{ij} &= \{k \neq i,j \,|\, (i,k),(k,j) \in EV\} \end{aligned} \tag{5}$$

where MV is the set of incomparable pairs of alternatives (missing values), EV is the set of pairs of alternatives for which the expert provides preference values (expert values), and H_{ij} is the set of intermediate alternatives x_k $(k \neq i, j)$ that can be used to estimate the preference value $p_{ij} (i \neq j)$ using (1).

The subset of missing values MV that can be estimated in step h is

$$EMV_h = \left\{ (i,j) \in MV \setminus \bigcup_{l=0}^{h-1} EMV_l \,|\, i \neq j \wedge \exists k \in \left\{ H_{ij}^h \right\} \right\} \tag{6}$$

with

$$H_{ij}^h = \left\{ k \neq i,j \,|\, (i,k),(k,j) \in EV \cup \left\{ \bigcup_{l=0}^{h-1} EMV_l \right\} \right\} \tag{7}$$

and $EMV_0 = \emptyset$ (by definition). The iterative procedure stops when $EMV_{maxIter} = \emptyset$, with $maxIter > 0$. If $\bigcup_{l=0}^{maxIter} EMV_l = MV$, then all missing values are estimated, and consequently, the procedure is said to be successful in the completion of the incomplete fuzzy preference relation. The estimated value for p_{ij}, with $(i,j) \in EMV_h$, is

$$ep_{ij} = \frac{\sum_{k \in H_{ij}^h} ep_{ij}^k}{\# H_{ij}^h}. \tag{8}$$

Example 1: Let us suppose that an expert provides the following incomplete fuzzy preference relation over a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$:

$$P = \begin{pmatrix} - & \mathbf{0.2} & \mathbf{0.6} & \mathbf{0.4} \\ \mathbf{0.8} & - & x & x \\ \mathbf{0.4} & x & - & x \\ \mathbf{0.6} & x & x & - \end{pmatrix}$$

where symbol x means an unknown value. Because the known values in P involve all four alternatives, then all the missing values can successfully be estimated. Indeed, in step 1, the set of elements that can be estimated is

$$EMV_1 = \{(2,3),(2,4),(3,2),(3,4),(4,2),(4,3)\}.$$

As an example, to estimate p_{34}, the procedure is given as follows:

$$H_{34}^1 = \{1\} \quad \Rightarrow \quad ep_{34} = ep_{34}^1 = p_{31} + p_{14} - 0.5 = 0.3.$$

After these elements have been estimated, we have

$$P = \begin{pmatrix} - & \mathbf{0.2} & \mathbf{0.6} & \mathbf{0.4} \\ \mathbf{0.8} & - & 0.9 & 0.7 \\ \mathbf{0.4} & 0.1 & - & 0.3 \\ \mathbf{0.6} & 0.3 & 0.7 & - \end{pmatrix}.$$

Note 1: When the information provided is completely consistent, then $ep_{ij}^k = p_{ij}\ \forall k$. However, because experts are not always fully consistent, the information given by an expert may not verify (1), and some of the estimated preference degree values ep_{ij}^k may not belong to the unit interval [0, 1]. We note, from (2), that the maximum value of any of the preference degrees ep_{ij}^k is 1.5, while the minimum one is -0.5. To normalize the expression domains in the decision model, the final estimated value of p_{ij} $(i \neq j)$, denoted as cp_{ij}, is defined as the median of the values 0, 1, and ep_{ij}

$$cp_{ij} = \text{med}\{0, 1, ep_{ij}\}. \tag{9}$$

The error in [0, 1] between a preference value p_{ij} and its final estimated one cp_{ij} is

$$\varepsilon p_{ij} = |cp_{ij} - p_{ij}|. \tag{10}$$

Reciprocity of $P = (p_{ij})$ implies reciprocity of $CP = (cp_{ij})$; therefore, $\varepsilon p_{ij} = \varepsilon p_{ji}$. We interpret $\varepsilon p_{ij} = 0$ as a situation of total consistency between p_{ij} (p_{ji}) and the rest of the information in P. Obviously, the higher the value of εp_{ij} is, the more inconsistent p_{ij} (p_{ji}) is with respect to the rest of the information in P. The following holds: $|ep_{ij} - p_{ij}| = |ep_{ij} - cp_{ij}| + |cp_{ij} - p_{ij}|, \forall i, k$, and consequently, $\varepsilon p_{ij} \leq |ep_{ij} - p_{ij}|, \forall i, k$.

B. Fedrizzi–Giove's Reconstruction Method

This method is based on the resolution of an optimization problem with an objective function measuring the "global [additive] inconsistency" of the incomplete fuzzy preference relation [11]. Indeed, based on (1), for each triplet of alternatives (x_i, x_j, x_k), Fedrizzi and Giove define its associated inconsistency contribution as

$$L_{ijk} = (p_{ik} + p_{kj} - p_{ij} - 0.5)^2. \tag{11}$$

It is worth noting that the error between a preference value p_{ij} and its local estimated one obtained using the intermediate alternative x_k, denoted as ep_{ij}^k, is the square root of L_{ijk}.

The global inconsistency index of a fuzzy preference relation P is defined as follows:

$$\rho = 6 \cdot \sum_{i<j<k} L_{ijk}. \tag{12}$$

The missing values in an incomplete fuzzy preference relation are treated as variables in the global consistency index. The stationary vector that minimizes the global inconsistency function is taken as the estimated values for the unknown preference values. Obviously, these estimated values are the most consistent with the available preference values.

Under reciprocity, if a preference value p_{ij} is missing, then the value p_{ji} is also missing. Therefore, it makes sense in this context to denote these two missing preference values as the missing comparison $\{x_i, x_j\}$. When a single comparison $\{x_i, x_j\}$ is missing, Fedrizzi–Giove's method produces the following linear equation:

$$(n-2)p_{ij} - \sum_{\substack{k=1 \\ k \neq j}}^{n} p_{ik} - \sum_{\substack{k=1 \\ k \neq i}}^{n} p_{kj} + \frac{n}{2} = 0 \tag{13}$$

with solution

$$\widehat{p}_{ij} = \frac{1}{n-2}\left[\sum_{\substack{k=1 \\ k \neq j}}^{n} p_{ik} + \sum_{\substack{k=1 \\ k \neq i}}^{n} p_{kj} - \frac{n}{2}\right]. \tag{14}$$

Example 2: Let us assume the same incomplete preference relation of Example 1. Denoting $p_{23} = x$, $p_{24} = y$, and $p_{34} = z$, the global inconsistency index of P is

$$\rho = 6 \cdot \left[(0.9 - x)^2 + (0.7 - y)^2 + (0.3 - z)^2 + (0.5 - x + y - z)^2\right].$$

The optimal solution corresponds to $x = 0.9$, $y = 0.7$, and $z = 0.3$. These values coincide with the estimated values obtained in Example 1 via Herrera-Viedma *et al.*'s method.

To establish the condition under which this method can guarantee the successful reconstruction of an incomplete fuzzy preference relation, the authors introduce the concept of an independent/dependent set of missing comparisons.

1) A set of missing comparisons is called independent when no alternative is shared between any two of their missing comparisons.
2) A set of missing comparisons is called dependent when for every partition of it into two subsets, there exists at least one alternative that is in both subsets.

Each set of missing comparisons can be expressed as a disjoint union of independent and/or dependent sets of missing comparisons.

The maximum cardinality of an independent set of missing comparisons is $\text{int}(n/2) : n/2$ or $(n-1)/2$ for an even or odd number of alternatives, respectively. After reordering the set of alternatives, the maximum cardinality would correspond to the following set of missing comparison values:

$$\left\{\{x_1, x_{\text{int}(n/2)+1}\}, \{x_2, x_{\text{int}(n/2)+2}\}, \{x_3, x_{\text{int}(n/2)+3}\}, \ldots, \{x_{\text{int}(n/2)}, x_{\text{int}(n/2)+int(n/2)}\}\right\}.$$

Fedrizzi and Giove show that the optimal values of an independent set of missing comparisons exist and are computed by solving each one of the corresponding linear equations independently of the rest. However, in the case of a dependent set of missing comparisons, the optimal values exist and are unique if its cardinality is lower than $n-1$. When this is not the case, i.e., the cardinality of a set of missing comparisons is greater than or equal to $n-1$, Fedrizzi–Giove's method does not guarantee the existence nor the uniqueness of estimated values for the missing comparisons.

Note 2: Again, in this method, the optimal values might not belong to the unit interval, and consequently, Fedrizzi and Giove propose, as done by Herrera-Viedma *et al.*, their truncation using the same median function given in Note 1.

Note 3: In [7], the consistency of reciprocal preference relations is modeled via a functional equation, and it is shown that when such a function is almost continuous and monotonic (increasing), then it must be a representable uninorm. Consistency when represented by the conjunctive representable cross-ratio uninorm is equivalent to Tanino's multiplicative transitivity property. In this case, the above problem between the additive consistency property and the [0, 1] scale used for providing the preference values disappears.

III. Comparison Between the Two Methods

Herrera-Viedma *et al.*'s method estimates the missing comparison $\{x_i, x_j\}$ as an average of the local estimated values that can be calculated via all possible intermediate alternatives x_k for which the indirect comparisons $\{x_i, x_k\}$ and $\{x_k, x_j\}$ exist. This method seems to be radically different to the second one, which obtains the estimated values as the vector of values that minimizes the global inconsistency function with variables of the unknown preference values. However, because both methods are driven by the same concept, i.e., the additive consistency property, they share similarities, and therefore, they are not as different as they seem to be.

Following the line of reasoning of Fedrizzi and Giove, we start by showing that both methods provide the same estimated values for the single-missing-comparison case. In general, we show that for independent sets of missing comparisons, both methods derive the same estimated values. It is only for dependent sets of missing comparisons that both methods differ. However, we show that Herrera-Viedma *et al.*'s method is identical to Fedrizzi–Giove's method if the overall estimated values are computed taking into account all $n-2$ intermediate alternatives, regardless of the existence or absence of the indirect comparisons. Numerical examples are used to illustrate these results.

The differences between both reconstruction methods reside not only in the different sets of estimated values that are derived from their application but also in their successful application in reconstructing the original incomplete fuzzy preference relation. When Herrera-Viedma *et al.*'s method is unsuccessful, Fedrizzi–Giove's method is as well. However, in those cases when Fedrizzi–Giove's method cannot guarantee the successful reconstruction of an incomplete preference relation, we have that Herrera-Viedma *et al.*'s method can. These results lead us to claim that both methods should be seen as complementary in their application, and as such, we propose a reconstruction policy of incomplete fuzzy preference relations using both methods.

A. Independent-Missing-Comparison Case

Two missing comparisons $\{x_i, x_j\}$ and $\{x_s, x_t\}$ are called independent if they do not share any alternatives, i.e., $i, j \notin \{s, t\}$. A set of missing comparisons is independent when any two of its missing comparisons are independent. In general, the set of "missing comparisons can be divided in a certain number of independent missing comparisons and some disjoint sets of dependent comparisons" [11]. Obviously, given a missing comparison $\{x_i, x_j\}$ from a set of independent comparisons, the number of intermediate alternatives x_k $(k \neq i, j)$ that can be used to estimate the preference value p_{ij} $(i \neq j)$ using (1) is $n-2$, i.e., $\#H_{ij} = n-2$, and therefore, the overall estimated value ep_{ij} of p_{ij} is

$$ep_{ij} = \sum_{\substack{k=1 \\ k\neq i,j}}^{n} \frac{p_{ik} + p_{kj} - 0.5}{n-2}$$

$$= \frac{1}{n-2}\left[\sum_{\substack{k=1 \\ k\neq i,j}}^{n} (p_{ik} + p_{kj} - 0.5) + (p_{ii} - 0.5) + (p_{jj} - 0.5)\right]$$

which can be expressed as

$$ep_{ij} = \frac{1}{n-2}\left[\sum_{\substack{k=1 \\ k\neq i,j}}^{n} p_{ik} + \sum_{\substack{k=1 \\ k\neq i,j}}^{n} p_{kj} - \frac{n}{2}\right]. \quad (15)$$

The right-hand side of (15) is identical to the right-hand side of the estimated value expression (14) derived by Fedrizzi and Giove, i.e., $ep_{ij} = \widehat{p}_{ij}$ for an independent comparison $\{x_i, x_j\}$. Because the estimated values of independent missing comparisons "can be calculated *independently* from other missing comparisons" [11], we conclude that both reconstruction methods produce the same result in this case. Let us illustrate this fact with a numerical example.

Example 3: Assume the same numerical matrix used by Fedrizzi and Giove [11, p. 312]

$$P = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.8155 & 0.5 & 0.3423 \\ 0.5 & 0.5 & 0.6577 & 0.8155 & 0.5 & 34233 \\ 0.5 & 0.3423 & 0.5 & 0.8662 & 0.75 & 0.3423 \\ 0.1845 & 0.1845 & 0.1338 & 0.5 & 0.25 & 0.25 \\ 0.5 & 0.5 & 0.25 & 0.75 & 0.5 & 0.25 \\ 0.6577 & 0.6577 & 0.6577 & 0.75 & 0.75 & 0.5 \end{pmatrix}.$$

Assume as well that alternatives x_2 and x_3 are incomparable, i.e., the elements p_{23} and $p_{32} = 1 - p_{23}$ are unknown. The overall estimated value obtained applying Herrera-Viedma *et al.*'s method is

$$ep_{23} = \sum_{\substack{k=1 \\ k\neq 2,3}}^{6} \frac{p_{2k} + p_{k3} - 0.5}{6-2} = 0.424825 \approx 0.4248$$

which is identical to the estimated value obtained by Fedrizzi and Giove.

B. Dependent-Missing-Comparison Case

It is obvious that the two reconstruction methods differ in this case. Let us illustrate this fact with another numerical example. Using the same above numerical matrix, assume now that alternatives x_3 and x_5 are also incomparable. The estimated values obtained by applying Fedrizzi–Giove's method are (with four significant decimal places) $\widehat{p}_{23} = 0.4726$ $(\widehat{p}_{32} = 0.5274)$ and $\widehat{p}_{35} = 0.5590$ $(\widehat{p}_{35} = 0.4410)$. The overall estimated values using (8), i.e., taking into account only those intermediate alternatives for which indirect comparisons exist, are different, as given in the equations shown at the bottom of the page.

In general, for a particular set of m dependent missing comparisons, Fedrizzi and Giove obtain a linear system with m equations. For each one of the dependent missing comparisons, i.e., $\{x_i, x_j\}$, the corresponding equation of this system is

$$(n-2)p_{ij} - \sum_{\substack{k=1\\k\neq j}}^{n} pik - \sum_{\substack{k=1\\k\neq i}}^{n} pkj + \frac{n}{2} = 0.$$

This is exactly the same equation that is obtained by Herrera-Viedma *et al.*'s method if all intermediate variables were taken into account in calculating the overall estimated value, regardless of whether indirect preference values exist or not, i.e., if (3) is always used for estimating the missing preference values as shown in (15). To illustrate this case, we compute the overall estimated values ep_{23} and ep_{35} taking into account all $(n-2)$ intermediate alternatives and will show that they are identical to the above $\widehat{p}_{23}$ and $\widehat{p}_{35}$

$$\begin{aligned} ep_{23} &= \frac{1}{4}\left((p_{21}+p_{13}-0.5)+(p_{24}+p_{43}-0.5)\right.\\ &\qquad \left.+\,(p_{25}+ep_{53}-0.5)+(p_{26}+p_{63}-0.5)\right)\\ &= \frac{1.4493+ep_{53}}{4}\\ ep_{53} &= \frac{1}{4}\left((p_{51}+p_{13}-0.5)+(p_{52}+ep_{23}-0.5)\right.\\ &\qquad \left.+\,(p_{54}+p_{43}-0.5)+(p_{56}+p_{63}-0.5)\right)\\ &= \frac{1.2915+ep_{23}}{4}. \end{aligned}$$

The solution to this system of equations is $ep_{23} = 0.47258$ $(ep_{32} = 0.52742)$ and $ep_{53} = 0.44102$ $(ep_{35} = 0.55898)$.

C. Successful Reconstruction of Incomplete Fuzzy Preference Relations

In this section, we will analyze the conditions under which each reconstruction method can successfully be applied in estimating all missing values in an incomplete fuzzy preference relation.

1) Herrera-Viedma *et al.*'s method is unable to estimate all missing values only when there is at least one alternative x_i for which all comparisons in the set $\{\{x_i, x_j\} \mid j = 1, \ldots, n \wedge j \neq i\}$ are missing. This has the effect of having row and column i of a fuzzy preference relation with no entries.
2) Fedrizzi–Giove's method can only guarantee the existence and uniqueness of estimated values for the missing comparisons when cardinalities of dependent sets of missing comparisons are all lower than $n-1$. Therefore, when there is a dependent set of missing comparisons with a cardinality greater than or equal to $n-1$, this method does not guarantee the existence nor the uniqueness of estimated values for the missing comparisons.

The set of missing comparisons $\{\{x_i, x_j\} \mid j = 1, \ldots, n \wedge j \neq i\}$ is dependent with a cardinality $n-1$. Consequently, when Herrera-Viedma *et al.*'s reconstruction method is unsuccessful Fedrizzi–Giove's reconstruction method is unsuccessful as well. However, it is easy to see that there are many situations when Fedrizzi–Giove's method is unsuccessful but Herrera-Viedma *et al.*'s method is successful in estimating all missing comparisons. Take, for example, the case of an incomplete fuzzy preference with the minimal necessary information Herrera-Viedma *et al.*'s methods require to be successful: the presence in the incomplete fuzzy preference relation of just the following set of comparisons $\{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}\}$. When the number of alternatives n is greater than or equal to four, Fedrizzi–Giove's method cannot guarantee the existence nor the uniqueness of estimated values for the $(n-1)\cdot(n-3)$ (dependent) missing comparisons. To illustrate this, we use the above 6×6 fuzzy preference relation and assume that the only comparisons known are $\{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_5, x_6\}\}$. The set of missing comparisons is

$$\begin{gathered}\{\{x_1,x_3\},\{x_1,x_4\},\{x_1,x_5\},\{x_1,x_6\},\{x_2,x_4\},\\ \{x_2,x_5\},\{x_2,x_6\},\{x_3,x_5\},\{x_3,x_6\},\{x_4,x_6\}\}.\end{gathered}$$

This set does not admit any decomposition in a number of disjoint sets of dependent missing comparisons (of cardinality lower than four), and consequently, Fedrizzi–Giove's method cannot guarantee the existence nor the uniqueness of their estimated values.

D. Reconstruction Policy of Incomplete Fuzzy Preference Relations

Given an incomplete reciprocal fuzzy preference relation, this should be reconstructed by first checking if Fedrizzi–Giove's method can guarantee the existence and uniqueness of the most consistent estimated values with the set of available preference values EV; if not, Herrera-Viedma *et al.*'s method should be checked if applicable, and the new estimated values obtained are added to the set of available preference values $(EV \leftarrow EV + EMV)$. Afterward, Fedrizzi–Giove's method is checked again if it can guarantee the existence and uniqueness of the most consistent estimated values for the new set of available preference values. By using this application policy, all incomplete reciprocal preference relations would be reconstructed except when there is at least one alternative x_i such that (p_{ij}, p_{ji}) are unknown for all j. We note that some strategies for dealing with this type of ignorance situation can be found in [3].

$$\begin{aligned} ep_{23} &= \frac{(p_{21}+p_{13}-0.5)+(p_{24}+p_{43}-0.5)+(p_{26}+p_{63}-0.5)}{3}\\ &= 0.4831\\ ep_{35} &= \frac{(p_{31}+p_{15}-0.5)+(p_{34}+p_{45}-0.5)+(p_{36}+p_{65}-0.5)}{3}\\ &= 0.5675. \end{aligned}$$

The proposed reconstruction method pseudocode is given in Algorithm 1.

Algorithm 1 Reconstruction policy using both methods
Input: number of alternatives n; set of available comparisons, EV; set of missing comparisons MV

1. **if** maximum of cardinality of sets of dependent missing comparisons $\leq n-2$ **then**
2. Apply Fedrizzi–Giove's reconstruction method
3. End
4. **else if** $EMV \neq \emptyset$ **then**
5. Apply one iteration of Herrera-Viedma *et al.*'s reconstruction method
6. $MV \leftarrow MV \setminus EMV$
7. $EV \leftarrow EV \cup EMV$
8. Go to 1.
9. **end if**

Output: EV, MV

IV. Conclusion

This note has presented and compared two methods for calculating the missing values of an incomplete fuzzy preference relation. Both methods are driven by the additive consistency property. Both methods, as originally presented, provide the same set of solutions for independent sets of missing comparisons but not for dependent missing comparisons. It has also been shown that a modification of Herrera-Viedma *et al.*'s coincides with Fedrizzi–Giove's method. However, the main difference between both methods resides in their successful application in reconstructing an incomplete fuzzy preference relation. Fedrizzi–Giove's method performs worse than Herrera-Viedma *et al.*'s method for a large number of alternatives. This latter method fails (as well as the former) to complete an incomplete fuzzy preference relation only when no preference values are known for at least one of the alternatives. All of these together lead us to consider both methods as complementary, rather than antagonistic, in their application, and as such, we have proposed a new policy for reconstructing incomplete fuzzy preference relations that make use of both methods.

References

[1] S. Alonso, F. J. Cabrerizo, F. Chiclana, F. Herrera, and E. Herrera-Viedma, "An interactive decision support system based on consistency criteria," *J. Mult.-Valued Log. Soft Comput.*, vol. 14, no. 3–5, pp. 371–386, 2008.

[2] S. Alonso, F. Chiclana, F. Herrera, and E. Herrera-Viedma, "A learning procedure to estimate missing values in fuzzy preference relations based on additive consistency," in *Proc. MDAI*, 2004, vol. 3131, pp. 227–238.

[3] S. Alonso, E. Herrera-Viedma, F. Chiclana, and F. Herrera, "Individual and social strategies to deal with ignorance situations in multiperson decision making," *J. Inf. Technol. Decis. Making*, Dec. 2008. to be published.

[4] J. Bezdek, B. Spillman, and R. Spillman, "A fuzzy relation space for group decision theory," *Fuzzy Sets Syst.*, vol. 1, no. 4, pp. 255–268, Oct. 1978.

[5] F. J. Carmone, Jr., A. Kara, and S. H. Zanakis, "A Monte Carlo investigation of incomplete pairwise comparison matrices in AHP," *Eur. J. Oper. Res.*, vol. 102, no. 3, pp. 533–553, Nov. 1997.

[6] F. Chiclana, F. Herrera, and E. Herrera-Viedma, "Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations," *Fuzzy Sets Syst.*, vol. 97, no. 1, pp. 33–48, Jul. 1998.

[7] F. Chiclana, E. Herrera-Viedma, S. Alonso, and F. Herrera, "Cardinal consistency of reciprocal preference relations: A characterization of multiplicative transitivity," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 1, pp. 14–23, Feb. 2009.

[8] F. Chiclana, F. Mata, L. Martinez, E. Herrera-Viedma, and S. Alonso, "Integration of a consistency control module within a consensus decision making model," *Int. J. Uncertain. Fuzziness Knowl.-Based Syst.*, vol. 16, pp. 35–53, Apr. 2008. suppl. 1.

[9] D. Dubois and H. Prade, "The three semantics of fuzzy sets," *Fuzzy Sets Syst.*, vol. 90, no. 2, pp. 141–150, Sep. 1997.

[10] D. H. Ebenbach and C. F. Moore, "Incomplete information, inferences, and individual differences: The case of environmental judgements," *Org. Behav. Human Decis. Process.*, vol. 81, no. 1, pp. 1–27, Jan. 2000.

[11] M. Fedrizzi and S. Giove, "Incomplete pairwise comparison and consistency optimization," *Eur. J. Oper. Res.*, vol. 183, no. 1, pp. 303–313, Nov. 2007.

[12] P. C. Fishburn, *Utility Theory for Decision Making*. Melbourne, FL: Krieger, 1979.

[13] E. Herrera-Viedma, F. Chiclana, F. Herrera, and S. Alonso, "Group decision-making model with incomplete fuzzy preference relations based on additive consistency," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 37, no. 1, pp. 176–189, Feb. 2007.

[14] E. Herrera-Viedma, F. Herrera, F. Chiclana, and M. Luque, "Some issues on consistency of fuzzy preference relations," *Eur. J. Oper. Res.*, vol. 154, no. 1, pp. 98–109, Apr. 2004.

[15] S. H. Kim and B. S. Ahn, "Group decision making procedure considering preference strength under incomplete information," *Comput. Oper. Res.*, vol. 24, no. 12, pp. 1101–1112, Dec. 1997.

[16] J. K. Kim and S. H. Choi, "A utility range-based interactive group support system for multiattribute decision making," *Comput. Oper. Res.*, vol. 28, no. 5, pp. 485–503, Apr. 2001.

[17] J. K. Kim, S. H. Choi, C. H. Han, and S. H. Kim, "An interactive procedure for multiple criteria group decision making with incomplete information," *Comput. Ind. Eng.*, vol. 35, no. 1/2, pp. 295–298, Oct. 1998.

[18] I. Millet, "The effectiveness of alternative preference elicitation methods in the analytic hierarchy process," *J. Multi-Criteria Decis. Anal.*, vol. 6, no. 1, pp. 41–51, 1997.

[19] H. Nurmi, "Approaches to collective decision making with fuzzy preference relations," *Fuzzy Sets Syst.*, vol. 6, no. 3, pp. 249–259, Nov. 1981.

[20] T. Tanino, "Fuzzy preference orderings in group decision making," *Fuzzy Sets Syst.*, vol. 12, no. 2, pp. 117–131, Feb. 1984.