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A note on uncertainty and discounting in models of economic growth

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Abstract The implications of uncertainty for appropriate discounting in models of economic growth have been studied at some length, notably, (*Review of Economic Studies*, 36:153–163; 1969) and (*Journal of Public Economics*, *85*:149–166; 2002). A detailed account has now appeared in *Journal of Risk and Uncertainty*, *37*:141–169; 2008, sections 4 and 5 (pp. 160–166). One interesting, if perhaps minor, aspect is that under certain circumstances, there appeared to be no solution or at least no satisfactory one. More importantly, the formulas are usually given for the log normal case and are somewhat complicated and hard to interpret intuitively. I show here that assuming a general distribution for returns to capital gives simpler and more understandable results.

Keywords Uncertainty · Discount rate · Intertemporal optimization · Relative risk aversion

JEL C61 · D9 · H43 · O41

1 The general model

I follow Dasgupta's notation in general. Here, r_t is the net return on capital committed for one period, so that the gross return is $1+r_t$ per unit of capital committed. At the beginning of time t, the capital is K_t. The individual saves a fraction, s_t, of that capital so that consumption in period t is, $(1-s_t)\,K_t$ and $s_t K_t$ is

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available for investment. One unit of investment yields a random gross return, $1+r_t$, so that,

$$K_{t+1} = s_t\, Kt(1 + r_t), \tag{1}$$

and,

$$c_t = (1 - s_t)K_t. \tag{2}$$

Note that, under Eq. 1, all investments are, in effect, made for one period; this is a "circulating capital" model, in a somewhat old-fashioned terminology. It differs from the assumption made by Gollier (2008, p.174) in which an investment yields return only after a fixed period of time.

Assume that,

$$\text{the random variables, } \{\mathrm{r_t}\}, \text{ are i.i.d.} \tag{3}$$

I make here the usual assumption that felicity, U, is given by,

$$U(c) = (1 - \eta)^{-1} c^{1-\eta},\ \eta > 1, \tag{4}$$

and welfare, V_t, at time t by,

$$V_t = E\left\{\sum_{s=t}^{\infty} (1 + \delta)^{-(s-t)} U(c_s)\right\}. \tag{5}$$

where $1+\delta$ is a discount factor. I comment on the assumption (4) in Section 6 below.

Problem Maximize V_0 with respect to the savings ratios, s_t, for a given value of K_0 (In general, s_t can be a function of the history up to time t, i.e., the values of K_0 and of $r_s (0 \le s < t)$.).

From the homotheticity of U and the homogeneity of degree 1 of the production relations (1–2), it is obvious that, if an optimum exists, s_t must be a constant, independent of t and of history. Then, Eqs. 1 and 2 can be written,

$$K_{t+1} = s\, K_t (1 + r_t), \tag{1'}$$

$$c_t = (1 - s)\, K_t. \tag{2'}$$

From Eq. 1′,

$$K_t = s^t \left[\prod_{s=0}^{t-1} (1 + r_s)\right] K_0.$$

Then, from Eqs. 2′, 3, and 4,

$$E[U(c_t)] = (1 - \eta)^{-1}(1 - s)^{1-\eta}\left(s^{1-\eta}\right)^t \left\{E\left[(1 + r)^{1-\eta}\right]^t K_0^{1-\eta}.\right. \tag{6}$$

and therefore, from Eq. 5,

V_0 is a (negative) constant times a geometric series

$$\text{with factor, } s^{1-\eta} E\left[(1+r)^{1-\eta}\right] \Big/ (1+\delta). \tag{7}$$

2 Non-existence or catastrophe

Dasgupta (2008, Proposition 4, p. 165) states a condition under which, "no optimum policy exists." This seems to be a somewhat unclear conclusion. Of course, there are conditions under which there is a supremum which is not an optimum, i.e., a sequence of policies with increasing payoffs which are eventually better than any given payoff. This interpretation does not appear relevant to the present case.

It is clear that the would-be optimization depends on three factors (call them *parameters*): η, δ, and the density of $1+r$, call it ϕ. For any given set of parameters, either $V_0=-\infty$ for all strategies or V_0 is finite for some strategies. (A strategy is a function giving the savings rate as a function of the capital stock.) The following is surely true:

$$\begin{aligned}&\text{If the parameters are such that } V_0 \text{ is finite}\\ &\text{for some choice of strategies, then there}\\ &\text{exists an optimal strategy (which must, of course, be a fixed savings rate).}\end{aligned} \tag{8}$$

If the parameters are such that $V_0=-\infty$ for all strategies, one might say that all strategies are optimal. It would perhaps be better to call this condition, a *catastrophe.* In fact, this term accords with Dasgupta's characterization [2008, p. 165]: "To put it crudely, every saving policy yields an infinitely awful outcome."

Now, from Eq. 7, catastrophe holds if and only if,

$$s^{1-\eta} E\left[(1+r)^{1-\eta]}\right) \Big/ (1+\delta) \geq 1. \tag{9}$$

Since the left-hand side is a decreasing function of s, and, necessarily, $s \leq 1$, catastrophe holds for a given set of parameters if and only if Eq. 9 holds for $s=1$ (in that case, it holds for all s, while if Eq. 9 fails to hold for some s, so that, by Eq. 7, there is an optimal policy, it fails to hold for $s=1$).

Theorem 1 *If felicity is characterized by constant relative risk aversion greater than 1, then welfare equals* $-\infty$ *for all possible saving strategies if and only if,*

$$E\left[(1+r)^{1-\eta}\right] \geq 1+\delta.$$

The case where $\eta<1$ yields parallel results. The word, "negative," in Eq. 7 has to be changed to, "positive," and the references to "catastrophe" changed to, "bliss," the possibility that welfare can be positively infinite. Note that the left-hand side of Eq. 9 is now an increasing function of s.

Theorem 2 *If felicity is characterized by constant relative risk aversion less than 1, then welfare can reach $+\infty$ by some savings ratio if and only if,*

$$E\left[(1+r)^{1-\eta}\right] \geq 1+\delta.$$

The case of a logarithmic felicity function, where the relative risk aversion is 1 everywhere, gives an interesting result. The reasoning leading to Eq. 6 now yields,

$$E[U(c_t)] = \ln(1-s) + \{\ln s + E[\ln(1+r)]\}\, t + \ln K_0.$$

Using the identity,

$$\sum_{t=o}^{\infty} t(1+\delta)^{-t} = (1+\delta)/\delta^2,$$

yields, from Eq. 5,

$$V_0 = [(1+\delta)/\delta]\ln(1-s) + \left[(1+\delta)/\delta^2\right]\ln s + \left[(1+\delta)/\delta^2\right]E[\ln(1+r)] + [(1+\delta)/\delta]K_o. \tag{10}$$

The policy variable, s, appears in only the first two terms, so its optimal value depends only on the time preference parameter, δ; its value is given by,

$$s = l/(1+\delta),$$

which lies between 0 and 1. However, the value of the welfare might be $+\infty$ or $-\infty$ with $E[\ln(1+r)]$.

Theorem 3 *If felicity is characterized by a constant relative risk aversion equal to 1 and if E[log (1+r)] is finite, then welfare can never equal $+\infty$, and the savings rate can always be chosen so that welfare is not equal to $-\infty$.*

3 Some interpretative remarks

(a) Raise both sides of the condition in Theorem 1 to the power $1/(1-\eta)$ which is negative. The condition becomes,

$$\mu_{1-\eta}(1+r) \leq (1+\delta)^{1/(1-\eta)}, \tag{11}$$

where $\mu_{1-\eta}(1+r)$ is the mean of order $1-\eta$ of the random variable, $1+r$. Hence, in some sense, at least, catastrophe requires that a suitably chosen average of $1+r$ is sufficiently small. Note that the right-hand side of Eq. 11 can easily be seen to be less than 1, so this can hold only if, on the average, there is a negative return to capital.

(b) There is nothing in Theorem 1 or Eq. 11 that demands uncertainty. If r is non-stochastic, Eq. 11 becomes the condition,

$$1+r \leq (1+\delta)^{1/(1-\eta)},$$

so there can be catastrophe even without uncertainty. As just remarked, this can only hold if there is a negative return to capital, in this case, for certain.

(c) However, uncertainty does matter. How shall we measure an increase in uncertainty? A frequently-used definition is that of a *mean-preserving spread* (e.g., Rothschild and Stiglitz (1970)), interpreted as an increase in dispersion: If x is a random variable and $x'=x+y$, where y is non-degenerate and independent of x and $E(y)=0$, then x' is a *mean-preserving spread* of x. We then consider x' to have more uncertainty than x.

This definition clearly doesn't work for non-negative variables, so we apply it to their logarithms. I.e., we consider $1+r'$ to be more variable than $1+r$ if $\ln(1+r')$ is a mean-preserving spread of $\ln(1+r)$.

Let us examine the moments of a non-degenerate random variable, y. (We assume that all moments are finite.) Define the function,

$$F(m) = E(y^m),$$

where m can take on any real value, positive or negative. Then, by differentiation,

$$F'(m) = E(y^m \ln y),\ F''(m) = E\left[y^m (\ln y)^2\right].$$

Then $F''(m)>0$ for all m, so that F is a strictly concave function. Now assume,

$$E(\ln y) = 0. \tag{12}$$

The minimum of F will occur when $F'=0$; by Eq. 12, this holds when $m=0$. Since $F(0)=1$, $F(m)>1$ for all $m\neq 0$. Further, since F is strictly convex, it must be that,

$$\lim_{m\to\infty} F(m) = +\infty,\ \lim_{m\to-\infty}, F(m) = +\infty,$$

Lemma 1 *If y is a non-degenerate non-negative random variable for which $E(\ln y)=0$ and $E(y^m)$ is finite for all m, then $E(y^m)>1$ for all $m\neq 0$, and*

$$\lim_{m\to-\infty} F(m) = +\infty.$$

If $ln(1+r')$ is a mean-preserving spread of $ln(1+r)$, then, by definition, $y=(+r')/(1+r)$ is independent of $1+r$; y is non-degenerate and non-negative, and $E(ln\ y)=0$. From Lemma 1, $E(y^m)>1$. But,

$$E[(1+r')^m] = E\{[(1+r)y)]^m\} = E[(1+r)^m y^m] = E[(1+r)]^m E(y^m),$$

since $1+r$ and y are independent and therefore $(1+r)^m$ and y^m are independent. Therefore,

$$E[(1+r')^m] > E[(1+r)^m).$$

Hence, a mean-preserving spread increases any moment (other than the zeroth), and in particular, the moment defined by $m=1-\eta$. It therefore makes more likely the satisfaction of the condition of Theorem 1 for a catastrophe.

We can say more. Suppose we consider a family of mean-preserving spreads obtained by multiplying a given spread by a positive parameter, i.e., define,

$$ln\left(1+r_p\right) = ln(1+r) + p\, ln\, y, \tag{13}$$

where, as before, y is a non-degenerate non-negative random variable, independent of $1+r$, and p a positive parameter. Then $1+r_p=(1+r)\, y^p$, and, for any m,

$$E\left[\left(1+r_p\right)^m\right] = E[(1+r)^m]E(y^{pm}).$$

Set $m=1-\eta<0$. Then, by Lemma 1,

$$\lim_{p\to\infty} E(y^{pm}) = +\infty,$$

and therefore,

$$\lim_{p\to\infty} E\left[\left(1+r_p\right)^{1-\eta}\right] = +\infty.$$

Theorem 4 *If felicity is characterized by a constant relative risk aversion greater than 1, then, for any distribution of r (productivity of capital), there is a sufficiently wide mean-preserving spread of ln (1+r) such that catastrophe holds.*

(d) For the record, it is easy to derive the optimal savings ratio when there is no catastrophe. When the condition of Theorem 1 does not hold, it can easily be seen from (5) and (6) that,

$$V_0 = (1-\eta)^{-1}(1-s)^{=1}\left\{1-(1+\delta)^{-1}s^{1-\eta}E\left[(1+r)^{1-\eta}\right]\right\}^{-1}.$$

By differentiation, the optimal value of s can easily be found.

Theorem 5 *If* $E\left[(1+r)^{1-\eta}\right] < 1+\delta$, *then the optimal ratio of consumption to capital is given by,*

$$s = \left\{E\left[(1+r)^{1-\eta}\right]\Big/(1+\delta)\right\}^{1/\eta}.$$

4 Thick-tailed distributions

Although much of the literature concentrates on the case where the distribution of gross return is log-normal, there has been a recent emphasis on the possibility that the distribution has thick tails, either because it is the result of a Bayesian inference about the parameters or because of serial correlation over time, here excluded by assumption (3); see Weitzman (2009) and Gollier (2008) (which also cites some earlier literature). These papers raise some very important issues beyond the scope of this note. I simply point out some direct implications of the present model when the distribution is thick-tailed.

Theorem 1 shows immediately that if $E\left[(1+r)^{1-\eta}\right]=+\infty$, catastrophe holds for any δ. For this to happen, it must be that,

$$\int_x (1+r)^{1-\eta}\phi(1+r)\,d(1+r)\;approaches+\infty\;as\,x\,approaches\,0.$$

(Recall that ϕ is the density of $1+r$.) To see how this can happen, suppose that ϕ is thick-tailed at the origin, specifically that it is asymptotic to a power function, say $(1+r)^a$, i.e., $(1+r)^{-a}\,\phi(1+r)$ is bounded away from zero and infinity as r approaches -1. Note that the lower the value of a, the more thick-tailed the distribution is. Then the integral of $\phi(1+r)$ is of the order of $(1+r)^{1+a}$ and therefore converges as the lower limit approaches 0 if and only if $a>-1$ ($a=-1$ integrates to the logarithm). Of course, since ϕ is a density, the integral does converge. Similarly, $E\left[(1+r)^{1-\eta}\right)$ is finite if and only if $0<2-\eta+a$. Equivalently,

Theorem 6 *If felicity is characterized by a constant relative risk aversion, $\eta>1$, and if, for some a, the density $\phi(1+r)$ is asymptotic to $(1+r)^a$ as $1+r$ approaches 0, then $a>-1$. Catastrophe holds for all δ if and only if $a\leq\eta-2$.*

Thus, the existence of catastrophe holds if the distribution of returns is sufficiently thick-tailed *relative to the coefficient of relative risk aversion*. Thick tails are not alone sufficient for catastrophe.

5 Reflections on the modeling

I am inclined to the view that the problem of possible catastrophe is a defect of the modeling, rather than a genuine issue calling for unlimited precautions. The problem is that, when $\eta>1$, the utility function approaches $-\infty$ as consumption approaches zero. Can we seriously discuss infinite values of utility, positive or negative? The case of positive infinities is the one addressed in the famous St. Petersburg paradox. The paradox implied more than that a player would prefer the St. Petersburg game to any finite certainty. It also implied that, given two certain outcomes, one better than the other, getting the lesser outcome with probability $1-p$ and entering the St. Petersburg game with probability p is preferred to the better certain outcome now matter how small p is. Hence, preferences among gambles become discontinuous.

Despite its famous apparent resolution through Daniel Bernoulli's expected-utility theory (1738, 1954), a deeper understanding was achieved only with Karl Menger's paper (1934). He showed that for *any* unbounded utility function (such as the logarithmic proposed by Daniel Bernoulli), a St. Petersburg-like paradox would hold.

What has escaped attention is that there is also a downward St. Petersburg paradox. If utility is unbounded below and approaches $-\infty$ as consumption approaches zero, then there are consequences entirely parallel to those in the case of utility unbounded above. Any certain outcome, no matter how bad, is preferred to a gamble which yields utility $A2^{-n}$ with probability 2^{-n}, no matter how large A is.

Also, as before, preferences among gambles become discontinuous in the probabilities.

Specifically, when $U(0)=-\infty$, an individual will take no risk of increasing the chance of achieving 0 for any payoff whatever. If, as is common, zero consumption is taken to be equivalent to death, then it follows that the value of statistical life is infinite, a conclusion clearly contrary to all empirical evidence and to everyday observation.

To achieve boundedness above, relative risk aversion must approach a limit greater than 1 as consumption goes to infinity; to achieve boundedness below, relative risk aversion must approach a limit less than 1 as consumption approaches 0. So long as we are basically interested in growth, the standard assumption that $\eta>1$ is a good approximation; if we are interested in avoiding catastrophe, the alternative assumption, $\eta<1$ would seem more appropriate.

What is true is that the logarithmic utility function is, to some extent, a useful compromise. It permits unboundedness in both directions but to the minimal possible degree.

However, the concern with climate change, where both technological progress and the possibility of unexpectedly large negative responses to climate change are important considerations, it appears that the very convenient assumption of an isoelastic utility function must regretfully be abandoned. With current computing capacity, solving stochastic growth models with felicity functions with increasing relative risk aversion (a now standard assumption in the economics of finance and insurance) offers no great difficulties.

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