

A NOTE ON UNIFORM CONVERGENCE OF STOCHASTIC PROCESSES¹

BY NARESH C. JAIN AND G. KALLIANPUR

University of Minnesota

0. Introduction. Our aim in this note is to extend Theorems 5.1 and 5.2 of [4]. Let $R(\cdot, \cdot)$ be the covariance kernel of a Gaussian process with index set S , here S will always mean a compact metric space. R is assumed throughout to be continuous on $S \times S$. Let $H(R)$ be the reproducing kernel Hilbert space of R . It is a Hilbert space of continuous functions k on S with the following properties:

$$(0.1) \quad R(\cdot, t) \in H(R) \quad \text{for each } t \in S;$$

$$(0.2) \quad \langle k, R(\cdot, t) \rangle = k(t),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $H(R)$. For a discussion of reproducing kernel Hilbert spaces and their application to the study of Gaussian processes we refer to [1] and [5]. In what follows $C(S)$ will denote the Banach space of real-valued continuous functions on S with the sup norm, and \mathcal{C} the σ -algebra of Borel sets of $C(S)$. x will denote a generic element of $C(S)$.

Before stating the main results we would like to record here for later reference the fact that if $\{X_t, t \in S\}$ is a Gaussian process on some probability space (Ω, \mathcal{F}, P) , then there is an isometric isomorphism between $H(R)$ and the closure of the linear space spanned by $\{X_t, t \in S\}$ in $L_2(\Omega, \mathcal{F}, P)$. We shall denote this closure by $\mathcal{L}_2(X_t, t \in S)$ and this isometric isomorphism by θ , where for $t \in S$ we have $\theta(R(\cdot, t)) = X_t$. We now state the main results.

THEOREM 1. *Let $\{X_t, t \in S\}$ be a Gaussian process with covariance R and almost all paths continuous on a complete probability space (Ω, \mathcal{F}, P) . Let $\{\psi_j\}_{j=1}^\infty$ be a complete orthonormal system (CONS) in $H(R)$ and let $\{\xi_j\}_{j=1}^\infty$ be the sequence of independent random variables on (Ω, \mathcal{F}, P) each distributed normally with mean 0 and variance 1, given by $\xi_j = \theta(\psi_j)$. Then the partial sums*

$$(0.3) \quad \sum_{j=1}^n \xi_j(\omega) \psi_j(t) = S_n(t, \omega)$$

converge uniformly in $t \in S$ to $X_t(\omega)$ as $n \rightarrow \infty$ a.e. (P).

THEOREM 2. *Let $\{\eta_j\}_{j=1}^\infty$ be a sequence of independent random variables on a complete probability space (Ω, \mathcal{F}, P) , each distributed normally with mean 0 and variance 1. Let R be a covariance such that there exists a Gaussian process with this covariance and with almost all sample paths continuous (on some probability space). Let $\{\psi_j\}_{j=1}^\infty$ be a CONS in $H(R)$. If $S = [0, 1]$, then the partial sums*

$$(0.4) \quad \sum_{j=1}^n \eta_j(\omega) \psi_j(t) = S_n'(t, \omega)$$

converge uniformly in $t \in [0, 1]$ to a Gaussian process on (Ω, \mathcal{F}, P) whose covariance is R and almost all of whose sample paths are continuous as $n \rightarrow \infty$ a.e. (P).

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We would like to remark here that in the case of standard Brownian motion $H(R)$ is the space of absolutely continuous functions $f(t) = \int_0^t f'(u) du$ with $\int_0^1 f'^2(u) du < \infty$. If $\{\varphi_n\}_{n=1}^\infty$ is a CONS in $L_2[0, 1]$, then $\int_0^t \varphi_n(u) du, n = 1, 2, \dots$, is a CONS in $H(R)$ for Brownian motion and so Theorems 5.1 and 5.2 [4] become particular cases of our theorems 1 and 2 respectively. It should be observed, however, that in Theorem 5.2 [4] it is not presumed that standard Brownian motion on $[0, 1]$ can be defined to have continuous paths; this fact is automatically proved there. However, in a general result such as Theorem 2, one would need some conditions on R to guarantee that a Gaussian process with covariance R and having continuous paths exists.

1. Proofs. The following lemma is given for later use.

LEMMA 1. *Let $\{\psi_n\}_{n=1}^\infty$ be a CONS in $H(R)$, then $\sum_{n=1}^\infty \psi_n^2(t)$ converges uniformly in $t \in S$ to $R(t, t)$.*

PROOF. Given $t \in S, R(\cdot, t) \in H(R)$, hence by the Parseval relation $\langle R(\cdot, t), R(\cdot, t) \rangle = \sum_{n=1}^\infty \langle R(\cdot, t), \psi_n \rangle^2 = \sum_{n=1}^\infty \psi_n^2(t)$. On the other hand $\langle R(\cdot, t), R(\cdot, t) \rangle = R(t, t)$. Hence $\sum_{n=1}^\infty \psi_n^2(t)$ converges to $R(t, t)$ for every $t \in S$. Dini's theorem applied to $f_n(t) = R(t, t) - \sum_{j=1}^n \psi_j^2(t)$ now shows that $f_n(t) \rightarrow 0$ uniformly in $t \in S$ as $n \rightarrow \infty$.

Before proceeding with the proofs of Theorems 1 and 2 we introduce some more notation. z will stand for a generic element of the topological dual of $C(S)$; thus z is a finite signed Borel measure on S . For $x \in C(S)$, (z, x) will denote the value z takes at x . We define by $S_n(S'_n)$ the mapping of Ω into $C(S)$ ($C[0, 1]$) given for $\omega \in \Omega$ by $S_n(\omega) = S_n(t, \omega), t \in S$ ($S'_n(\omega) = S_n(t, \omega), 0 \leq t \leq 1$). Then $S_n(S'_n)$ are strongly measurable $C(S)$ ($C[0, 1]$)-valued random variables on (Ω, \mathcal{F}, P) .

PROOF OF THEOREM 1. $R(\cdot, t)$ has an expansion in $H(R)$ given by

$$\sum_{j=1}^\infty \langle \psi_j, R(\cdot, t) \rangle \psi_j(\cdot) = \sum_{j=1}^\infty \psi_j(t) \psi_j(\cdot).$$

By the isometric isomorphism between $H(R)$ and $\mathcal{L}_2(X_t, t \in S)$ we conclude that $\sum_{j=1}^\infty \psi_j(t) \xi_j$ converges in $\mathcal{L}_2(\Omega, \mathcal{F}, P)$ to X_t for every $t \in S$. Since $\sum_{j=1}^\infty \psi_j(t) \xi_j$ is a series of independent random variables, it follows that it converges to X_t a.e. (P). Let X denote the mapping of Ω into $C(S)$ corresponding to $(X_t, t \in S)$ defined the same way as S_n and S'_n . We now show that $S_n \rightarrow X$ a.e. (P) as $C(S)$ -valued random variables. The argument is essentially the same as in [4] page 45. It is enough by Theorem 4.1 [4] ((e) \Rightarrow (a)) that the random variables (z, S_n) converge in probability to (z, X) for every continuous linear functional z on $C(S)$. We have

$$\begin{aligned} E[|(z, S_n) - (z, X)|] &= E[|\int_S z(dt)(S_n(t) - X_t)|] \\ &\leq \int_S |z|(dt) E|S_n(t) - X_t|, \\ &\leq \int_S |z|(dt) E^\frac{1}{2}[S_n(t) - X_t]^2 \\ &= \int_S |z|(dt) (\sum_{j=n+1}^\infty \psi_j^2(t))^\frac{1}{2}, \end{aligned}$$

(where $|z|$ = total variation of z).

But $\sum_{j=n+1}^\infty \psi_j^2(t) \rightarrow 0$ uniformly in $t \in [0, 1]$ as $n \rightarrow \infty$ by Lemma 1, hence the last expression above tends to 0 as $n \rightarrow \infty$. This completes the proof.

PROOF OF THEOREM 2. We will reduce the proof of this theorem to that of Theorem 1. We first prove that for a fixed $t \in [0, 1]$, $S_n'(t, \omega)$ converges a.e. (P) to a random variable $Y_t(\omega)$, where $\{Y_t, t \in [0, 1]\}$ is a Gaussian process on (Ω, \mathcal{F}, P) with covariance R . The proof of this is similar to that of Theorem 4 [6]. Note first that an application of Parseval's relation as in Lemma 1 shows that the series $\sum_{j=1}^{\infty} \psi_j(s)\psi_j(t)$ converges to $R(s, t)$. For $t \in [0, 1]$, the a.e. (P) convergence of $S_n'(t, \omega)$ follows from the fact that the random variable $\eta_j(\omega)\psi_j(t)$, $j = 1, 2, \dots$, are independent with mean 0 and variance $\psi_j^2(t)$ and by Lemma 1 we have $\sum_{j=1}^{\infty} \psi_j^2(t) < \infty$. We denote this limit of $S_n'(t, \omega)$ by $Y_t(\omega)$, defining Y_t arbitrarily on the P -null set where the limit may not exist. It is now clear that the process $\{Y_t, t \in [0, 1]\}$ is a Gaussian process on (Ω, \mathcal{F}, P) with covariance R . Let $\{Z_t, t \in [0, 1]\}$ be a separable version of $\{Y_t, t \in [0, 1]\}$, which exists by Theorem 2.4 [3] on the same probability space (Ω, \mathcal{F}, P) . Noting that there exists a Gaussian process on some probability space with covariance R and almost all paths continuous and the fact that the Z_t process is separable, we conclude from Theorem 9.2 [2] that there is a set $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$ such that for $\omega \in \Omega_0$ the Z_t process has continuous paths. Let Z be the corresponding $C[0, 1]$ -valued strongly measurable random variable on (Ω, \mathcal{F}, P) . For a discussion of such facts as Z defining a strongly measurable random variable we refer to page 57 [2]; what is needed is that Z map Ω into $C[0, 1]$ and Z_t be a random variable for each $t \in [0, 1]$. This remark applies also to the definition of S_n, S_n' and X . We thus have proved so far that for each $t \in [0, 1]$ the random variables $S_n'(t, \omega)$ converge a.e. (P) to $Z_t(\omega)$, where the Z_t process is a Gaussian process on $[0, 1]$ with almost all paths continuous and covariance R . We now show exactly as in the proof of Theorem 1 that for any continuous linear functional z on $C[0, 1]$ the random variables (Z, S_n') converge in probability to the random variable (z, Z) ; all we have to do is to replace S by $[0, 1]$ everywhere in the very last part of the proof of Theorem 1. This proves Theorem 2.

REMARK. Let $S = [0, 1]$. Let $\{\lambda_j\}_{j=1}^{\infty}$, $\lambda_j > 0$, be the eigenvalues of R and $\{\varphi_j\}_{j=1}^{\infty}$ the corresponding normalized (in $L_2[0, 1]$) eigenfunctions. Then it is known that $\{\lambda_j^{\frac{1}{2}}\varphi_j\}_{j=1}^{\infty}$ is a CONS in $H(R)$. Thus whenever Theorem 1 or Theorem 2 applies, the uniform convergence in t of the "Karhunen-Loève" expansion for the process follows as a special case.

REFERENCES

- [1] ARONSZAJN, N. (1950). Theory of reproducing kernels. *Trans. Amer. Math. Soc.* **68**.
- [2] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.
- [3] DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York.
- [4] ITO, K. and NISIO, M. (1968). On the convergence of sums of independent Banach space valued random variables. *Osaka J. Math.* **5** 35-48.
- [5] PARZEN, E. T. (1959). Statistical inference on time series by Hilbert space methods. **1**. Dept. of Statistics, Stanford Univ. Tech. Report No. 23.
- [6] SHEPP, L. A. (1966). Radon-Nikodym derivatives of Gauss measures. *Ann. Math. Statist.* **37** 321-354.