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A NOTE ON UNIQUELY MAXIMAL BANACH SPACES

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Let X be a real or complex Banach space with norm $\|.\|$. Let G denote the set of all isometric automorphisms on X. Then G is a bounded subgroup of the group of all invertible operators $GL(X)$ in $B(X)$. We shall call G the group of isometries with respect to the norm $\|.\|$. A bounded subgroup of $GL(X)$ is said to be maximal if it is not contained in any larger bounded subgroup. The Banach space X has maximal norm if G is maximal. Hilbert spaces have maximal norm. For the (real or complex) spaces c_0, l_p $(1 \leqq p < \infty)$, $L_p[0,1]$ $(1 \leqq p < \infty)$, Pelczynski and Rolewicz have shown that the standard norms are maximal ([**3**], pp. 252–265). In finite dimensional spaces the only maximal groups of isometries are the groups of orthogonal transformations. Given any bounded group H in $B(X)$, X can be renormed equivalently so that each $T \in H$ is an isometry, by $\|x\|_1 = \sup\{\|Tx\|;\ T \in H\}$. Therefore corresponding to every maximal subgroup G there is at least one maximal norm for which G is the group of isometries. In this paper we shall investigate those maximal groups G for which there is only one maximal norm with G as its group of isometries.

We have the following definition:

Definition 1. The Banach space X has *uniquely maximal* norm if it has maximal norm and there is no equivalent norm, not a linear multiple of the original norm, with the same group of isometries.

Uniquely maximal and maximal are not equivalent for norms on a Banach space. Consider the following examples.

Example 2. The standard norm in the real Banach space l_1 is maximal. The isometries in l_1 are of the form $U(\{x_n\}) = \{\alpha_n x_{\sigma(n)}\}$ where $\alpha_n = \pm 1$ and σ is a permutation ([**1**], p. 178). Define

$$\|x_n\|_0 = \sup\{\textstyle\sum x_n y_n;\ |y_n| \leqq 1, |y_n - y_m| \leqq 1, |y_n + y_m| \leqq 1\}.$$

Then $\|.\|_0$ is an equivalent norm on l_1 with the same group of isometries as the usual norm $\|.\|_1$, and is not a linear multiple of the original norm. Hence the standard norm in l_1 is not uniquely maximal.

Example 3. In [**2**], Kalton and Wood showed that the uniform norm is a maximal norm for the complex Banach space $C[0,1]$. The isometries in $C[0,1]$ are of the form $U(f)(t) = \alpha(t) f(\phi(t))$ for all $f \in C[0,1]$, $t \in [0,1]$ where $\alpha(t)$ is a continuous function such that $|\alpha(t)|=1$ for all $t\in[0,1]$ and ϕ is a homeomorphism of $[0,1]$ (see [1], p. 173). Define $\|f\|_1=\|f\|+|f(0)|+|f(1)|$ for all $f\in C[0,1]$. Then $\|.\|_1$ is an equivalent norm with the same group of isometries but is not a linear multiple of the original norm. Hence the uniform norm on $C[0,1]$ is not uniquely maximal.

We shall now obtain a characterisation of uniquely maximal norms. They turn out to be exactly those norms which are convex transitive. A norm is called convex transitive if $\overline{\text{co}}\{Ux;\ U\in G\}=\{y;\ \|y\|\leqq 1\}$ for each $x\in X$ with $\|x\|=1$.

In order to prove this result we shall require the following lemma.

Lemma 4. *Let X have a uniquely maximal norm. Then $\|f\|=\sup\{|f(Ux)|;\ U\in G\}$ for each $x\in X$ with $\|x\|=1$ and each $f\in X^*$.*

Proof. Fix $f\in X^*$. Define

$$\|x\|_1=\|x\|+\sup\{|f(Ux)|;\ U\in G\} \text{ for all } x\in X.$$

Then $\|.\|_1$ is an equivalent norm on X.

If $V\in G$, then

$$\|Vx\|_1=\|Vx\|+\sup\{|f(UVx)|;\ U\in G\}=\|x\|+\sup\{|f(Ux)|;\ U\in G\}=\|x\|_1.$$

Therefore $\|.\|_1$ has at least the same isometries as $\|.\|$. Hence as the norm is uniquely maximal,

$$\|x\|_1=\|x\|+\sup\{|f(Ux)|;\ U\in G\}=k\|x\|$$

for all $x\in X$ and some constant $k>0$. We have $\sup\{|f(Ux)|;\ U\in G\}=r\|x\|$ for all $x\in X$ and some constant $r>0$.

Now $|f(Ux)|\leqq\|f\|\ \|Ux\|=\|f\|\ \|x\|$ for all $U\in G$. Therefore $r\leqq\|f\|$. Given $\varepsilon>0$ there exists $y\in X$ with $\|y\|=1$ such that $|f(y)|\geqq\|f\|-\varepsilon$. Hence

$$r=\sup\{|f(Uy)|;\ U\in G\}\geqq|f(y)|\geqq\|f\|-\varepsilon.$$

We have proved $\quad \sup\{|f(Ux)|;\ U\in G\}=\|f\|\ \|x\|$ for all $x\in X$.

Theorem 5. *For a Banach space X, the norm is uniquely maximal if and only if the norm is convex transitive.*

Proof. Assume X has a convex transitive norm. Fix $x\in X$ with $\|x\|=1$. Then $\overline{\text{co}}\{Ux;\ U\in G\}=\{y;\ \|y\|\leqq 1\}$. Suppose there exists an equivalent norm $\|.\|_1$ on X with isometries G_1 such that $G\subseteq G_1$. Let $y\in X$ with $\|y\|=1$. Given $\varepsilon>0$ there exists $\{U_1,\ldots,U_n\}\subseteq G$ and $\{\lambda_1,\ldots,\lambda_n\}\subseteq\mathbb{R}^+$ such that

$$\left\|y-\sum_1^n\lambda_m U_m(x)\right\|<\varepsilon, \text{ and } \sum_1^n\lambda_m=1.$$

We have

$$\|y\|_1 \leqq \left\| y - \sum_1^n \lambda_m U_m(x) \right\|_1 + \left\| \sum_1^n \lambda_m U_m(x) \right\|_1$$

$$\leqq K \left\| y - \sum_1^n \lambda_m U_m(x) \right\| + \sum_1^n \lambda_m \|U_m(x)\|_1$$

$$\leqq K\varepsilon + \|x\|_1 \text{ for some constant } K \text{ by equivalence.}$$

Hence $\|y\|_1 \leqq \|x\|_1$, and similarly $\|x\|_1 \leqq \|y\|_1$. Therefore $\{x; \|x\| = 1\} \subseteq \{y; \|y\|_1 = r\}$ for some $r > 0$, that is, $r\|x\| = \|x\|_1$ for all $x \in X$. Hence the norm is uniquely maximal.

The above proof is essentially the proof given by Rolewicz ([3], p. 256) that a convex transitive norm is a maximal norm.

Suppose that the norm is not convex transitive. Then there exists $x \in X$ with $\|x\| = 1$ such that

$$B = \overline{\text{co}}\{Ux;\ U \in G\} \subsetneqq \{y;\ \|y\| \leqq 1\}.$$

Let $z \in \{y;\ \|y\| \leqq 1\} \backslash B$. By the Hahn Banach separation theorem (see [4], p. 60) there exists $f \in X^*$, the dual space of X, such that $|f(x)| \leqq 1$ for all $x \in B$ and $|f(z)| > 1$. But by Lemma 4, B is a norming set for f, which is a contradiction.

By the results of Rolewicz ([3], §6 and 7) on convex transitive norms, the spaces $L_p[0,1]$ $(1 \leqq p < \infty)$ and the space $C_0[0,1]$ (all continuous complex valued functions vanishing at the end points, see Example 2) have uniquely maximal norms.

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