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A Note on Unlinking Numbers of Montesinos Links

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ABSTRACT. Let K (resp. L) be a Montesinos knot (resp. link) with at least four branches. Then we show that the unknotting number (resp. unlinking number) of K (resp. L) is greater than 1.

1. INTRODUCTION

The unknotting number (resp. unlinking number) of a knot K (resp. link L) in S^3, $u(K)$ (resp. $u(L)$) is the minimum number of crossing changes needed to create the unknot (resp. unlink). The minimum being taken over all possible sets of changes in all possible presentations of K (resp. L).

These numbers are very intuitive invariant and not easy to calculate. In [14], Scharlemann proved that unknotting number one knots are

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prime. An alternative proof was given by Zhang [18]. The analogous result for links (i.e., unlinking number one links are prime) was proved by Eudave-Muñoz [3] and Gordon-Luecke [4] in different methods. For two bridge knots, Kanenobu-Murakami [6] determined two bridge knots with unknotting number one. Later Kohn [7] determined two bridge links with unlinking number one. Recently Menasco [9] determined the unknotting (resp. unlinking) number of torus knots (resp. torus links). A survey of methods of calculation of unknoting numbers is given by Nakanishi [13].

In this paper, we study unknotting numbers (resp. unlinking numbers) of Montesinos knots (resp. Montesinos links).

Let $M(e;(\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r))$ be a Montesinos knot or link with r branches (see Figure 1), where a box $\boxed{\alpha_i,\beta_i}$ stands for a so-called "rational tangle" of type (α_i,β_i) ([11], [12], [19] and [2]).

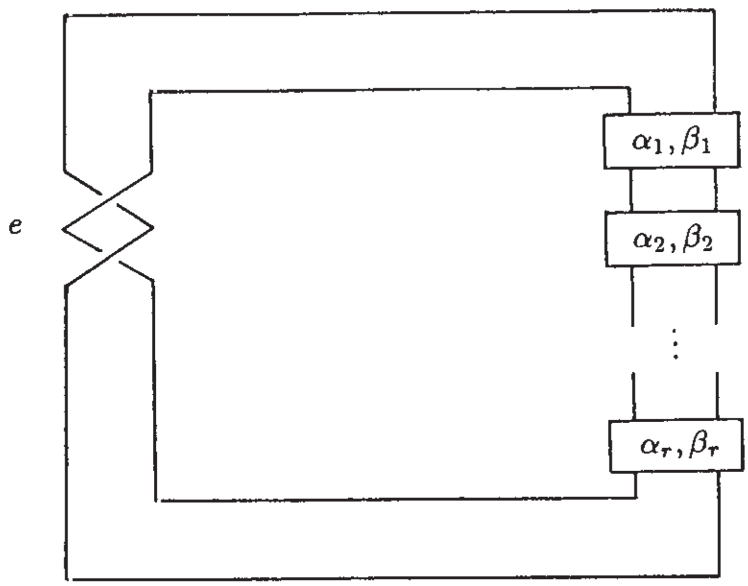
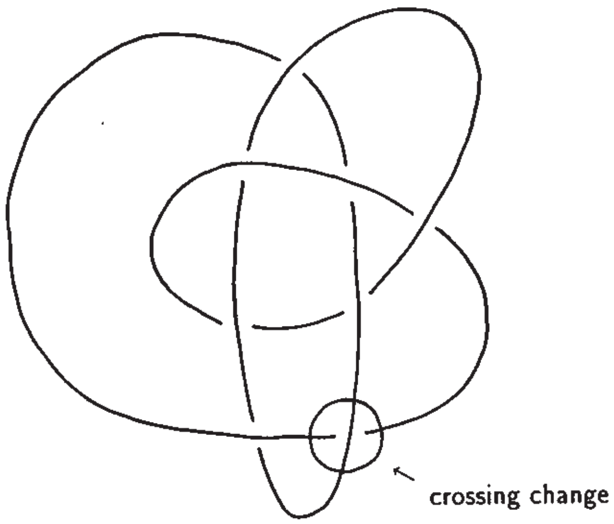


Figure 1

In the following we assume that $\alpha_i > 1$. (If for some $i, \alpha_i = 1$, then the knot or link would have a simpler form.)

Montesinos knot with $r \leq 3$ can have unknotting number one. For example $8_{20} = M(1;(2,1),(3,1),(3,2))$ has unknotting number one (see Figure 2).



$8_{20} = M(1;(2,1),(3,1),(3,2))$

Figure 2

On the other hand if $r \geq 4$, we prove the following.

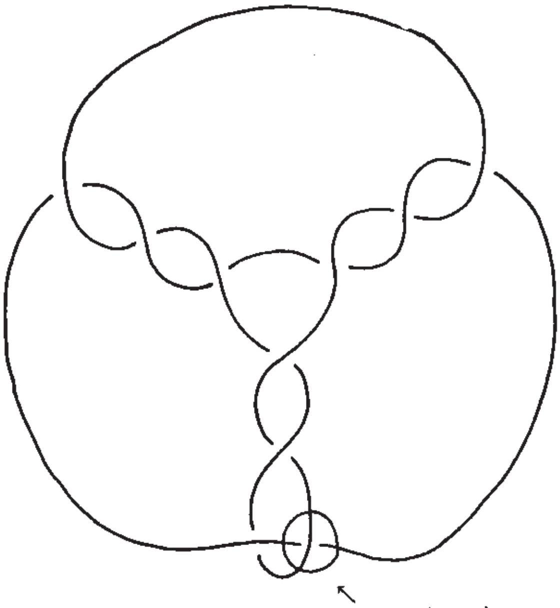
Theorem 1.1. *Let* $K = M(e;(\alpha_1,\beta_1),\ldots(\alpha_r,\beta_r))$ *be a Montesinos knot with* $r \geq 4$. *Then* $u(K) \geq 2$.

The two components Montesinos link $L = M(0;\ (3,1),\ (3,-1),\ (5,2))$ illustrated by Figure 3 has $u(L) = 1$.

If $r \geq 4$, we have:

Theorem 1.2. *Let* $L = M(e;(\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r))$ *be a Montesinos link with* $r \geq 4$. *Then* $u(L) \geq 2$.

The present pro^fs of Theorems 1.1 and 1.2 follow the same philosophy of [6], [7], [18] and [4], except for the case where L has more than two components (Proposition 4.6).



$L = M(0;(3,1),(3,-1),(5,2))$

Figure 3

2. PRELIMINARIES

Let k be a knot in the interior of an orientable 3-manifold M. Let $N(k)$ be a tubular neighborhood of k in M. For the isotopy class (slope) α of an essential simple closed curve on $\partial N(k)$, $M(k;\alpha)$ denotes the manifold obtained from M by α-surgery on k, i.e., the result of attaching a solid torus V to M-int$N(k)$ by identifying ∂V with $\partial N(k)$ so that α bounds a disk in V. If α and β are two slopes on $\partial N(k)$, then $\Delta(\alpha, \beta)$ denotes their minimal geometric intersection number.

If K (resp. L) is a knot (resp. link) in S^3, we use M_K (resp. M_L) to denote the two-fold branched covering of S^3 branched over the knot K (resp. the link L).

Lemma 2.1 ([11], [8] and [7]). *(1) Let K be a knot in S^3 with $u(K) = 1$, then M_K is homeomorphic to $S^3(k;\gamma)$ for some knot $k \subset S^3$ and γ with $\Delta(\gamma,\mu) = 2$, where μ is a meridian slope of k.*

(2) Let L be a two components link in S^3 with $u(L) = 1$, then M_L is homeomorphic to $S^2 \times S^1(k;\gamma)$ for some knot $k \subset S^2 \times S^1$ and γ with $\Delta(\gamma,\mu)$, where μ is a meridian slope of k.

Lemma 2.2 ([11], [12], [19], [2]). *The two-fold branched covering of S^3 branched over a Montesinos knot or link $M(e;(\alpha_1,\beta_1),\dots,(\alpha_r,\beta_r))$ is a Seifert fibred manifold with the 2-sphere S^2 as base, obstruction invariant e and r exceptional fibres of types (α_i,β_i).*

Lemma 2.3 ([1], [10]). *Let k be a non-hyperbolic knot in S^3. If $S^3(k;\gamma)$ is a Seifert fibred manifold over S^2 with at least four exceptional fibres, then $\Delta(\gamma,\mu) = 1$.*

Remark. In [10] it is also proved that if there are two such surgery slopes γ_1, and γ_2, then $\Delta(\gamma_1,\gamma_2) \leq 1$.

A 3-manifold M is a *cable* on a manifold M_1, if $M = C \cup_T M_1$ where C is a cable space [5], $\partial M \subset \partial C$ and $T = \partial C \cap \partial M_1$ is an incompressible torus in M_1.

Lemma 2.4 ([1, Theorems 0.5 and 0.6]). *Let M be a closed orientable 3-manifold and k a knot in M. Assume that M-int$N(k)$ is irreducible and is neither a Seifert fibred manifold nor a cable on a (boundary-irreducible) Seifert fibred manifold. If $M(k;\gamma_1)$ is a Seifert fibred manifold over S^2 with at least four exceptional fibres and $M(k;\gamma_2)$ has a cyclic fundamental group, then $\Delta(\gamma_1,\gamma_2) \leq 1$.*

In particular the above lemma implies,

Corollary 2.5 ([1]). *Let k be a hyperbolic knot in S^3. If $S^3(k;\gamma)$ is a Seifert fibred manifold over S^2 with at least four exceptional fibres, then $\Delta(\gamma,\mu) = 1$, where μ is a meridian slope of k.*

3. PROOF OF THEOREM 1.1

Let $K = M(e;(\alpha_1,\beta_1),\dots,(\alpha_r,\beta_r))$ be a Montesinos knot with $r \geq 4$. Assume for contradiction that K has unknotting number one.

From Lemma 2.1 (1), we see that M_K (the two-fold branched covering of S^3 branched covering over K) is homeomorphic to $S^3(k;\gamma)$ for some knot $k(\subset S^3)$ and γ with $\Delta(\gamma,\mu)=2$, where μ is a meridian slope of k. Since K is a Montesinos knot with $r(\geq 4)$ branches, M_K is a Seifert fibred manifold over S^2 with $r(\geq 4)$ exceptional fibres. Therefore Lemma 2.3 and Corollary 2.5 imply that $\Delta(\gamma,\mu)=1$, a contradiction. Hence K cannot have unknotting number one. ∎

4. PROOF OF THEOREM 1.2.

To prove Theorem 1.2, we divide into two cases : (1) the link L has exactly two components, or (2) L has more than two componentes.

First we consider the case (1).

Proposition 4.1. *Let* $L = M(e;(\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r))$ *be a two components Montesinos link with* $r \geq 4$. *Then* $u(L) \geq 2$.

We prepare some lemmas to prove this proposition.

Lemma 4.2. *Let k be a knot in $S^2 \times S^1$. If $S^2 \times S^1$–int$N(k)$ is reducible, then k is a local knot, i.e., there exists a 3-ball B^3 in $S^2 \times S^1$ such that $B^3 \supset k$.*

Proof. Let Σ be an essential 2-sphere in $S^2 \times S^1$–int$N(k)$. If Σ separates $S^2 \times S^1$–int$N(k)$, then since $S^2 \times S^1$ is prime it bounds a 3-ball in $S^2 \times S^1$ containing k. Thus k is a local knot.

If Σ does not separate $S^2 \times S^1$–int$N(k)$, then we take a simple loop J in $S^2 \times S^1$–int$N(k)$ meeting Σ transversely in a single point. The boundary Σ' of a tubular neighborhood of $\Sigma \cup J$ is a 2-sphere which separates $S^2 \times S^1$ into $X_1 = N(\Sigma \cup J)$ and $X_2 = S^2 \times S^1$–int$N(\Sigma \cup J)$. Since $S^2 \times S^1$ is prime and X_1 is not a 3-ball, $X_2(\supset k)$ is a 3-ball. Hence k is a local knot in $S^2 \times S^1$. ∎

Lemma 4.3 . *Let k be a local knot in $S^2 \times S^1$. If $S^2 \times S^1(k;\gamma)$ is Seifert fibred, then $S^2 \times S^1(k;\gamma) \cong S^2 \times S^1$. (In particular $S^2 \times S^1(k;\gamma)$ is not a Seifert fibred manifold over S^2 with at least four exceptional fibres for auy slope γ.)*

Proof. Since k is local, $S^2 \times S^1(k;\gamma)$ has $S^2 \times S^1$ as a connected summand. A reducible Seifert fibred manifold is homeomorphic to $S^2 \times S^1$ or $P^3 \# P^3$, P^3 is a real projective space and the result follows. ■

In the following S^3 and $S^2 \times S^1$ are not considered as lens spaces.

Lemma 4.4. *Let k be a knot in $S^2 \times S^1$ such that $S^2 \times S^1$-int$N(k)$ is a Seifert fibred manifold or a cable on a Seifert fibred manifold. Then $S^2 \times S^1(k;\gamma)$ cannot be a Seifert fibred manifold over S^2 with at least four exceptional fibres for any slope γ.*

Proof. Suppose for contradiction that $S^2 \times S^1(k;\gamma)$ admits a Seifert fibration over S^2 with at least four exceptional fibres. Then the Seifert fibration is unique [5, VI.17] (because $S^2 \times S^1(k;\gamma)$ is not the double of a twisted I-bundle over the Klein bottle), and any incompressible torus is isotopic to a vertical one (i.e., a union of fibres) ([16]).

Case 1. $S^2 \times S^1$-int$N(k)$ is Seifert fibred.

In this case from [7, Lemma 4] we see that k is a regular fibre in some Seifert fibration of $S^2 \times S^1$. Since any Seifert fibration of $S^2 \times S^1$ has S^2 as base with zero or two exceptional fibres, $S^2 \times S^1$-int$N(k)$ is Seifert fibred over the disk D^2 with zero or two exceptional fibres. If the surgery slope γ coincides with a regular fiber of $S^2 \times S^1$-int$N(k)$, then the result $S^2 \times S^1(k;\gamma)$ is the 3-sphere S^3 or a connected sum of two lens spaces, which cannot admit a Seifert fibration over S^2 with at least four exceptional fibres. If γ is not a regular fibre of $S^2 \times S^1$-int$N(k)$, then $S^2 \times S^1(k;\gamma)$ admits a Seifert finration extending that of $S^2 \times S^1$-int$N(k)$. Hence the result $S^2 \times S^1(k;\gamma)$ is Seifert fibred over S^2 with at most three exceptional fibres. It follows that $S^2 \times S^1(k;\gamma)$ cannot admit a Seifert fibration over S^2 with at least four exceptional fibres.

Case 2. $S^2 \times S^1$-int$N(k)$ is not Seifert fibred : $S^2 \times S^1$-int$N(k)$ is a cable on a (boundary-irreducible) Seifert fibred manifold.

Let $C(\subset S^2 \times S^1$-int$N(k))$ be the cable space and $M_1(\subset S^2 \times S^1$-int$N(k))$ the Seifert fibred manifold. Let μ be the slope of a meridian of k in $S^2 \times S^1$ and τ the slope of a regular fibre of the cable space C.

Claim 4.5. $\Delta(\tau, \mu) = 1$.

Proof of Claim 4.5. If $\tau = \mu(i.e., \Delta(\tau, \mu) = 0)$, then $C \cup N(k)$ ($\subset S^2 \times S^1$) and hence $S^2 \times S^1$ has a lens space summand, a contradiction. If $\Delta(\tau, \mu) \geq 2$, then the Seifert fibration of the cable space C can be extended to that of $C \cup N(k)$, which is boundary-irreducible. Since M_1 is also boundary-irreducible, $S^2 \times S^1$ contains an incompressible torus. This is a contradiction. ■

It follows that $C \cup N(k)$ is a solid torus in $S^2 \times S^1$, whose core is the exceptional fibre f of the cable spce C. Thus we can regard $C \cup N(K)$ as a tubular neighborhood $N(f)$ of f in $S^2 \times S^1$.

If the surgery slope γ coincides with τ (i.e., $\Delta(\gamma, \tau) = 0$), then $C \cup_\gamma V$, where V denotes the filling solid torus, has a lens space summand. This implies that $S^2 \times S^1(k; \gamma)$ has a lens space summand. Hence it cannot be a Seifert fibred manifold over S^2 with at least four exceptional fibres. Now we consider the case where the surgery slope γ does not coincide with τ. In this case the Seifert fibration of C can be extended to that of $C \cup_T V$. Suppose that $\Delta(\gamma, \tau) = 1$. Then $C \cup_\gamma V$ becomes a solid torus whose core is the exceptional fibre f in the cable space C. Therefore $S^2 \times S^1(k; \gamma) \cong S^2 \times S^1(f; \gamma')$ for some slope γ' on $\partial N(f)$. Since the exterior $S^2 \times S^1\text{-int}N(f) = M_1$ is Seifert fibred, we can conclude that $S^2 \times S^1(f; \gamma')$ cannot have a Seifert fibration over S^2 with at least four exceptional fibres by Case 1. Let us assume that $\Delta(\gamma, \tau) \geq 2$. In this case $C \cup_\gamma V$ admits a Seifert fibration over D^2 with just two exceptional fibres by extending the Seifert fibration of C. Since both M_1 and $C \cup_\gamma V$ are boundary-irreducible, $S^2 \times S^1(k; \gamma)$ contains the incompressible torus ∂M_1, which can be assumed to be vertical by isotoping the Seifert fibration. If $C \cup_\gamma V$ is not a twisted I-bundle over the Klein bottle, then the Seifert fibration is unique up to isotopy ([5, VI.18.Theorem]). Therefore the Seifert fibration of $C \cup_\gamma V$ which extends that of C is isotopic to the Seifert fibration of $C \cup_\gamma V$ which is the restriction of that of $S^2 \times S^1(k; \gamma)$. Hence $S^2 \times S^1\text{-int}N(k) = C \cup M_1$ is Seifert fibred, a contradiction. We assume that $C \cup_\gamma V$ is a twisted I-bundle over the Klein bottle. Then it has just two Seifert fibrations up to isotopy ([17]) : the extended Seifert fibration of the cable space C or a Seifert fibration over Möbius band with no exceptional fibre. In the first case the above argument implies that $S^2 \times S^1$–int$N(k) = C \cup M_1$ is Seifert fibred, a contradiction. In the latter case $S^2 \times S^1(k;\gamma)$ is Seifert fibred over a non-orientable surface, and hence cannot admit a desired Seifert fibration. ■

Proof of Proposition 4.1. Let $L = M(e;(\alpha_1,\beta_1),\dots,(\alpha_r,\beta_r))$ be a two components Montesinos link with $r \geq 4$. Assume for contradiction that $u(L) = 1$. From Lemma 2.1(2), we see that the two-fold branched covering M_L of S^3 branched over L is homeomorphic to $S^2 \times S^1(k;\gamma)$ for some knot k in $S^2 \times S^1$ and γ with $\Delta(\gamma,\mu) = 2$, where μ is a meridian slope of k in $S^2 \times S^1$. Since L is a Montesinos link with $r(\geq 4)$ branches, M_L is a Seifert fibred manifold over S^2 with $r(\geq 4)$ exceptional fibres. If $S^2 \times S^1$–int$N(k)$ is reducible, then by Lemma 4.2, k is a local knot and $S^2 \times S^1(k;\gamma)$ cannot be a Seifert fibred manifold over S^2 with at least four exceptional fibres by Lemma 4.3. So we may assume $S^2 \times S^1$–int$N(k)$ is irreducible. Suppose that $S^2 \times S^1$–int$N(k)$ is Seifert fibred manifold or a cable on a Seifert fibred manifold. In this special case, by Lemma 4.4 $S^2 \times S^1(k;\gamma)$ is not a desired Seifert fibred manifold. It follows from Lemma 2.4 that we have $\Delta(\gamma,\mu) \leq 1$, this is a contradiction. Therefore $u(L) \geq 2$. ■

As for the case (2) : the link L has more than two components, we can prove the following proposition.

Proposition 4.6. *Let $L = M(e;(\alpha_1,\beta_1),\dots,(\alpha_r,\beta_r))$ be a Montesinos link with more than two components. Then $u(L) \geq 2$.*

Proof. In the following we use indices modulo r. Let $C_{i,1}$ and $C_{i,2}$ be parallel arcs in L connecting two rational tangles $\boxed{\alpha_i,\beta_i}$ and $\boxed{\alpha_{i+1},\beta_{i+1}}$ (see Figure 4).

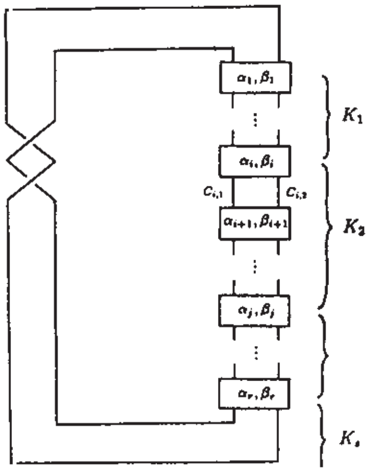


Figure 4

Claim 4.7. For each i, two arcs $C_{i,1}$ and $C_{i,2}$ are contained in the same component of L.

Proof of Claim 4.7. If for some $j, C_{j,1}$ and $C_{j,2}$ are contained in distinct components of L, then $C_{j,1}$ and $C_{j+1,k}$ ($k = 1$ or 2) are contained in the same component, and hence $C_{j,2}$ and $C_{j+1,3-k}$ are also contained in the same component. Thus $C_{j+1,i}$ and $C_{j+1,2}$ are contained in distinct components. Inductively we can observe that for each i, $C_{i,1}$ and $C_{i,2}$ are contained in distinct components. Hence L has exactly two components, a contradiction. ■

By Claim 4.7, components of L are positioned as in Figure 4 , i.e., components $K_1, \ldots, K_s$ of L appear in clockwise order.

Suppose for contradiction that L has unlinking number one. There are two possibilities: a crossing change on the same component of L converts L into the unlink or a crossing change on distinct components of L converts L into the unlink.

Suppose that a crossing change on a component K_i transforms L into a trivial link. Then since the link type of $K_{i+1} \cup K_{i+2}$ is not changed under the crossing change, the sublink $L' = K_{i+1} \cup K_{i+2}$ is trivial. Next we consider the case where a crossing change on distinct components K_i and K_j ($i \neq j$) converts L into a trivial link. Then we can take a component $K_{j^*} (= K_{j-1}$ or $K_{j+1})$ so that $K_{j^*} \neq K_i$. Since the crossing change does not change the link type of $K_j \cup K_{j^*}$, the sublink $L' = K_j \cup K_{j^*}$ is a trivial link.

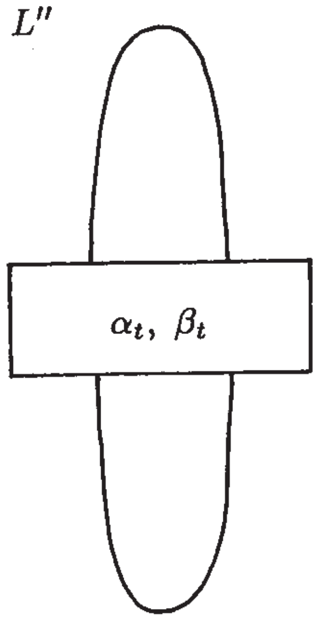


Figure 5

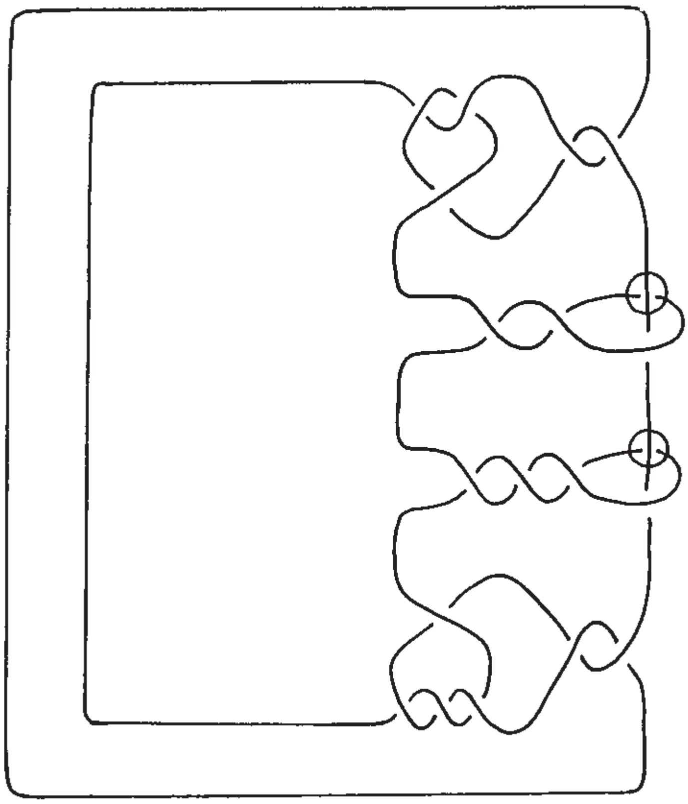
In any case each component of L' intersects a rational tangle $\boxed{\alpha_t, \beta_t}$ for some t $(1 \leq t \leq r)$. Therefore L' has a connected summand L'' given by Figure 5.

Since $\alpha_t > 1$, the factor link L'' is non-trivial (see [15]). Hence L' is also non-trivial, a contradiction. This completes the proof of Proposition 4.6. ■

Theorem 1.2 follows from Propositions 4.1 and 4.6.

5. EXAMPLES

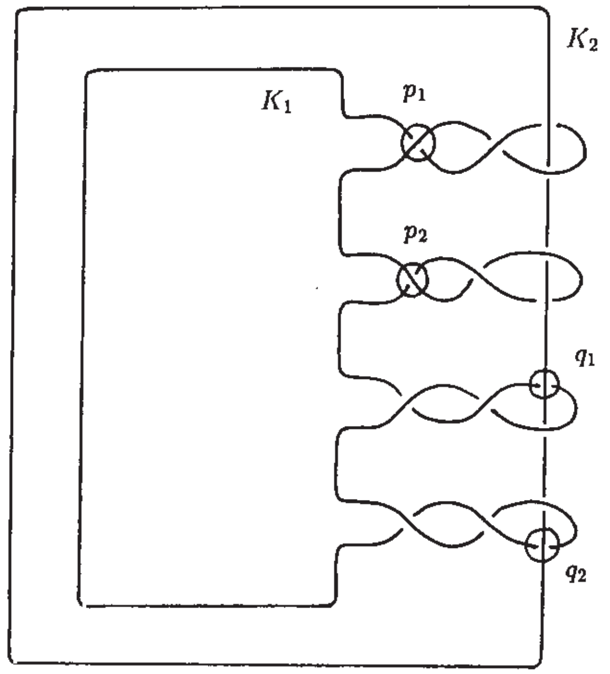
Example 5.1. Let K be a Montesinos knot $M(0;\ (4,3),\ (3,2),\ (5,2),\ (5,-4))$ (see Figure 6). Then by changing the indicated crossings in Figure 6, we obtain a trivial knot. Thus $u(K) \leq 2$. On the other hand Theorem 1.1 implies that $u(K) \geq 2$ and hence $u(K) = 2$



$K = M(0; (4,3), (3,2), (5,2)(5,-4))$
Figure 6

Example 5.2. Let L be a Montesinos link $M(0;\ (5,-2),\ (5,2),\ (5,-2),\ (5,2))$ with two components K_1 and K_2 (see Figure 7). If we change crossings at $\{p_1, p_2\}$ or $\{q_1, q_2\}$, we obtain a trivial link. Thus $u(L) \leq 2$. Hence we see that $u(L) = 2$ by Theorem 1.2.

We note that the crossing change at p_i $(i = 1, 2)$ is a crossing change on K_1 and the crossing change at q_i $(i = 1, 2)$ is a crossing change on K_1 and K_2.



$L = M(0;(5,-2),(5,2),(5,-2),(5,2))$
Figure 7

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References

[1] Boyer, S. and Zhang, X., *The semi-norm and Dehn filling*, (preprint).

[2] Burde, G. and Zieschang, H., *Knots*, de Gruyter Studies in Mathematics, no.5, Walter de Gruyter, Berlin, 1985.

[3] Eudave-Muñoz, M.,*Primeness and sums of tangles*, Trans. Amer. Math. Soc. 306 (1988), 773-790.

[4] Gordon, C.McA. and Luecke, J., *Links with unlinking number one are prime*, Proc. Amer. Math. Soc. 120 (1994), 1271-1274.

[5] Jaco, W., *Lectures on three manifold topology*, CBMS Regional Conference Series in Math. 43, Amer. Math. Soc., 1980.

[6] Kanenobu, T. and Murakami, H., *Two-bridge knots with unknotting number one*, Proc. Amer. Math. Soc. 98 (1986), 499-502.

[7] Kohn, P., *Two-bridge links with unlinking number one*, Proc. Amer. Math. Soc. 113 (1991), 1135-1147.

[8] Lickorish, W.B.R., *The unknotting number of a classical knot*, Contemp. Math., vol. 44, Amer. Math. Soc., Providence, RI, 1985, pp. 117-121.

[9] Menasco, W., *The Bennequin-Milnor unknotting conjectures*, C. R. Acad. Sci. Paris, Série I 318 (1994), 831-836.

[10] Miyazaki, K. and Motegi, K., *Seifert fibred manifolds and Dehn surgery*, (to appear in Topology).

[11] Montesinos, J.M., *Surgery on links and double branched coverings of S^3*, Ann. of Math. Stud., no. 84, Princeton Univ. Press, Princeton, NJ, 1975, pp. 227-259.

[12] Montesinos, J.M., *Variedades de Seifert que son recubridadores cíclicos ramificados de dos hojas,* Bol. Soc. Mat. Mex. 18 (1973), 1-32.

[13] Nakanishi, Y., *A note on unknotting number*, Math. Sem. Notes, Kobe Univ. 9 (1981), 99-108.

[14] Scharlemann, M., *Unknotting number one knots are prime*, Invent. Math. 82 (1985), 37-55.

[15] Schubert, H., *Knoten mit zwei Brücken*, Math. Zeit. 66 (1956), 133-170.

[16] Waldhausen, F., *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten I, II*, Invent. Math. 3, 4 (1967), 308-333, 87-117.

[17] Wang, S. and Wu, Y.-Q., *Covering invariants and cohopficity of 3-manifold groups*, Proc. London Math. Soc. 68 (1994), 203-224.

[18] Zhang, X., *Unknotting number one knots are prime : a new proof*, Proc. Amer. Math. Soc. 113 (1991), 611-612.

[19] Zieschang, H., *Classification of Montesinos knots, Proc. Leningrad 1982*, Lect. Notes in Math., vol. 1060, 378-389, Springer-Verlag, 1984.

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