

## A NOTE ON VERY WEAK $p$ -HARMONIC MAPPINGS

JUHA KINNUNEN & SHULIN ZHOU

ABSTRACT. We prove a new *a priori* estimate for very weak  $p$ -harmonic mappings when  $p$  is close to two. This sheds some light on a conjecture posed by Iwaniec and Sbordone.

### 1. INTRODUCTION

Let  $\Omega$  be a bounded regular domain (Definition 2.1 in [IS]) in  $\mathbf{R}^n$  and  $1 < p < \infty$ . We are interested in the  $p$ -harmonic system

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \tag{1.1}$$

under minimal assumptions on  $u$ . System (1.1) is understood in the weak sense: a weak  $p$ -harmonic mapping is defined as  $u \in W_{\text{loc}}^{1,p}(\Omega, \mathbf{R}^m)$  satisfying

$$\int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla \phi \, dx = 0 \tag{1.2}$$

for every  $\phi \in C_0^\infty(\Omega, \mathbf{R}^m)$ . However, equation (1.2) makes sense when  $u \in W_{\text{loc}}^{1,r}(\Omega, \mathbf{R}^m)$  with  $r \geq \max\{1, p-1\}$ . Such solutions are called *very weak  $p$ -harmonic mappings*.

Iwaniec and Sbordone showed in [IS] that there exist  $r_1 = r_1(p, m, \Omega)$  and  $r_2 = r_2(p, m, \Omega)$  satisfying

$$1 < r_1 < p < r_2 < \infty \tag{1.3}$$

such that every very weak  $p$ -harmonic mapping  $u \in W_{\text{loc}}^{1,r_1}(\Omega, \mathbf{R}^m)$  belongs to  $W_{\text{loc}}^{1,r_2}(\Omega, \mathbf{R}^m)$ . They conjectured (Conjecture 1 in [IS]) that every  $r_1 > \max\{1, p-1\}$  would do for the  $p$ -harmonic system, but their estimate for  $r_1$  is very close to  $p$ .

The objective of our note is to study the case when  $p$  is close to two. We show that  $r_1$  in (1.3) can be chosen arbitrarily close to one if  $p$  is close to two. This does not solve the conjecture by Iwaniec and Sbordone, but we are able to obtain estimates when  $r_1$  is arbitrarily close to  $\max\{1, p-1\}$ .

---

1991 Mathematics Subject Classifications: 35J60, 35B45.

Key words and phrases: Very weak  $p$ -harmonic mapping.

©1997 Southwest Texas State University and University of North Texas.

Submitted November 27, 1997. Published December 19, 1997.

The second author would like to thank NSF of China for their support, and the University of Helsinki for its hospitality.

**Theorem 1.4.** *Let  $0 < \eta < 1$ . For every exponent  $r_1 \geq 1 + \eta$  there are  $\delta = \delta(\eta, m, \Omega) > 0$  and  $r_2 = r_2(p, m, \Omega) > p$  such that every very weak  $p$ -harmonic mapping  $u \in W_{\text{loc}}^{1,r_1}(\Omega, \mathbf{R}^m)$  belongs to  $W_{\text{loc}}^{1,r_2}(\Omega, \mathbf{R}^m)$  provided  $|p - 2| < \delta$ .*

Our method is a sharpening of [IS]. If  $u$  belongs to the Sobolev space with the natural exponent  $p$ , then we may use the classical method of reverse Hölder inequalities, see [ME]. However, if  $u$  is assumed to belong to a Sobolev space below the natural exponent, then we cannot choose  $\eta u$ , where  $\eta$  is a cut-off function, as a test function in (1.2) and hence it is difficult to obtain any *a priori* estimates. Iwaniec and Sbordone used the Hodge decomposition ingeniously to build a test function for equation (1.2). Another argument has been given by Lewis in [L]. The main new feature in our proof is that we use the Hodge decomposition twice and also employ the fact that the matrix field involved is divergence free. This trick works only for  $p$ -harmonic operator. As Serrin's example in [S] shows, our result is not true for  $\mathcal{A}$ -harmonic operators studied by [IS] and [L].

## 2. MAIN RESULTS

For the convenience of the reader we recall the formulation of the Hodge decomposition (Theorem 3 in [IS]).

**Theorem 2.1.** *Let  $\Omega$  be a regular domain in  $\mathbf{R}^n$  and  $w \in W_0^{1,r}(\Omega, \mathbf{R}^m)$  with  $r > 1$ , and let  $-1 < \varepsilon < r - 1$ . Then there exist  $\phi \in W_0^{1,r/(1+\varepsilon)}(\Omega, \mathbf{R}^m)$  and a divergence free matrix field  $H \in L^{r/(1+\varepsilon)}(\Omega, \mathbf{R}^{m \times n})$  such that*

$$|\nabla w|^\varepsilon \nabla w = \nabla \phi + H, \quad (2.2)$$

and

$$\|H\|_{r/(1+\varepsilon), \Omega} \leq C_r(\Omega, m) |\varepsilon| \|\nabla w\|_{r, \Omega}^{1+\varepsilon}. \quad (2.3)$$

An examination of [IS] reveals that all their results are based on Theorem 5.1 in [IS]. Instead of rewriting all results in [IS], we only prove a sharpening of Theorem 5.1 in [IS] when  $p$  is close to two.

We consider the very weak solutions  $w \in W_0^{1,r}(\Omega, \mathbf{R}^m)$  with  $r > \max\{1, p - 1\}$  of the non homogeneous system

$$\operatorname{div} (|g + \nabla w|^{p-2} (g + \nabla w)) = \operatorname{div} h \quad (2.4)$$

where  $g \in L^r(\Omega, \mathbf{R}^{m \times n})$  and  $h \in L^{r/(p-1)}(\Omega, \mathbf{R}^{m \times n})$  are matrix fields. Equation (2.4) is understood in the weak sense, that is,

$$\int_{\Omega} |g + \nabla w|^{p-2} (g + \nabla w) \cdot \nabla \phi \, dx = \int_{\Omega} h \cdot \nabla \phi \, dx \quad (2.5)$$

for every  $\phi \in W_0^{1,r/(r-p+1)}(\Omega, \mathbf{R}^m)$ .

**Theorem 2.6.** *Let  $0 < \eta < 1$ . Suppose that  $w \in W_0^{1,r}(\Omega, \mathbf{R}^m)$  with  $r \geq 1 + \eta$  satisfies (2.3). Then there is  $\delta = \delta(\eta, m, \Omega) > 0$  such that if  $|p - 2| < \delta$ , then*

$$\int_{\Omega} |\nabla w|^r \, dx \leq C(\eta, p, m, \Omega) \int_{\Omega} (|g|^r + |h|^{r/(p-1)}) \, dx. \quad (2.7)$$

*Proof.* Using Theorem 2.1 with  $\varepsilon = r - p$  we obtain functions  $\phi_1 \in W_0^{1,r/(r-p+1)}(\Omega, \mathbf{R}^m)$  and  $H_1 \in L^{r/(r-p+1)}(\Omega, \mathbf{R}^{m \times n})$  such that

$$|\nabla w|^{r-p} \nabla w = \nabla \phi_1 + H_1, \quad (2.8)$$

$$\int_{\Omega} H_1 \cdot \nabla \varphi \, dx = 0, \quad \text{for every } \varphi \in W_0^{1,r/(p-1)}(\Omega, \mathbf{R}^m), \quad (2.9)$$

and

$$\|H_1\|_{r/(r-p+1),\Omega} \leq c_1 |r - p| \|\nabla w\|_{r,\Omega}^{r-p+1}, \quad c_1 = C_r(\Omega, m). \quad (2.10)$$

In particular, we have

$$\|\nabla \phi_1\|_{r/(r-p+1),\Omega} \leq (c_1 + 1) \|\nabla w\|_{r,\Omega}^{r-p+1}. \quad (2.11)$$

Since  $\phi_1$  can be used as a test function in (2.5), we obtain

$$\int_{\Omega} |\nabla w + g|^{p-2} (\nabla w + g) \cdot \nabla \phi_1 \, dx = \int_{\Omega} h \cdot \nabla \phi_1 \, dx.$$

Inserting (2.8) we arrive at

$$\begin{aligned} \int_{\Omega} |\nabla w|^r \, dx &= \int_{\Omega} |\nabla w|^{p-2} \nabla w \cdot H_1 \, dx + \int_{\Omega} h \cdot \nabla \phi_1 \, dx \\ &\quad + \int_{\Omega} (|\nabla w|^{p-2} \nabla w - |\nabla w + g|^{p-2} (\nabla w + g)) \cdot \nabla \phi_1 \, dx \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (2.12)$$

We begin with estimating  $I_1$ . This is the crucial step of our argument. By using Theorem 2.1 again with  $\varepsilon = p - 2$ , we obtain  $\phi_2 \in W_0^{1,r/(p-1)}(\Omega, \mathbf{R}^m)$  and  $H_2 \in L^{r/(p-1)}(\Omega, \mathbf{R}^{m \times n})$  such that

$$|\nabla w|^{p-2} \nabla w = \nabla \phi_2 + H_2 \quad (2.13)$$

$$\int_{\Omega} H_2 \cdot \nabla \varphi \, dx = 0, \quad \text{when } \varphi \in W_0^{1,r/(r-p+1)}(\Omega, \mathbf{R}^m), \quad (2.14)$$

and

$$\|H_2\|_{r/(p-1),\Omega} \leq c_1 |p - 2| \|\nabla w\|_{r,\Omega}^{p-1}, \quad c_1 = C_r(\Omega, m). \quad (2.15)$$

Using (2.13), (2.14), (2.8), (2.9) and (2.15) we have

$$\begin{aligned} I_1 &= \int_{\Omega} (\nabla \phi_2 + H_2) \cdot H_1 \, dx = \int_{\Omega} H_1 \cdot H_2 \, dx \\ &= \int_{\Omega} (|\nabla w|^{r-p} \nabla w - \nabla \phi_1) \cdot H_2 \, dx \\ &= \int_{\Omega} |\nabla w|^{r-p} \nabla w \cdot H_2 \, dx \\ &\leq c_1 |p - 2| \|\nabla w\|_{r,\Omega}^r. \end{aligned}$$

The same reasoning shows that

$$I_1 \leq c_1 |r - p| \|\nabla w\|_{r,\Omega}^r$$

and hence

$$I_1 \leq c_1 \min\{|p - 2|, |r - p|\} \|\nabla w\|_{r,\Omega}^r. \quad (2.16)$$

By virtue of (2.11) we may estimate  $I_2$  and  $I_3$  in the same way as in [IS]. That is

$$\begin{aligned} I_2 + I_3 &\leq C \left( \|\nabla w\|_{r,\Omega}^{r-p+1} \|h\|_{r/(p-1),\Omega} \right. \\ &\quad \left. + \|g\|_{r,\Omega}^{p-1} \|\nabla w\|_{r,\Omega}^{r-p+1} + \|g\|_{r,\Omega} \|\nabla w\|_{r,\Omega}^{r-1} \right). \end{aligned}$$

Recalling Young's inequality we conclude that, for every  $\theta > 0$ ,

$$I_2 + I_3 \leq \theta \|\nabla w\|_{r,\Omega}^r + C_\theta \left( \|g\|_{r,\Omega}^r + \|h\|_{r/(p-1),\Omega}^{r/(p-1)} \right). \quad (2.17)$$

Using (2.12), (2.16) and (2.17) we obtain

$$\left(1 - c_1 \min\{|p - 2|, |r - p|\} - \theta\right) \|\nabla w\|_{r,\Omega}^r \leq C_\theta \left( \|g\|_{r,\Omega}^r + \|h\|_{r/(p-1),\Omega}^{r/(p-1)} \right).$$

In particular, if  $c_1 |p - 2| < 1$ , then (2.5) holds. Estimates for the constant  $c_1 = C_r(\Omega, m)$  can be found in [I] and formula (2.10) in [IS]. Using these estimates it is easy to see that we may choose  $c_1 = c(\eta, m, \Omega)$ . This completes the proof.

#### REFERENCES

- [GLS] D. Giachetti, F. Leonetti and R. Schianchi, *On the regularity of very weak minima*, Proc. Roy. Soc. Edinburgh Sect. A **126** (1996), 287–296.
- [GS] D. Giachetti and R. Schianchi, *Boundary higher integrability for the gradient of distributional solutions of nonlinear systems*, Studia Math. **123** (2) (1997), 175–184.
- [I] T. Iwaniec,  *$p$ -harmonic tensors and quasiregular mappings*, Ann. of Math. **136** (1992), 589–624.
- [IS] T. Iwaniec and C. Sbordone, *Weak minima of variational integrals*, J. Reine Angew. Math. **454** (1994), 143–161.
- [L] J. Lewis, *On very weak solutions of certain elliptic equations*, Comm. Partial Differential Equations **18** (9&10) (1993), 1515–1537.
- [M] N. Meyers, *An  $L^p$ -estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 189–206.
- [ME] N. Meyers and A. Elcrat, *Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions*, Duke Math. J. **42** (1975), 121–136.
- [S] J. Serrin, *Pathological solutions of elliptic equations*, Ann. Scuola Norm. Sup. Pisa **18** (1964), 385–387.

JUHA KINNUNEN

DEPARTMENT OF MATHEMATICS, P.O.BOX 4, FIN-00014 UNIVERSITY OF HELSINKI, FINLAND.

*E-mail address:* Juha.Kinnunen@Helsinki.Fi

SHULIN ZHOU

DEPARTMENT OF MATHEMATICS, PEKING UNIVERSITY, BEIJING 100871, P. R. CHINA

*E-mail address:* zsl@sxx0.math.pku.edu.cn