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A NOTE ON VON NEUMANN RHO-INVARIANT OF SURFACE BUNDLES OVER THE CIRCLE

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Abstract. In this short note, we give a formula for the von Neumann rho-invariant of surface bundles over the circle S^1. As a corollary, we describe a relation among the von Neumann rho-invariant, the first Morita-Mumford class and the Rochlin invariant in a framework of the bounded cohomology.

1. Introduction. Let M be an oriented closed Riemannian 3-manifold. Then we can define the η-invariant $\eta(M)$ of the signature operator. If we are given a surjective homomorphism from $\pi_1 M$ to a discrete group Γ, we have a Γ-covering $\bar{M} \to M$ and we can lift the metric and the signature operator to $\bar{M}$. In this situation, the von Neumann η-invariant $\eta^{(2)}(\bar{M})$ is defined for $\bar{M}$. Cheeger and Gromov showed in [3] that the difference $\eta^{(2)} - \eta$ is independent of a Riemannian metric. This topological invariant $\eta^{(2)} - \eta$ is called the *von Neumann rho-invariant* and is denoted by $\rho^{(2)}(\bar{M})$.

Recently, an approximation theorem of the η-invariants is shown by Vaillant [11] and Lück-Schick [7]. To be more precise, for a sequence of normal subgroups $\Gamma \rhd \Gamma_1 \rhd \Gamma_2 \rhd \cdots$ such that $[\Gamma : \Gamma_k] < \infty$ and $\bigcap_k \Gamma_k = \{e\}$, and Γ/Γ_k-coverings $M^k = \bar{M}/\Gamma_k \to M$, it holds that

$$\eta^{(2)}(\bar{M}) = \lim_{k\to\infty} \frac{\eta(M^k)}{[\Gamma : \Gamma_k]}\,.$$

Applying this formula to surface bundles over the circle S^1, we can describe the von Neumann rho-invariant by virtue of Meyer's signature cocycle [8] (see Proposition 2.1).

Let Σ_g be an oriented closed surface of genus g ($g \geq 2$) and $\mathcal{M}_g$ its mapping class group. Namely, $\mathcal{M}_g$ is the group of all isotopy classes of orientation preserving diffeomorphisms of Σ_g. Then the *first Morita-Mumford class* $e_1 \in H^2(\mathcal{M}_g, \mathbf{Z})$ (see [10]) is defined to be the Gysin image (integration along the fiber) of the square of the Euler class of the central extension

$$0 \to \mathbf{Z} \to \mathcal{M}_{g,1} \to \mathcal{M}_{g,*} \to 1\,.$$

Here $\mathcal{M}_{g,1}$ is the mapping class group of Σ_g relative to an embedded disc $D \subset \Sigma_g$ and $\mathcal{M}_{g,*}$ denotes the one relative to a base point $* \in D$. It is also known that e_1 is a bounded

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cohomology class. For a technical reason (see Section 3), we consider e_1 on $\mathcal{M}_{g,*}$ rather than on $\mathcal{M}_g$.

In general, the pull back of e_1 via a holonomy homomorphism $f : \pi_1 S^1 \to \mathcal{M}_{g,*}$ of a Σ_g-bundle over S^1 is automatically vanishing, because $H^2(S^1, \boldsymbol{Z})$ is trivial. However, Kitano showed in [6] that $e_1/48$ makes sense as a bounded cohomology class in $H_b^2(S^1, \boldsymbol{Z})$ and furthermore it is essentially given by the Rochlin invariant μ, if the image of f is contained in the Torelli subgroup $\mathcal{I}_{g,*} = \mathrm{Ker}\{\mathcal{M}_{g,*} \to \mathrm{Sp}(2g, \boldsymbol{Z})\}$. Combining our formula for $\rho^{(2)}$ with a result of Kitano, we have the following result on the level 2 subgroup $\mathcal{M}_{g,*}(2) = \mathrm{Ker}\{\mathcal{M}_{g,*} \to \mathrm{Sp}(2g, \boldsymbol{Z}/2)\} \supset \mathcal{I}_{g,*}$.

THEOREM 1.1. *Let $f : \boldsymbol{Z} \to \mathcal{M}_{g,*}(2)$ be a holonomy homomorphism. Then the first Morita-Mumford class $f^* e_1/48 \in H_b^2(\boldsymbol{Z}, \boldsymbol{Z})$ is represented by $\mu f - \rho^{(2)} f/16 \in H^1(\boldsymbol{Z}, \boldsymbol{R}/\boldsymbol{Z})$.*

REMARK 1.2. It is known that $H_b^2(\boldsymbol{Z}, \boldsymbol{Z}) \cong H_b^1(\boldsymbol{Z}, \boldsymbol{R}/\boldsymbol{Z}) \cong H^1(\boldsymbol{Z}, \boldsymbol{R}/\boldsymbol{Z}) \cong \boldsymbol{R}/\boldsymbol{Z}$. See [4], [6] for the proof.

As was mentioned in [6], $e_1/48$ and μ depend on a fixed spin structure on the fiber Σ_g. In principle, our theorem implies that the difference between the first Morita-Mumford class and the Rochlin invariant does not depend on a spin structure, and it is given by the von Neumann rho-invariant. In particular, we see from Corollary 2.4 that vanishing of the von Neumann rho-invariant gives a description of the first Morita-Mumford class via the Rochlin invariant on the Torelli group $\mathcal{I}_{g,*}$.

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2. A formula of $\rho^{(2)}$.

In this section, we give a formula for the von Neumann rho-invariant $\rho^{(2)}$ of surface bundles over S^1. As for the precise definition, see Cheeger-Gromov [3]. We remark that it does not depend on a Riemannian metric on the manifold.

Let M_φ be the mapping torus $\Sigma_g \times \boldsymbol{R}/(x, t) \sim (\varphi(x), t+1)$ corresponding to a holonomy $\varphi \in \mathcal{M}_{g,*}$. Let $\boldsymbol{Z} \to \bar{M}_\varphi \to M_\varphi$ be the $\boldsymbol{Z}$-covering associated to the surjective homomorphism $p : \pi_1 M_\varphi \to \pi_1 S^1 \cong \boldsymbol{Z}$.

PROPOSITION 2.1. *The von Neumann rho-invariant of $\bar{M}_\varphi$ is given by*

$$\rho^{(2)}(\bar{M}_\varphi) = -\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k-1} \mathrm{sign}(\varphi, \varphi^i)\,,$$

where sign *is Meyer's signature* 2*-cocycle* [8] *of the mapping class group* $\mathcal{M}_{g,*}$.

PROOF. Set $M_\varphi^k = \bar{M}_\varphi / k!\boldsymbol{Z}$. It is easy to see that $\boldsymbol{Z}/k! \to M_\varphi^k \to M_\varphi$ is the $\boldsymbol{Z}/k!$-covering associated to a homomorphism $p_k : \pi_1 M_\varphi \to \boldsymbol{Z}/k!$. We then apply an approximation theorem of the η-invariants, due to Vaillant [11] and Lück-Schick [7], to the sequence $\boldsymbol{Z} \triangleright 2!\boldsymbol{Z} \triangleright 3!\boldsymbol{Z} \triangleright \cdots$. It follows that

$$\rho^{(2)}(\bar{M}_\varphi) = \eta^{(2)}(\bar{M}_\varphi) - \eta(M_\varphi) = \lim_{k\to\infty} \frac{\eta(M_\varphi^k)}{k!} - \eta(M_\varphi)$$
$$= \lim_{k\to\infty} \left\{ \frac{1}{k!} \eta(M_{\varphi^{k!}}) - \eta(M_\varphi) \right\} .$$

Since $M_{\varphi^{k!}} \to M_\varphi$ is a $k!$-fold cyclic covering, we can directly apply our previous results to this covering (see [9] Propositions 2.2 and 3.1). Hence we obtain

$$\rho^{(2)}(\bar{M}_\varphi) = \lim_{k\to\infty} \frac{1}{3k!} \sum_{i=1}^{k!-1} c(\varphi, \varphi^i) ,$$

where c denotes Atiyah's 2-cocycle [1] of the mapping class group defined by the canonical 2-framing of 3-manifolds. Furthermore it is known that the cocycle c coincides with $-3 \times$ Meyer's signature cocycle sign (see [10] for instance). Therefore we have

$$\rho^{(2)}(\bar{M}_\varphi) = -\lim_{k\to\infty} \frac{1}{k!} \sum_{i=1}^{k!-1} \mathrm{sign}(\varphi, \varphi^i) = -\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k-1} \mathrm{sign}(\varphi, \varphi^i) . \qquad \square$$

REMARK 2.2. By definition, Meyer's cocycle is a bounded 2-cocycle, so that the above limit exists. Moreover this defines a class function on $\mathcal{M}_{g,*}$. We also remark that the above formula holds for torus bundles (that is, $g = 1$).

EXAMPLE 2.3. Let us consider the genus one case. As is well-known, in this case $\mathcal{M}_{1,*} \cong SL(2, \boldsymbol{Z})$ holds. An element $A \in SL(2, \boldsymbol{Z})$ is classified by its trace into the following three cases:

(i) Elliptic case (namely, $|\mathrm{tr}\, A| < 2$). Let $A_n \in SL(2, \boldsymbol{Z})$ have the order n ($n = 3, 4, 6$). We can take

$$A_3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} , \quad A_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_6 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} .$$

An easy calculation shows that $\mathrm{sign}(A_n, A_n) = \cdots = \mathrm{sign}(A_n, A_n^{n-2}) = -2$ and $\mathrm{sign}(A_n, A_n^{n-1}) = \mathrm{sign}(A_n, A_n^n) = 0$ (see [8]). Hence we have

$$\rho^{(2)}(\bar{M}_{A_n}) = \begin{cases} 2/3 , & n = 3 , \\ 1 , & n = 4 , \\ 4/3 , & n = 6 . \end{cases}$$

It should be noted that $\rho^{(2)}(\bar{M}_\varphi) = 0$ for any involution $\varphi \in \mathcal{M}_{g,*}$ (see [9]).

(ii) Parabolic case (namely, $|\mathrm{tr}\, A| = 2$). We can take

$$A_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad (b \in \boldsymbol{Z}) .$$

Then we obtain $\rho^{(2)}(\bar{M}_{A_b}) = -\mathrm{sgn}(b)$, where $\mathrm{sgn}(b) = b/|b|$ if $b \neq 0$ and 0 if $b = 0$.

(iii) Hyperbolic case (namely, $|\mathrm{tr}\, A| > 2$). Since Meyer's function of genus one, that is a class function $\phi : SL(2, \boldsymbol{Z}) \to (1/3)\boldsymbol{Z}$ such that $\delta\phi = \mathrm{sign}$, satisfies $\phi(A^k) = k\phi(A)$ for a hyperbolic element A (see [8]), we have

$$\rho^{(2)}(\bar{M}_A) = -\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k-1}\operatorname{sign}(A, A^i) = -\lim_{k\to\infty}\left\{\phi(A) - \frac{1}{k}\phi(A^k)\right\} = 0\,.$$

COROLLARY 2.4. *If φ is an element of the Torelli group $\mathcal{I}_{g,*}$, then $\rho^{(2)}(\bar{M}_\varphi) = 0$.*

PROOF. Meyer's 2-cocycle sign is originally defined on the Siegel modular group $\mathrm{Sp}(2g, \boldsymbol{Z})$, and $\mathcal{I}_{g,*}$ is the kernel of the homomorphism $\mathcal{M}_{g,*} \to \mathrm{Sp}(2g, \boldsymbol{Z})$, so that $\operatorname{sign}(\varphi, \varphi^i)$ vanishes for any i. □

3. Morita-Mumford class and Rochlin invariant. In this section, we summarize a work of Kitano [6], which gives a description of the Rochlin invariant as a secondary characteristic class within a framework of the bounded cohomology H_b^*. As for the definition of H_b^*, see Gromov [5].

Let (M, α) be an oriented spin 3-manifold with a spin structure α. It is a classical result that there exists a compact oriented spin 4-manifold (W, β) such that $\partial W = M$ and $\beta|_M = \alpha$. The *Rochlin invariant* $\mu(M, \alpha) \in \boldsymbol{Q}/\boldsymbol{Z}$ is defined by

$$\mu(M, \alpha) = \frac{\operatorname{Sign} W}{16} \mod \boldsymbol{Z}\,,$$

where Sign W denotes the signature of a 4-manifold W. By Rochlin's theorem, $\mu(M, \alpha)$ does not depend on the choice of W.

Let us fix a spin structure α of Σ_g $(g \geq 2)$. For each $\varphi \in \mathcal{M}_{g,*}(2)$, there exists a spin structure $\tilde{\alpha}$ on M_φ such that the restriction on each fiber is α. If we require that the restriction of $\tilde{\alpha}$ to the S^1-orbit of $* \in \Sigma_g$ is the bounding spin structure (namely, not the Lie group spin structure), then $\tilde{\alpha}$ is uniquely determined. This is the reason why we consider $\mathcal{M}_{g,*}(2)$ rather than $\mathcal{M}_g$.

Now consider the set of pairs $\{(\varphi, W)\}$, where $\varphi \in \mathcal{M}_{g,*}(2)$ and W is an oriented spin 4-manifold. Of course, $\partial W = M_\varphi$ and the induced spin structure on M_φ is $\tilde{\alpha}$. Two pairs (φ, W) and (φ, W') are said to be equivalent if Sign W = Sign W'. The set of equivalence classes, which we denote by $\mathcal{M}_{g,*}^\alpha(2)$, has a group structure defined by the fiber connected sum (see [6] for details), and there is a natural surjective homomorphism

$$\mathcal{M}_{g,*}^\alpha(2) \to \mathcal{M}_{g,*}(2)$$

given by $(\varphi, W) \mapsto \varphi$. Moreover we introduce a map $\tau : \mathcal{M}_{g,*}^\alpha(2) \to \boldsymbol{Q}$ by

$$\tau(\varphi, W) = \frac{1}{16}\operatorname{Sign} W \in \frac{1}{16}\boldsymbol{Z} \subset \boldsymbol{Q}\,.$$

PROPOSITION 3.1 (Kitano [6]). *Under the setting above, the following hold.*

(i) *The Euler class e_α of the extension $\mathcal{M}_{g,*}^\alpha(2) \to \mathcal{M}_{g,*}(2)$ is a bounded cohomology class and is given by $e_1/48$ on the level 2 subgroup $\mathcal{M}_{g,*}(2)$.*

(ii) *Let $f : \boldsymbol{Z} \to \mathcal{M}_{g,*}(2)$ be a homomorphism. The pull back $f^*e_\alpha \in H_b^2(\boldsymbol{Z}, \boldsymbol{Z})$ is described by $\bar{\tau}_\infty f \in H^1(\boldsymbol{Z}, \boldsymbol{R}/\boldsymbol{Z})$, where $\bar{\tau}_\infty : \mathcal{M}_{g,*}(2) \to \boldsymbol{R}/\boldsymbol{Z}$ is the reduction mod $\boldsymbol{Z}$ of* *a map*

$$\tau_\infty(\varphi, W) = \lim_{k\to\infty} \frac{\tau((\varphi, W)^k)}{k} \in \boldsymbol{R}\,.$$

From the additivity of the signature, we obtain

$$\tau_\infty(\varphi, W) = \tau(\varphi, W) + \frac{1}{16}\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k-1}\mathrm{sign}(\varphi, \varphi^i)\,.$$

Taking the reduction mod $\boldsymbol{Z}$ and using Proposition 2.1, we have

$$\bar{\tau}_\infty(\varphi) = \mu(M_\varphi, \tilde{\alpha}) - \frac{1}{16}\rho^{(2)}(\bar{M}_\varphi) \in \boldsymbol{R}/\boldsymbol{Z}$$

for $\varphi \in \mathcal{M}_{g,*}(2)$. Therefore Theorem 1.1 follows from Proposition 3.1 and the proof is completed.

REFERENCES

[1] M. ATIYAH, On framings of 3-manifolds, Topology 29 (1990), 1–7.

[2] J. BARGE AND E. GHYS, Cocycles d'Euler et de Maslov, Math. Ann. 294 (1992), 235–265.

[3] J. CHEEGER AND M. GROMOV, Bounds on the von Neumann dimension of L^2-cohomology and the Gauss-Bonnet theorem for open manifolds, J. Differential Geom. 21 (1985), 1–34.

[4] E. GHYS, Groupes d'homéomorphismes du cercle et cohomologie bornée, The Lefschetz centennial conference, III (Mexico City, 1984), 81–106, Contemp. Math. 58, Amer. Math. Soc., Providence, R.I., 1987.

[5] M. GROMOV, Volume and bounded cohomology, Inst. Hautes Études Sci. Publ. Math. 56 (1982), 5–100.

[6] T. KITANO, On the first Morita-Mumford class of surface bundles over S^1 and the Rochlin invariant, J. Knot Theory Ramifications 9 (2000), 179–186.

[7] W. LÜCK AND T. SCHICK, Approximating L^2-signatures by their finite-dimensional analogues, Forum Math. 17 (2005), 31–65.

[8] W. MEYER, Die Signatur von Flächenbündeln, Math. Ann. 201 (1973), 239–264.

[9] T. MORIFUJI, The η-invariant of mapping tori with finite monodromies, Topology Appl. 75 (1997), 41–49.

[10] S. MORITA, Structure of the mapping class groups of surfaces: a survey and a prospect, Proceedings of the Kirbyfest (Berkeley, CA, 1998), 349–406 (electronic), Geom. Topol. Monogr. 2, Geom. Topol. Publ., Coventry, 1999.

[11] B. VAILLANT, Indextheorie für Überlagerungen, Diplomarbeit, Universität Bonn, 1997.

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