A Note on Weak and Strong Linearizations of Regular Matrix Polynomials*

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1 Introduction

A paper of Tan and Pugh [TP] raises the question of ambiguity in a frequently used form of linearization when applied to regular matrix polynomials. Here, further insight into this question is provided, as well as a reminder of a stronger form of linearization (for which ambiguities are removed) introduced by Gohberg et al. [GKL].

Let $A_0, A_1, \dots, A_n \in \mathbb{C}^{n\times n}$, and define the matrix polynomial $L(\lambda) = \sum_{i=0}^{l} \lambda^i A_i$. Then $L(\lambda)$ is said to be *regular* if $\det L(\lambda)$ is not identically equal to zero. We consider only regular matrix polynomials, and notice that $A_l = 0$ is admitted. The *degree* of $L(\lambda)$ is the largest j for which $A_j \neq 0$. Thus, it may happen that $l > \deg(L)$.

Some important ideas for this discussion are as follows: Two regular matrix polynomials $A(\lambda)$, $B(\lambda)$ of the same size are said to be *equivalent* if there are unimodular[1] matrix polynomials $E(\lambda)$, $F(\lambda)$ such that $A(\lambda) = E(\lambda)B(\lambda)F(\lambda)$. The canonical form under equivalence is the well-known *Smith form*, and it reveals the structure of the invariant polynomials and (finite) elementary divisors.

There is also a *local* Smith form in which the transforming matrices $E(\lambda)$ and $F(\lambda)$ are invertible near an eigenvalue λ_0 and the elementary divisor structure of the single

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[1] i.e. invertible with non-vanishing determinant independent of λ

eigenvalue λ_0 is revealed (see [BGR], for example).

Define the *reverse* polynomial of $L(\lambda)$ to be the regular polynomial $L^{\#}(\lambda) = \lambda^l L(\lambda^{-1})$. Now the elementary divisors of $L(\lambda)$ at infinity are defined via $L^{\#}(\lambda)$ as follows: if the local Smith form for $L^{\#}(\lambda)$ at $\lambda = 0$ is

$$\operatorname{diag}[\lambda^{\kappa_1},\ \lambda^{\kappa_2},\ \cdots,\lambda^{\kappa_n}] \tag{1}$$

with $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_n \geq 0$, then these integers are, by definition, the degrees of the elementary divisors of $L(\lambda)$ at infinity. The eigenvalue at infinity is said to have algebraic multiplicity $\kappa = \sum_{j=1}^{n} \kappa_j$.

Two regular linear pencils $\lambda A_1 - B_1$ and $\lambda A_2 - B_2$ are said to be *strictly equivalent* if there are nonsingular matrices E and F such that $\lambda A_1 - B_1 = E(\lambda A_2 - B_2)F$. This relation holds if and only if the two pencils have the same (finite) elementary divisors and the same elementary divisors at infinity. Note, in particular, that equivalence may not preserve the elementary divisors at infinity.

These notions are classical (with some more modern features), and the reader is referred to the Appendix of Gohberg et al. [GLR] for a complete account (including derivation of the Weierstrass and Kronecker canonical forms for regular and singular pencils, respectively). Extensions to analytic and meromorphic functions can be found in the first two chapters of [BGR]. For the reduction of pairs of matrices by strict equivalence see Lancaster and Rodman [LR].

Two notions of linearization are to be examined. The weaker form is most frequently used and is defined as follows: An $ln \times ln$ linear matrix pencil $\lambda A - B$ is a *linearization (of order ln)* of $L(\lambda)$ if there are unimodular matrix polynomials $E(\lambda),\ F(\lambda)$ such that

$$\begin{bmatrix} L(\lambda) & 0 \\ 0 & I_{l(n-1)} \end{bmatrix} = E(\lambda)(\lambda A - B)F(\lambda), \tag{2}$$

and note that this is an equivalence in the sense of our definition.

Following Gohberg et al. [GKL], an $ln \times ln$ linear matrix pencil $\lambda A - B$ is a *strong linearization* of $L(\lambda)$ if, in addition to the condition (2), there are unimodular matrix polynomials $H(\lambda),\ K(\lambda)$ such that

$$\begin{bmatrix} L^{\#}(\lambda) & 0 \\ 0 & I_{l(n-1)} \end{bmatrix} = H(\lambda)(A - \lambda B)K(\lambda); \tag{3}$$

an equivalence relation connecting the reverse polynomial and the reverse pencil.

It is apparent from the definitions that, for a strong linearization the degrees of elementary divisors of $L(\lambda)$ at infinity are precisely the degrees of the (finite) elementary divisors of $L^{\#}(\lambda)$ at $\lambda = 0$. It will be shown that, in contrast, given only the first form of linearization, the elementary divisors of $\lambda A - B$ at infinity may be *any* set of integers $k_n \geq k_{n-1} \geq \cdots \geq k_1 \geq 0$ for which $\sum_{j=1}^{n} k_j = \kappa$, the algebraic multiplicity of the eigenvalue of $L(\lambda)$ at infinity (see Proposition 1 below).

2 The Weierstrass form

Denote a primitive $m \times m$ Jordan matrix with eigenvalue λ_0 by

$$J_m(\lambda_0) = \begin{bmatrix} \lambda_0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_0 \end{bmatrix}. \tag{4}$$

Let $\lambda A - B$ be a linearization of $L(\lambda)$ in the (weak) sense that there is an equivalence of the form (2). Then this pencil is also regular and is strictly equivalent to a (block diagonal) Weierstrass canonical form $W(\lambda)$; all blocks are built from (4) for some m and λ_0. Thus,

$$\begin{aligned} W(\lambda) \;\; = \;\; & (I_{k_1} + \lambda J_{k_1}(0)) \oplus \cdots \oplus (I_{k_p} + \lambda J_{k_p}(0)) \oplus \\ & \oplus(\lambda I_{l_1} + J_{l_1}(\lambda_1)) \oplus \cdots \oplus (\lambda I_{l_q} + J_{l_q}(\lambda_q)), \end{aligned} \tag{5}$$

(see Theorem A.5.3 of [GLR]). The first p blocks determine the structure of the eigenvalue at infinity and the remaining q blocks are associated with the finite eigenvalues (and there may be repetitions in the listing $\{\lambda_1, \ldots, \lambda_q\}$). This (finite) structure is uniquely defined by the equivalence (2), i.e. it is common to all linearizations of $L(\lambda)$

Observe that the pencil $W(\lambda)$ is also a linearization of $L(\lambda)$ (in the weak sense) and that $\sum_{j=1}^{p} k_j = \kappa$, the algebraic multiplicity of the eigenvalue of $L(\lambda)$ (or of $\lambda A - B$) at infinity. By absorbing constant nonsingular matrices into $E(\lambda)$ and $F(\lambda)$ of (2), it can be assumed, without loss of generality, that a linearization in Weierstrass form is obtained, and is further abbreviated to

$$W(\lambda) = \begin{bmatrix} I_\kappa + \lambda W_\infty & 0 \\ 0 & W_F(\lambda) \end{bmatrix}, \tag{6}$$

where

$$W_\infty = J_{k_1}(0) \oplus \cdots \oplus J_{k_p}(0)$$

and

$$W_F(\lambda) = (\lambda I_{l_1} + J_{l_1}(\lambda_1)) \oplus \cdots \oplus (\lambda I_{l_q} + J_{l_q}(\lambda_q)).$$

Now it may be assumed that

$$\begin{bmatrix} L(\lambda) & 0 \\ 0 & I_{l(n-1)} \end{bmatrix} = E(\lambda)W(\lambda)F(\lambda)$$

for some matrix polynomials $E(\lambda)$ and $F(\lambda)$ with constant nonzero determinants. Clearly $I_\kappa + \lambda W_\infty$ is a polynomial in λ with constant nonzero determinant, and it is easily verified that the inverse of any one of the blocks of $I_\kappa + \lambda W_\infty$ has the same property. Consequently, $(I_\kappa + \lambda W_\infty)^{-1}$ is also a unimodular polynomial in λ.

By absorbing appropriate factors into the multipliers $E(\lambda)$ and $F(\lambda)$ in the equivalence relations it is apparent that, while maintaining equivalence, the initial block $I_\kappa + \lambda W_\infty$ can be replaced by another such block determining arbitrary elementary divisors at infinity subject only to the constraint that the sum of their degrees is κ.

For example, two canonical forms (at infinity) with $k_1 = 3$ ($p = 1$) and $k_1 = 1,\ k_2 = 2$ ($p = 2$) are connected by the relation

$$\begin{bmatrix} 1 & \lambda & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \lambda & -\lambda^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{bmatrix}.$$

The canonical form on the right (with two canonical blocks) is transformed into that on the left by a unimodular polynomial multiplier (the first factor on the right). This polynomial factor (or its inverse) can be absorbed into a polynomial matrix associated with an equivalence transformation.

In contrast to this discussion, the *strong linearization* ensures that the elementary divisors at infinity for the linearization are precisely those of $L(\lambda)$ itself, as defined above. To summarise, our conclusion is:

Proposition 1 *Let $L(\lambda)$ be a regular matrix polynomial with an eigenvalue at infinity of algebraic multiplicity $\kappa > 0$. Then, given any partition of κ into positive integers, $\kappa = \sum_{j=1}^{p} k_j$, there is a linearization of $L(\lambda)$ with an eigenvalue at infinity having p elementary divisors of degrees $k_1, \ldots, k_p$.*

EXAMPLE: Let $L(\lambda) = \begin{bmatrix} \lambda^2 - 1 & 3 \\ 1 & 1 \end{bmatrix}$ and $l = 2$. Then $\det L(\lambda) = (\lambda - 2)(\lambda + 2)$,

$$L^{\#}(\lambda) = \begin{bmatrix} -\lambda^2 + 1 & 3\lambda^2 \\ \lambda^2 & \lambda^2 \end{bmatrix},$$

and $\det L^{\#}(\lambda) = (-\lambda^2)(4\lambda^2 - 1)$. Since

$$L^{\#}(\lambda) = \begin{bmatrix} 1 - 4\lambda^2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \lambda^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

(a local Smith form at $\lambda = 0$) it follows that $\kappa_1 = 2,\ \kappa_2 = 0$, and a corresponding Weierstrass linearization is:

$$W(\lambda) = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} \lambda - 2 & 0 \\ 0 & \lambda + 2 \end{bmatrix}.$$

However, by absorbing the polynomial

$$\begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

into one of the unimodular multipliers, another linearization of Weierstrass type is obtained, namely,

$$\hat{W}(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} \lambda - 2 & 0 \\ 0 & \lambda + 2 \end{bmatrix}.$$

3 Additional comments

For completeness, two basic results about strong linearizations are recalled from Gohberg et al. [GKL].

Proposition 2 *If $L(\lambda)$ is a regular polynomial, then the (so-called) first and second companion linearizations*

$$
\begin{aligned}
C_{1,l} &= \lambda \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ 0 & 0 & \cdots & 0 & A_l \end{bmatrix} - \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I \\ -A_0 & -A_1 & \cdots & -A_{l-2} & -A_{l-1} \end{bmatrix}, \\
C_{2,l} &= \lambda \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ 0 & 0 & \cdots & 0 & A_l \end{bmatrix} - \begin{bmatrix} 0 & 0 & \cdots & 0 & -A_0 \\ I & 0 & \cdots & 0 & -A_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -A_{l-2} \\ 0 & 0 & \cdots & I & -A_{l-1} \end{bmatrix},
\end{aligned}
$$

are strong linearizations of $L(\lambda)$ of order nl.

Proposition 3 *If $L(\lambda)$ is a regular polynomial, then all strong linearizations of $L(\lambda)$ are strictly equivalent.*

It follows that the two companion forms are strictly equivalent, of course. However, it is noteworthy that this is not simply a result of the fact that these forms are the "block transpose" of each other - the sparse block structures must also play a role.
EXAMPLE: In the above example the two companion forms

$$
\lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -3 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}, \quad \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & -3 \\ 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},
$$

are strong linearizations.

References

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