

A NOTE ON WEAK-STAR AND NORM BOREL SETS IN THE DUAL OF THE SPACE OF CONTINUOUS FUNCTIONS

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ABSTRACT. Let $\text{Bo}(T, \tau)$ be the Borel σ -algebra generated by the topology τ on T . In this paper we show that if K is a Hausdorff compact space, then every subset of K is a Borel set if and only if

$$\text{Bo}(C^*(K), w^*) = \text{Bo}(C^*(K), \|\cdot\|),$$

where w^* denotes the weak-star topology and $\|\cdot\|$ is the dual norm with respect to the sup-norm on the space of real-valued continuous functions $C(K)$. Furthermore, we study the topological properties of the Hausdorff compact spaces K such that every subset is a Borel set. In particular, we show that if the axiom of choice holds true, then K is scattered.

1. INTRODUCTION

Due to the presence of several important topologies on a Banach space it is natural to ask if there is any relationship between the Borel σ -algebras generated by these different topologies. Many authors have studied the relationship between the Borel sets generated by the weak topology and the one generated by the norm topology (see for example [2, 3, 14, 17]) and have found various conditions for the coincidence of these two classes. In particular, it is shown that the coincidence of the above σ -algebras is related to the existence of special types of equivalent norms (see [7–9, 13, 14, 17] for a study of these types of equivalent renormings).

The subject of this paper is to understand the topology of a compact space K such that the weak-star and norm Borel structure of $C^*(K)$ agree.

2. NOTATION AND PRELIMINARIES

A family \mathcal{N} of subsets of a topological space (T, τ) is a *network* for T if for every point $t \in T$ and every neighbourhood U of t there exists $N \in \mathcal{N}$ such that $t \in N \subseteq U$ (see [5] for more information). We remark that the definition of network differs from the definition of basis of a topological space, indeed it is not required for the sets of a network to be open.

We will denote by $P(X)$ the power set of a set X .

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Let τ be a topology on the set T ; then we will use the symbol $\text{Bo}(T, \tau)$ to denote the Borel σ -algebra generated by the topology τ on T , while we will denote with $\text{Ba}(T, \tau)$ the Baire σ -algebra, i.e., the smallest σ -algebra with respect to which all the τ -continuous real-valued functions are measurable. Finally, the symbol $\text{Ba}_p(T, \tau)$ will denote the σ -algebra of the sets with the Baire property in (T, τ) , i.e., all sets of the form $U\Delta M$, where U is open and M is of first category (see [15, Theorem 4.1]).

Given two topological spaces (X, τ_X) and (Y, τ_Y) , then a map $F : X \rightarrow P(Y)$ is said to be upper semicontinuous at a point $x \in X$ (*usc at x*, for short) if, for every open set V containing $F(x)$, there exists a neighbourhood U of x such that

$$F(U) = \bigcup \{F(u) \mid u \in U\} \subseteq V.$$

We say that F is upper semicontinuous (*usc*, for short) if it is upper semicontinuous at x for every point $x \in X$. We say that a map F is *usco* if it is *usc* and takes non-empty compact values.

We will denote by \aleph_0 and \mathfrak{c} the cardinality of the set of the natural numbers and of its power set, respectively. We will use the symbol ω_1 to denote the first uncountable ordinal.

3. THE MAIN RESULTS

Throughout this section (K, τ) will denote a Hausdorff compact space and $C(K)$ the space of real-valued continuous functions on K endowed with the sup-norm. In the main theorem we prove that if $\text{Bo}(C^*(K), w^*) = \text{Bo}(C^*(K), w)$, then

$$\text{Bo}(C^*(K), w^*) = \text{Bo}(C^*(K), \|\cdot\|).$$

Proposition 3.1. *Let K be a Hausdorff compact space containing a non-Borel subset; then $\text{Bo}(C^*(K), w^*) \neq \text{Bo}(C^*(K), w)$.*

Proof. It is a well-known fact that (K, τ) is homeomorphic to (K_0, w^*) , where $K_0 = \{\delta_k \mid k \in K\}$, the set of the Dirac measures concentrated in $k \in K$. So K_0 is a compact subset of $(C^*(K), w^*)$ and, in particular, it is closed with respect to the weak topology.

We claim that K_0 is discrete with respect to the weak topology. Indeed consider the family of functions $\mathcal{A} = \{f_k : K \rightarrow \mathbb{R} \mid k \in K\}$ defined as follows:

$$f_k(x) = \begin{cases} 1, & x = k, \\ 0, & x \neq k, \end{cases} \quad k, x \in K.$$

For every $k \in K$, f_k is a Borel function and can be seen as an element of $C^{**}(K)$ in the following way:

$$C^*(K) \langle \mu, f_k \rangle_{C^{**}(K)} = \int_K f_k d\mu, \quad \mu \in C^*(K).$$

Observe that

$$\left\{ \mu \in K_0 \mid f_k(\mu) > \frac{1}{2} \right\} = \{\delta_k\}.$$

So K_0 is discrete and closed with respect to the weak topology, which implies that $\text{Bo}(K_0, w) = P(K_0)$. Finally, if it holds that $\text{Bo}(C^*(K), w^*) = \text{Bo}(C^*(K), w)$, then $\text{Bo}(K_0, w^*) = \text{Bo}(K_0, w) = P(K_0)$, a contradiction. \square

We are now interested in the topological properties of compact spaces K such that every subset of K is a Borel set. As one may expect, these properties are strictly related to some set-theoretic axioms.

Proposition 3.2. *Let the continuum hypothesis hold true and let (X, τ) be a Hausdorff space with a countable network. If every subset of X is a Borel set, then X is finite or countable.*

Proof. By [10, paragraph 4A3F], $|\text{Bo}(X, \tau)| \leq \mathfrak{c}$. We have

$$2^{|X|} = |\mathcal{P}(X)| = |\text{Bo}(X, \tau)| \leq \mathfrak{c}.$$

By the continuum hypothesis it follows that $|X| \leq \aleph_0$, since $\mathfrak{c} = 2^{\aleph_0}$. □

Theorem 3.3. *Let the axiom of choice hold true and let (K, τ) be a Hausdorff compact space. If every subset of K is a Borel set, then K is scattered.*

Proof. By [16] a Hausdorff compact space K is scattered if and only if $[0, 1]$ is not a continuous image of K . By contradiction, let $f : K \rightarrow [0, 1]$ be a continuous surjection and consider the multifunction

$$F = f^{-1} : [0, 1] \longrightarrow \mathcal{K}^*(K),$$

where $\mathcal{K}^*(K)$ is the collection of all compact non-empty subsets of K . F is a compact and non-empty valued multifunction, and recalling that for every $B \in P([0, 1])$

$$F^{-1}(B) := \{x \in [0, 1] \mid F(x) \cap B \neq \emptyset\} = f(B)$$

and that continuous function from a compact space to a Hausdorff space is closed (see [5, p. 169]), we obtain that F is a usco map. Since $[0, 1]$ is a compact second countable space, in particular a Baire space, and K is a completely regular space (see [5, p. 196]) we can apply [11, Theorem 6] and get

$$f_0 : [0, 1] \longrightarrow K$$

a $\text{Ba}_p([0, 1], \tau_{\mathbb{R}})$ - $\text{Bo}(K, \tau)$ -selection of F .

We claim that f_0 is injective. Indeed if $x, y \in [0, 1]$ and $f_0(x) = f_0(y)$, then $f_0(x) \in f^{-1}(x)$ and $f_0(y) \in f^{-1}(y)$. But we know that $f^{-1}(x) \cap f^{-1}(y) = \emptyset$, so $x = y$. Recalling that for injective function it holds

$$f_0^{-1}(f_0(A)) = A \quad \text{for every } A \in P([0, 1]),$$

and every subset of K is a Borel set we have $\text{Ba}_p([0, 1], \tau_{\mathbb{R}}) = P([0, 1])$, which is a contradiction by [15, Theorem 5.5]. □

One may think that a space with only Borel subsets should be meager, but by [19] the existence of a measurable cardinal is equiconsistent with the existence of a non-meager T_4 space with no isolated point in which every subset is the union of an open and a closed set.

Corollary 3.4. *If every subset of K is a Borel set, then*

$$\text{Bo}(C^*(K), w) = \text{Bo}(C^*(K), \|\cdot\|).$$

Moreover if $\text{Bo}(C^(K), w^*) = \text{Bo}(C^*(K), w)$, then $\text{Bo}(C^*(K), w^*) = \text{Bo}(C^*(K), \|\cdot\|)$.*

Proof. By Theorem 3.3 K is scattered and by [1, Lemma 8.3 of Chapter VI] $C(K)$ is an Asplund space. So by [6] $C^*(K)$ admits an LUR renorming, and using [3, Corollary 2.4] one obtains the thesis. □

We want to stress that the furthermore part of Corollary 3.4 is not typical for a generic dual space. Indeed by [4, Proposition 8] the dual of the James space $J(\omega_1)$ has a weak Borel set which is not a weak-star Borel set, but since the dual of $J(\omega_1)$ is a Asplund space then, by [1, Lemma 8.3 of Chapter VI] and [3, Corollary 2.4], it holds $\text{Bo}(J^*(\omega_1), w) = \text{Bo}(J^*(\omega_1), \|\cdot\|)$.

Remark 3.5. By [18, Corollary 4.4] $C^*(K)$ admits a w^* -Kadets norm if and only if K is a countable union of relatively discrete subsets, so every subset of K is a Borel set. But if we assume the Martin axiom and the negation of the continuum hypothesis, then there exists an uncountable subset X of \mathbb{R} such that every subset of X is a relative \mathcal{F}_σ ; see [12]. In particular, if we consider the one-point compactification $\alpha(X)$ of X , then every discrete subset of $\alpha(X)$ is countable. So $\alpha(X)$ is not the countable union of relative discrete subsets.

REFERENCES

- [1] Robert Deville, Gilles Godefroy, and Václav Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 64, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1993. MR1211634
- [2] G. A. Edgar, *Measurability in a Banach space*, Indiana Univ. Math. J. **26** (1977), no. 4, 663–677, DOI 10.1512/iumj.1977.26.26053. MR487448
- [3] G. A. Edgar, *Measurability in a Banach space. II*, Indiana Univ. Math. J. **28** (1979), no. 4, 559–579, DOI 10.1512/iumj.1979.28.28039. MR542944
- [4] G. A. Edgar, *A long James space*, Measure theory, Oberwolfach 1979 (Proc. Conf., Oberwolfach, 1979), Lecture Notes in Math., vol. 794, Springer, Berlin, 1980, pp. 31–37. MR577956
- [5] Ryszard Engelking, *General topology*, PWN—Polish Scientific Publishers, Warsaw, 1977. Translated from the Polish by the author; Monografie Matematyczne, Tom 60. [Mathematical Monographs, Vol. 60]. MR0500780
- [6] Marián Fabián and Gilles Godefroy, *The dual of every Asplund space admits a projectional resolution of the identity*, Studia Math. **91** (1988), no. 2, 141–151, DOI 10.4064/sm-91-2-141-151. MR985081
- [7] S. Ferrari, L. Oncina, J. Orihuela, and M. Raja, *Metritzation theory and the Kadec property*, Banach J. Math. Anal. **10** (2016), no. 2, 281–306, DOI 10.1215/17358787-3492809. MR3474840
- [8] S. Ferrari, J. Orihuela, and M. Raja, *Weakly metrizable spheres and renormings of Banach spaces*, Q. J. Math. **67** (2016), no. 1, 15–27, DOI 10.1093/qmath/hav038. MR3471268
- [9] Simone Ferrari, José Orihuela, and Matías Raja, *Generalized metric properties of spheres and renorming of Banach spaces*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), no. 3, 2655–2663, DOI 10.1007/s13398-019-00652-1. MR3956274
- [10] D. H. Fremlin, *Measure theory. Vol. 4*, Torres Fremlin, Colchester, 2006. Topological measure spaces. Part I, II; Corrected second printing of the 2003 original. MR2462372
- [11] Siegfried Graf, *A measurable selection theorem for compact-valued maps*, Manuscripta Math. **27** (1979), no. 4, 341–352, DOI 10.1007/BF01507290. MR534799
- [12] D. A. Martin and R. M. Solovay, *Internal Cohen extensions*, Ann. Math. Logic **2** (1970), no. 2, 143–178, DOI 10.1016/0003-4843(70)90009-4. MR0270904
- [13] Aníbal Moltó, José Orihuela, Stanimir Troyanski, and Manuel Valdivia, *A nonlinear transfer technique for renorming*, Lecture Notes in Mathematics, vol. 1951, Springer-Verlag, Berlin, 2009. MR2462399
- [14] L. Oncina, *The JNR property and the Borel structure of a Banach space*, Serdica Math. J. **26** (2000), no. 1, 13–32. MR1767031
- [15] John C. Oxtoby, *Measure and category*, 2nd ed., Graduate Texts in Mathematics, vol. 2, Springer-Verlag, New York-Berlin, 1980. A survey of the analogies between topological and measure spaces. MR584443
- [16] A. Pelczyński and Z. Semadeni, *Spaces of continuous functions. III. Spaces $C(\Omega)$ for Ω without perfect subsets*, Studia Math. **18** (1959), 211–222, DOI 10.4064/sm-18-2-211-222. MR107806

- [17] M. Raja, *Kadec norms and Borel sets in a Banach space*, *Studia Math.* **136** (1999), no. 1, 1–16, DOI 10.4064/sm-136-1-1-16. MR1706603
- [18] M. Raja, *On dual locally uniformly rotund norms*, *Israel J. Math.* **129** (2002), 77–91, DOI 10.1007/BF02773154. MR1910933
- [19] Saharon Shelah, *A space with only Borel subsets*, *Period. Math. Hungar.* **40** (2000), no. 2, 81–84, DOI 10.1023/A:1010364023601. MR1805307

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