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A Note on Zero Divisor Graph Over Rings

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Abstract. In this article we discuss the graphs of the sets of zero-divisors of a ring. Now let R be a ring. Let G be a graph with elements of R as vertices such that two non-zero elements a, $b \in R$ are adjacent if ab = ba = 0. We examine such a graph and try to find out when such a graph is planar and when is it complete etc.

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1. Introduction

Zero divisor graphs have been of interest to authors, and a considerable work has been done in this direction, both in commutative as well as in noncommutative case. See for example [1, 6, 7, 8]. In this article we discuss the graphs on the sets of zero-divisors of a ring. We investigate such graphs and find their nature. In the first instance, we consider some of the rings that have been of interest in computer science and have applications as well. Now let R be a ring. Let G be a graph with elements of R as vertices such that two non-zero elements a, b R are adjacent if ab = ba = 0. We examine such a graph for the following rings:

1. $R = Z_n$, the set of integers modulo n with respect to addition modulo n and multiplication modulo n.
2. $R = P(X)$, the power set of a nonempty finite set with respect to addition '+' defined as A + B = $(A \cup B) - (A \cap B)$ and multiplication '.' defined as AB = $A \cap B$ where A, $B \in R$.
3. $R = M_2(S)$, the ring of all 22 matrices over a ring S with identity with respect to usual addition and multiplication of matrices. We discuss the graph for some subsets of R.
4. Particular case of (3) for S = Z_2.

All the above sets are of great interest in modern algebra as well as in discrete mathematics. In particular Z_2 (also known as Galois field modulo 2 and denoted by GF(2) and the ring of matrices over it are very interesting.

For the definitions of a ring, a subring of a ring, zero divisors, units, right/left ideal, field and other related results with examples, the reader is referred to Herstein [3]. For the definitions, examples and other related results of types of graphs, the reader is referred to [2, 5].

2. The ring Z_n

Consider Z_n, the set of integers modulo n. Z_n is a commutative ring with identity with respect to addition of integers modulo n and multiplication of integers modulo n. For n = p, p a prime number; the graph has no edges as Z_n in this case is a field and has no non-zero zero divisors. In case n is not a prime number. Say n = $(\alpha_1)^{p_1}.(\alpha_2)^{p_2}...(\alpha_k)^{p_k}$, p_i prime numbers, the possible adjacent vertices are $(\alpha, t\beta)$ with $t\beta < n$ and $\alpha\beta = n$. We illustrate for some n:

For n = 4, the only possible edge is (2, 2).

For n = 6, the possible edges are (2, 3) and (4, 3).

For n = 8, the possible edges are (2, 4); (4, 4); (4, 6).

For n = 9, the possible edges are (3, 3); (3, 6); (6, 6).

For n = 10, the possible edges are (2, 5); (4, 5); (6, 5); (8, 5).

For n=12, the possible edges are (2, 6); (4, 6); (6, 6); (8, 6); (10, 6); (3, 4); (9, 4).

For n = 25, the possible edges are (5, 5); (5, 10); (5, 15); (5, 20); (10, 10); (10, 20); (15, 10); (15, 15); (15, 20), (20, 20).

For n = 120, the possible edges are (2, 60); (4, 60); ... ; (118, 60); (3, 40); (6, 40); ... ; (117, 40); (4, 30); (8, 30); ... ; (116, 30); (5, 24); (10, 24); .... ; (115, 24); (6, 20); (12, 20); ... ; (114, 20); (8, 15); (16, 15); ... ; (112, 15); (12, 10); (24, 10); ... ;(108, 10); (20, 6); (40, 6); ... ;(100, 6); (24, 5); (48, 5); ... ;(96, 5); (30, 4); (60, 4); ... ;(90, 4); (40, 3); (80, 3).

Similarly one can find for other values of n. One can see that for n = p, a prime number, the graph is trivially planar. For n = 4, 6, 8, 9 , 12, 14, 15, 16, 18, 20, 21, 24 and 25, the graph is planar. From the above discussion, we have the following:

Theorem 2.1. *Let $n = pq$, where p and q are distinct prime numbers. Consider the ring Z_n as above. Define in Z_n a graph as ('a' is adjacent to 'b' if ab = 0, where a, $b \in Z_n$). If the isolated vertices are ignored, then the graph is bipartite.*

Proof. Here we see that n has only two prime factors. We arrange all multiples tp of p with $tp < n$ in one row and all multiples kq of q with $kq < n$ in another row. In this way we get a bipartite graph. □

Remark 2.2. The graph in Theorem 2.1 above is planar for $n < 35$. But for $n \geq 35$, it contains $K_{3,3}$. Therefore it is not planar.

Remark 2.3. For n = p^2, p a prime; the graph need not be planar. For example n = 49.

3. The Boolean Ring

Let X be a non-empty finite set. Consider P(X) the power set of X. Define in P(X) addition and multiplication as AB = $A \cap B$ and A + B = $(A \cup B) - (A \cap B)$ for any A, $B \in P(X)$. Then P(X) is a commutative ring with identity element X. The zero element of P(X) is ϕ, the empty set. We note in this ring every element is idempotent; i.e. A^2 = A for all $A \in P(X)$. This ring is called a Boolean ring. (We recall that a ring R is called a Boolean ring if each element of R is an idempotent. We also know that 'Every Boolean ring is a subring of P(X) for some X'. This result is known as the structure Theorem for Boolean rings). For details, the reader is referred to Musili [3]. If we define a graph with non-zero elements of P(X) as vertices such that A is adjacent to B if AB = ϕ, the zero element of P(X) where A, $B \in P(X)$. This graph is planar if number of elements of X is less than 4. For n = 4, 5, ... ; the graph is not planar but if we say that two sets A and B are adjacent if AB = ϕ, the zero element and A + B = X, the identity element. Then the graph is planar as two sets will be adjacent if and only if they are complements of each other.

For X = $\{a, b, c, d\}$, we have P(X) = $\{$ $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$; $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ $\}$, and the possible edges are ($\{a\}$, $\{b, c, d\}$); ($\{b\}$, $\{a, c, d\}$); ($\{c\}$, $\{a, b, d\}$); ($\{d\}$, $\{a, b, c\}$); ($\{a, b\}$, $\{c, d\}$); ($\{a, c\}$, $\{b, d\}$); ($\{a, d\}$, $\{b, c\}$). Thus we see that the set P(X) is divided into two subsets and these two subsets are in one to one correspondence with respect to this graph and this graph is bipartite. We end this section with the following:

Theorem 3.1. *Let X be a non empty set. Define a graph with non-zero elements of P(X) as vertices such that A is adjacent to B if AB = ϕ, the zero* *element of P(X) where A, $B \in P(X)$. This graph is planar if number of elements of X is less than 4.*

Illustrations: Let n(X) = k. For k = 1 is obvious. For k = 2, consider X = $\{a, b\}$. P(X) = $\{\ \{a\}; \{b\}; \{a, b\}; \phi\ \}$.
The possible edges are ($\{a\}$, $\{b\}$); ($\{a\}$, $\{c\}$); ($\{b\}$, $\{c\}$); ($\{a, b\}$, $\{c\}$); ($\{a, c\}$, $\{b\}$); ($\{b, c\}$, $\{a\}$).

For k = 4, consider X = $\{a, b, c, d\}$. Then P(X) = $\{\ \{a\}; \{b\}; \{c\}; \{d\}; \{a, b\}; \{b, c\}; \{a, c\}; \{a, d\}; \{b, d\}; \{c, d\}; \{a, b, c\}; \{a, b, d\}; \{a, c, d\}; \{b, c, d\}; \{a, b, c, d\}; \phi\ \}$. The graph contains a complete sub graph with four vertices $\{a\}$; $\{b\}$; $\{c\}$; $\{d\}$. The other possible edges are: ($\{a\}$, $\{b, c\}$); ($\{a\}$, $\{b, d\}$); ($\{a\}$, $\{c, d\}$); ($\{a\}$, $\{b, c, d\}$); ($\{b\}$, $\{a, c\}$); ($\{b\}$, $\{a, d\}$); ($\{b\}$, $\{c, d\}$); ($\{b\}$, $\{a, c, d\}$); ($\{c\}$, $\{a, b\}$); ($\{c\}$, $\{a, d\}$); ($\{c\}$, $\{b, d\}$); ($\{c\}$, $\{a, b, d\}$); ($\{d\}$, $\{a, b\}$); ($\{d\}$, $\{a, c\}$); ($\{d\}$, $\{b, c\}$); ($\{d\}$, $\{a, b, c\}$); ($\{a, b\}$, $\{c, d\}$); ($\{a, c\}$, $\{b, d\}$); ($\{a, d\}$, $\{b, c\}$) . Clearly this graph is not planar.

4. Matrix ring

Let S be a ring with identity 1. Let R = $M_2(S)$, the ring of all 2×2 matrices over S. In this case R is a non-commutative ring. Let G be a graph with non-zero elements of R as vertices such that two matrices A, $B \in R$ are adjacent if AB = BA = 0, the zero element of R. Then we have the following:

1. Let T be the set of matrices with non-zero entry only at $(1-1)^{th}$ place and K be the set of matrices with non-zero entry only at $(2-2)^{th}$ place. Then AB = BA = 0 for any $A \in T$ and for any $B \in K$. Therefore the graph is a complete bipartite.
2. Let U be the set of matrices with non-zero entry at $(1-2)^{th}$ place only. Then AB = BA = 0 for any A, $B \in U$. Therefore we see that in this set every element is nilpotent ($A^2 = 0$ for all $A \in U$) and the graph with vertices as elements of U is thus a complete graph.
3. Let V be the set of matrices with non-zero entry at $(2-1)^{th}$ place only. This set also has the same nature as U.
4. Let L be the set of all matrices with zero entries in second column and M be the set of all matrices with zero entries in first row. Then L is a left ideal of R and M is a right ideal of R. Also AB = 0 for any $A \in L$ and for any $B \in M$. Here BA need not be zero but if we consider the graph as directed one and say that A and B are adjacent if AB = 0, then we have an edge from each element of L to each element of M. Thus in this case the graph is a complete bipartite.
5. Let L be the set of all matrices with zero entries in first column and M be the set of all matrices with zero entries in second row. Then L and M have the same nature as in (4) above.

5. Matrix ring over Z_2

We now consider a special case of $M_2(S)$, when S = $Z_2 = \{0, 1\}$, the field of integers modulo 2. We consider R as in above section. We have the following notation:

A_{ij} denotes the matrix with 1 at $(ij)^{th}$ place and zero elsewhere. B_{ij} denotes the matrix with zero at $(ij)^{th}$ place and 1 elsewhere. R_i denotes the matrix with 1 in i^{th} row and zero elsewhere. C_i denotes the matrix with 1 in i^{th} column and zero elsewhere. A denotes the matrix with 1 at each place. We denote the zero matrix by 0 and the identity matrix by I. J denotes the matrix with diagonal entries zero and 1 elsewhere. With this we have the following:

1. I, J and B_{ij} are the units of R as
   $B_{22}B_{11} = B_{11}B_{22} = I$; $(B_{12})^2 = (B_{21})^2 = I$; $(J)^2 = I$.
2. $A_{11}A_{21} = A_{11}A_{22} = A_{11}R_2 = 0$; $R_1C_1 = R_1C_2 = R_1A = 0$;
   $R_2C_1 = R_2C_2 = R_2A = 0$; $C_1R_2 = C_2R_1 = 0$; $(A_{12})^2 = 0$.

We note that all non-units are zero divisors and the corresponding graph is planar. Thus we have the following:

Theorem 5.1. *Let $S = Z_2 = \{0, 1\}$, the field of integers modulo 2, and R = $M_2(S)$, the ring of all 22 matrices over S. Let G be a graph with non-zero elements of R as vertices such that two matrices $A, B \in R$ are adjacent if AB = 0, the zero of R. Then the di graph is planar.*

References

[1] D. F. Anderson and P. S. Livingston, The Zero-Divisor Graph of a Commutative Ring, Journal of Algebra, 1999, 217, 434-447.

[2] F. Harray, Graph Theory, Addison-Wesley Publishing Company, Inc., Reading, Mass., 1969.

[3] I. N. Herstein, Abstract Algebra, Macmillan, New York.

[4] C. Musili, Introduction to rings and modules, Revised Edition, Narosa Publishing House (1994).

[5] O. Ore, Theory of Graphs, American Mathematical Society, Providence, R.I., 1962.

[6] S. P. Redmond, Structure in the Zero-Divisor Graph of a Non-Commutative Ring, Houston Journal of Math, 2004, 30 (2), 345-355.

[7] N. O. Smith, Planar Zero-Divisor Graphs, International Journal of Commutative Rings, 2002, 2 (4), 177-188.

[8] Wu, Tongsuo, On Directed Zero-Divisor Graphs of Finite Rings, Discrete Mathematics, 2005, 296(1), 73-86.

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