A Novel Analytical Model to Design Piezoelectric Patches Used to Repair Cracked Beams

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ABSTRACT

In this paper, an analytical solution describing the deflection of a cracked beam repaired with piezoelectric patch is introduced. The solution is derived using perturbation method. A novel analytical model to calculate the proper dimensions of piezoelectric patches used to repair cracked beams is also introduced. This model shows that the thickness of the piezoelectric patch depends mainly on the thickness of the cracked beam, the electro-mechanical properties of the patch material, the applied load and the crack location. Furthermore, the model shows that the length of the piezoelectric patches depends on the thickness of the patch as well as it depends on the length of the cracked beam and the crack depth. The additional flexibility of the beam caused by crack is modeled depending on a dimensionless parameter identified from finite elements method. Different piezoelectric patches were designed and investigated using analytical and finite elements models. The results show that increasing the patch thickness enhances the beam resistance to crack and load effects, while increasing the length of the piezoelectric patch reduces the magnitude of the voltage required to repair the cracked beam.

Keywords: Repaired beam deflection, slope discontinuity and piezoelectric patch dimensions

نموذج تحليلي جديد لتصميم الصفائح الكهروضغطية المستخدمة لأصلاح العتبات الحاوية على تصدعات

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الخلاصة

يقدم هذا البحث النموذج التحليلي لوصف الانحراف الناتج في العتبات الحاوية على تصدع والمصلحة باستخدام صفائح كهروضغطية. تم تطوير هذا النموذج بالأعتماد على اضافة مفردات رياضية للحل التحليلي مشتقة من استخدام نموذج مبني بأستخدام طريقة العناصر المحددة. يقدم هذا البحث ايضاً نموذج تحليلي جديد لأيجاد الأبعاد المناسبة لصفائح الكهروضغطية المستخدمة لأصلاح العتبات الحاوية على تصدعات. ان النموذج المقترح اظهر ان سمك الصفيحة الكهروظغطية يعتمد بشكل كبير على سمك العتبة المراد اصلاحها و المواصفات الكهروميكانيكية للصفيحة و مقدار الحمل المسلط و موقع التصدع في العتبة. وايظا، اظهر النموذج ان طول الصفيحة الكهروضغطية يعتمد على سمك الصفيحة بالاضافة الى أنه يعتمد على طول العتبة و عمق التصدع. ان المرونه الاضافية للعتبة الناتجة بسبب وجود التصدع تم نمذجتها بالاعتماد على معامل بدون وحدات يمكن ان يوجد باستخدام طريقة العناصر المحددة. مختلف من الصفائح تم تصميمها ودراستها اعتماداً على النماذج التحليلية المقترحة و نموذج العناصر المحددة. أظهرت النتائج ان زيادة سمك الصفيحة الكهروضغطية سيحسن مقاومة العتبة لتاثير التصدع و تاثير الحمل الخارجي بينما زيادة طول الصفية الكروضغطية يقلل مقدار الفولطية المطلوبة لأجراء عملية الأصلاح

1. INTRODUCTION

Repair cracked structure using passive patch is achieved from bonding the patch at the crack location where it causes to local increasing in the structure stiffness. Thus, the structure will resist the crack propagation. Composite patch usually is preferable on the metal patch because it has a higher stiffness and a lighter weight. Many researches are investigated the repair of cracked structures using metal or composite patches (passive patches) e.g. **Rose, 1981, Chue et al., 1994, Sun et al., 1996, Doung et al., 2006**, **Ayatollahi and Hashemi, 2007, Hosseini-Toudeshky et al., 2011, Maligno et al., 2013,**. **Ramji et al., 2013,** and **Kwon and Hall, 2015**.
Passive patches become inefficient when the load conditions and/or the crack characteristics are changed. This motivates many researchers to study active patches made from a piezoelectric material instead passive patches. The active patch converts the applied voltage on it to a bending moment to counter the bending moment caused by the external load on the beam at the crack location. The magnitude and direction of the bending moment caused by the piezoelectric patch can be controlled by adjusting the external applied voltage. The technique of crack repair using piezoelectric patch is proposed first by **Wang et al., 2002**. The authors discussed the feasibility of using piezoelectric patch to repair a simply supported beam includes an open type crack subjected to static load. They were theoretically showed that using piezoelectric patch can effectively repair the cracked beam. Later **Wang et al., 2004** studied theoretically the repair of the cracked beam subjected to an external dynamic load. It was deduced that using piezoelectric patch may decrease the singularity at the crack tip. **Liu, 2007,** studied two-dimensional plane strain finite element analyses of the active repair for cracked structures using multi-layered piezoelectric patch. It was concluded that the better design choices for the piezoelectric patch are as follows: increasing the layer number and increasing the patch length. In additions, it is not a good idea to use higher input voltage that is larger than the required voltage because it will enlarge the crack open near the crack tip. Later **Liu, 2008, made a** comparison between two criteria which are used to repair beams with piezoelectric patches: the slope continuity and fracture mechanics criteria. The finite element analyses have been used to consider crack contact analyses and fracture mechanics in the crack tip. It was concluded that the fracture mechanics criterion is better for defining the required voltage for repair. **Ariaei et al., 2010,** introduced an approximated model describing the deflection of a beam contains a crack and is repaired with a piezoelectric patch; the beam is subjected to moving mass. The governor equation is derived basing on Timoshenko beam theory. The authors deduced that using piezoelectric patch can reduce the cracked beam deflection to be same as that when the beam is healthy i.e. has no crack. **Platz et al., 2011**, introduced a statistical approach to evaluate the reduction of the fatigue crack propagation within a simply supported beam which is repaired using piezoelectric patch. It was shown that repair the beam with piezoelectric patch leads to a significant reduction in the crack propagation within the beam.
In the present work, an analytical solution describing the deflection of a cracked beam is introduced. This solution is extended to describe a cracked beam repaired with piezoelectric patch. The solution is developed using perturbation method whereas the model describing the repaired beam includes a mathematical term identified from a finite elements model. Precisely, the crack is assumed to cause an additional flexibility to the beam. This flexibility makes discontinuity of the beam slope at the crack location. The slope discontinuity is described by a proposed relationship which contains a dimensionless parameter identified from finite elements models. Furthermore, the derived solution is developed to obtain a novel model which can be used to determine the proper dimensions of the piezoelectric patch depending on the magnitude of the subjected force, crack depth and location and the desired applied voltage. The patch

design is based on two important factors: factor of safety and the voltage factor. These factors are introduced to ensure that the piezoelectric patch can repair effectively the cracked beam depending on the designer preferences. A cantilever beam subjected to an external static load at its tip was used to validate the proposed model. The cantilever beam was studied at different conditions: different dimensions and locations of the crack and different dimensions of the piezoelectric patch. For each condition, a proper piezoelectric patch was designed and investigated using finite elements model. In all conditions a good agreement was obtained between the results and the designed piezoelectric patch satisfied the repair requirements.

2. MODELLING OF BEAM

Cantilever beam is chosen to investigate the parameters which affect the repair of a cracked beam with piezoelectric patch. The same procedure can be followed to obtain the solution for any other type of beam support. As shown in **Fig. 1**, three cases will be modeled: healthy cantilever beam, cantilever beam includes open type crack and cracked cantilever beam repaired with piezoelectric patch.

The crack is considered to be an open type crack as shown in **Fig. 2**. The width of the crack is b_c and the depth of the crack is t_c. It is assumed that the crack appears along the whole width of the beam.

2.1 Healthy Cantilever Beam

Healthy beam refers to that beam does not contain any crack; the equation of the deflection of such beams can be expressed as **(Hearn, 1985)**

$$Y_b I_b \frac{d^2 y}{dx^2} = M_b(x), \qquad (1)$$

where y the deflection of the beam, Y_b is the modulus of elasticity, I_b is the moment of inertia and $M(x)$ the external applied moment. Based on Equation (1), the deflection of the healthy cantilever beam shown in **Fig. 1a** can be expressed as

$$Y_b I_b \frac{d^2 y_h}{dx^2} = -Fx, \qquad 0 \le x \le l_b, \qquad (2)$$

where y_h is the deflection of the healthy beam and F is the applied external load. The slope θ_h and deflection y_h of the healthy cantilever beam can be obtained from integrating Equation (2) with using the following boundary conditions

$$\theta_h|_{x=l_b} = \left.\frac{dy_h}{dx}\right|_{x=l_b} = 0, \qquad (3a)$$

and

$$y_h|_{x=l_b} = 0, \qquad (3b)$$

where l_b is the length of the beam. Thus, the slope of this beam is can be written as

$$\theta_h = \frac{F}{2Y_b I_b}(l_b^2 - x^2), \qquad 0 \le x \le l_b. \tag{4}$$

The deflection of a healthy cantilever beam can be expressed as

$$y_h = \frac{F}{6Y_b I_b}(3l_b^2 x - 2l_b^3 - x^3), \qquad 0 \le x \le l_b. \tag{5}$$

More details can be found in any textbook of mechanics of materials e.g. **Hearn, 1985**.

2.2 Cracked Cantilever Beam

Cracked beam can be modeled basing on the piecewise approach introduced by **Krawczuk and Ostachowicz, 1995**. In this approach, the beam is divided into finite segments depending on the crack location. Referring to **Fig. 1b**, the slope of the cracked beam θ_c can written as

$$\theta_c = \begin{cases} \theta_{c1} & 0 \le x \le l_c, \\ \theta_{c2} & l_c \le x \le l_b. \end{cases} \tag{6}$$

The deflection of the cracked beam y_c can expressed as

$$y_c = \begin{cases} y_{c1} & 0 \le x \le l_c, \\ y_{c2} & l_c \le x \le l_b, \end{cases} \tag{7}$$

where l_c is the length of the beam at which the crack appears. For simplicity, l_c will be called crack location. The governing equations of the cracked beam can be written as

$$Y_b I_b \frac{d^2 y_{c1}}{dx^2} = -Fx, \qquad 0 \le x \le l_c, \tag{8a}$$

and

$$Y_b I_b \frac{d^2 y_{c2}}{dx^2} = -Fx, \qquad l_c \le x \le l_b. \tag{8b}$$

Integrating Equation (8a) and Equation (8b), thus

$$Y_b I_b \theta_{c1} = -\frac{F}{2}x^2 + c_3, \qquad 0 \le x \le l_c, \tag{9a}$$

$$Y_b I_b y_{c1} = -\frac{F}{6}x^3 + c_3 x + c_4, \qquad 0 \le x \le l_c, \tag{9b}$$

$$Y_b I_b \theta_{c2} = -\frac{F}{2}x^2 + c_1, \qquad l_c \le x \le l_b, \tag{9c}$$

$$Y_b I_b y_{c2} = -\frac{F}{6}x^3 + c_1 x + c_2, \qquad l_c \le x \le l_b. \tag{9d}$$

To find the constants (c_1 to c_4) in Equation (9a) to Equation (9d), it is required four boundary conditions; these conditions are:

$$\theta_{c2}|_{x=l_b} = 0, \tag{10a}$$

and

$$y_{c2}|_{x=l_b} = 0. \tag{10b}$$

It can be predicted that the crack causes a discontinuity of the beam slope at the crack location. This means that the slope of the beam to right of the crack differs from the slope of the beam to the left of the crack as shown in **Fig. 3**. Therefore, the following relationship between the slopes of the beam at the crack location can be assumed

$$\theta_{c1}|_{x=l_c} - \theta_{c2}|_{x=l_c} = \mu\, \theta_{c1}|_{x=l_c}. \tag{10c}$$

where μ is a dimensionless parameter has a value less than one as shown in **Fig. 3**. This parameter is introduced to count for the additional flexibility of the beam due to crack. Therefore, it will be called the crack flexibility parameter. It depends on beam characteristics (geometry and properties) and crack Characteristics (dimensions and location). This parameter can be identified either from experiments or from finite element model. In this paper, the crack flexibility parameter will be identified from solving the system with finite elements model. Because of the homogeneity in beam deflection, the following boundary condition can be assumed

$$y_{c1}|_{x=l_c} = y_{c2}|_{x=l_c}. \tag{10d}$$

These boundary conditions (Equation (10a) to Equation (10d)) are used to calculate the constants (c_1 to c_4) in Equation (9a) to Equation (9d); these constant found to be

$$c_1 = \frac{F}{2} l_b^2, \tag{11a}$$

$$c_2 = -\frac{F}{3} l_b^3, \tag{11b}$$

$$c_3 = \frac{F}{2}\left[\frac{(l_b^2 - l_c^2)}{(1-\mu)} + l_c^2\right], \tag{11c}$$

and

$$c_4 = \frac{F}{6}\left[3l_b^2 l_c - 2l_b^3 - 3l_c^3 - \frac{3}{1-\mu}(l_b^2 l_c - l_c^3)\right]. \tag{11d}$$

Substituting the expressions of the constants (Equation (11a) to Equation (11d)) into the equations of the slopes derived earlier (Equation (9a) and Equation (9c)) gives

$$\theta_{c1} = \frac{F}{2Y_b I_b}\left[\left(\frac{l_b^2 - l_c^2}{1-\mu}\right) + l_c^2 - x^2\right], \qquad 0 \le x \le l_c, \qquad (12a)$$

and

$$\theta_{c2} = \frac{F}{2Y_b I_b}(l_b^2 - x^2), \qquad l_c \le x \le l_b. \qquad (12b)$$

Also, from substituting Equation (11a) to Equation (11d) into Equation (9b) and Equation (9d) gives the deflections of the cracked beam as

$$y_{c1} = \frac{F}{6Y_b I_b}\left[3l_c^2 x + 3l_b^2 l_c - 2l_b^3 - 3l_c^3 - x^3 + \frac{3}{1-\mu}(l_b^2 - l_c^2)(x - l_c)\right], \qquad 0 \le x \le l_c, \qquad (13a)$$

and

$$y_{c2} = \frac{F}{6Y_b I_b}(3l_b^2 x - 2l_b^3 - x^3), \qquad l_c \le x \le l_b. \qquad (13b)$$

2.3 Cracked Cantilever Beam Repaired with Piezoelectric Patch

Now, same approach stated before will be used to model the repaired beam with patch. Referring to **Fig. 1c**, the slope of the repaired beam θ_r can be expressed as

$$\theta_r = \begin{cases} \theta_{r1}, & 0 \le x \le l_c - l_p, \\ \theta_{r2}, & l_c - l_p \le x \le l_c, \\ \theta_{r3}, & l_c \le x \le l_c + l_p, \\ \theta_{r4}, & l_c + l_p \le x \le l_b. \end{cases} \qquad (14)$$

Also, the deflection of the repaired beam y_r can be written as

$$y_r = \begin{cases} y_{r1}, & 0 \le x \le l_c - l_p, \\ y_{r2}, & l_c - l_p \le x \le l_c, \\ y_{r3}, & l_c \le x \le l_c + l_p, \\ y_{r4}, & l_c + l_p \le x \le l_b. \end{cases} \qquad (15)$$

The governing equations of the repaired beam can written as

$$Y_b I_b \frac{d^2 y_{r1}}{dx^2} = -Fx, \qquad 0 \le x \le l_c - l_p. \qquad (16a)$$

Two effects can be caused by piezoelectric patch at the bonding location on the beam: increasing local stiffness and applying local bending moment M_p. The local increasing of the beam stiffness at the bonding location can be addressed from considering the total flexural rigidity $Y_t I_t$, thus

$$Y_t I_t \frac{d^2 y_{r2}}{dx^2} = -Fx + M_p, \qquad l_c - l_p \le x \le l_c, \qquad (16b)$$

$$Y_t I_t \frac{d^2 y_{r3}}{dx^2} = -Fx + M_p, \qquad l_c \le x \le l_c + l_p, \qquad (16c)$$

and

$$Y_b I_b \frac{d^2 y_{r4}}{dx^2} = -Fx, \qquad l_c + l_p \le x \le l_b. \qquad (16d)$$

The total flexural rigidity $Y_t I_t$ is calculated and found to be

$$Y_t I_t = Y_p \left\{ \frac{b_p t_p^3}{12} + \left[b_p t_p \left(\frac{t_b + t_p}{2} \right)^2 \right] \right\} + Y_b \left(\frac{b_b t_b^3}{12} \right). \qquad (17)$$

where Y_p, t_p and b_p are the modulus of elasticity of the piezoelectric patch, the thickness of the piezoelectric patch and the width of the piezoelectric patch, respectively. t_b and b_b are the thickness and width of the beam, respectively. Again, integrating Equation (16a) to Equation (16d) gives

$$Y_b I_b \theta_{r1} = -\frac{Fx^2}{2} + r_7, \qquad 0 \le x \le l_c - l_p, \qquad (18a)$$

$$Y_b I_b y_{r1} = -\frac{Fx^3}{6} + r_7 x + r_8, \qquad 0 \le x \le l_c - l_p, \qquad (18b)$$

$$Y_t I_t \theta_{r2} = -\frac{Fx^2}{2} + M_p x + r_5, \qquad l_c - l_p \le x \le l_c, \qquad (18c)$$

$$Y_t I_t y_{r2} = -\frac{Fx^3}{6} + \frac{M_p x^2}{2} + r_5 x + r_6, \qquad l_c - l_p \le x \le l_c, \qquad (18d)$$

$$Y_t I_t \theta_{r3} = -\frac{Fx^2}{2} + M_p x + r_3, \qquad l_c \le x \le l_c + l_p, \qquad (18e)$$

$$Y_t I_t y_{r3} = -\frac{Fx^3}{6} + \frac{M_p x^2}{2} + r_3 x + r_4, \qquad l_c \le x \le l_c + l_p, \qquad (18f)$$

$$Y_b I_b \theta_{r4} = -\frac{Fx^2}{2} + r_1, \qquad l_c + l_p \le x \le l_b, \qquad (18g)$$

and

$$Y_b I_b y_{r4} = -\frac{Fx^3}{6} + r_1 x + r_2, \qquad l_c + l_p \le x \le l_b. \qquad (18h)$$

In order to calculate the constants (r_1 to r_8) in Equation (18a) to Equation (18h), the following boundary conditions are applied:

$$\theta_{r4}|_{x=l_b} = 0, \tag{19a}$$

$$y_{r4}|_{x=l_b} = 0, \tag{19b}$$

$$\theta_{r3}|_{x=l_c+l_p} = \theta_{r4}|_{x=l_c+l_p}, \tag{19c}$$

$$y_{r3}|_{x=l_c+l_p} = y_{r4}|_{x=l_c+l_p}, \tag{19d}$$

$$\theta_{r2}|_{x=l_c} - \theta_{r3}|_{x=l_c} = \mu\, \theta_{r2}|_{x=l_c}, \tag{19e}$$

$$y_{r2}|_{x=l_c} = y_{r3}|_{x=l_c}, \tag{19f}$$

$$\theta_{r2}|_{x=l_c-l_p} = \theta_{r1}|_{x=l_c-l_p}, \tag{19g}$$

and

$$y_{r1}|_{x=l_c-l_p} = y_{r2}|_{x=l_c-l_p}. \tag{19h}$$

Applying the above boundary conditions on Equation (18a) to Equation (18h), thus the constants can be expressed as (the constants are calculated depending on each other in order to keep short term expression)

$$r_1 = \frac{F l_b^2}{2}, \tag{20a}$$

$$r_2 = \frac{F l_b^3}{6} - r_1 l_b, \tag{20b}$$

$$r_3 = \frac{F}{2}(1-k)\left(l_c + l_p\right)^2 + k r_1 - M_p\left(l_c + l_p\right), \tag{20c}$$

$$r_4 = \frac{F}{6}(1-k)\left(l_c + l_p\right)^3 - \frac{M_p}{2}\left(l_c + l_p\right)^2 + (k r_1 - r_3)\left(l_c + l_p\right) + k r_2, \tag{20d}$$

$$r_5 = \frac{1}{(1-\mu)}\left[\mu\left(M_p l_c - \frac{F l_c^2}{2}\right) + r_3\right], \tag{20e}$$

$$r_6 = (r_3 - r_5) l_c + r_4, \tag{20f}$$

$$r_7 = \frac{1}{2k}\left[F(k-1)\left(l_c - l_p\right)^2 + 2M_p\left(l_c - l_p\right) + 2r_5\right], \tag{20g}$$

and finally

$$r_8 = \frac{1}{6k}\Big[F(k-1)\big(l_c - l_p\big)^3 + 3M_p\big(l_c - l_p\big)^2 + 6(r_5 - kr_7)\big(l_c - l_p\big) + 6r_6\Big]. \tag{20h}$$

The constant introduced in the above equations k is defined as

$$k = \frac{Y_t I_t}{Y_b I_b}, \tag{21}$$

3. OPERATION OF THE PIEZOELECTRIC PATCH

Adjusting the operation of the piezoelectric patch is the most important parameter which determines the achieving of the beam repair. The operation can be adjusted from controlling the voltage applied on the piezoelectric patch. This voltage should be calculated based on a certain operation strategy of the piezoelectric patch. The strategy should ensure that the applied voltage does not exceed the maximum voltage which the piezoelectric patch can withstand. Also, it should ensure that the moment induced by the piezoelectric patch not exceeds and counters the moment caused by the external force at the crack location.

3.1 Applied Voltage

The axial stress along piezoelectric layer σ_x induced by the applied voltage V_a can be written as (**Sun et al, 1999**)

$$\sigma_x = e_{31}\frac{V_a}{t_p}, \tag{22}$$

where t_p is the thickness of the piezoelectric layer and e_{31} the stress constant of the piezoelectric patch which can be calculated as (**Southin et al, 2001**)

$$e_{31} = \frac{d_{31}}{s_{11}^E + s_{12}^E}, \tag{23}$$

where d_{31} is the piezoelectric coefficient, s_{11}^E and s_{12}^E are elastic compliance of piezoelectric layer at constant electric field; these properties can be found in the datasheet of the used piezoelectric patch . The axial stress σ_x will cause a local bending moment effecting on the beam; this bending can be expressed as (**Wang et al, 2002**)

$$M_p = \sigma_x t_p\left(\frac{t_b + t_p}{2}\right). \tag{24}$$

The applied voltage V_a can be determined from using Equation (24) and Equation (22), thus

$$V_a = \frac{2M_p}{e_{31}\big(t_b + t_p\big)}. \tag{25}$$

The maximum moment generated by the piezoelectric patch $M_{p_{max}}$ equal to the moment caused by the external force at the crack location, otherwise the patch will bend the beam in the opposite and enlarge the crack open near the crack tip as stated by **Liu, 2008**. Therefore, the maximum moment induced by the piezoelectric patch $M_{p_{max}}$ can be expressed as

$$M_{p_{max}} = -Fl_c. \tag{26}$$

The minus sign refers to that the moment counting the moment caused by the external force F. The maximum applied voltage $V_{a_{max}}$ can be calculated from substituting Equation (26) into Equation (25), thus

$$V_{a_{max}} = -\frac{2Fl_c}{e_{31}(t_b + t_p)}. \tag{27}$$

In this paper, the repair is achieved if the piezoelectric patch makes the slope of the cracked beam return back to its original state when there is no crack; this means that the slope of the repaired beam equal to the slope of the healthy beam, thus

$$\theta_{r2}|_{x=l_c} = \theta_h|_{x=l_c}. \tag{28}$$

The slope $\theta_{r2}|_{x=l_c}$ can be found from substituting Equation (20a), Equation (20c) and Equation (20e) into Equation (18c) at $x = l_c$, thus

$$\theta_{r2}|_{x=l_c} = \frac{1}{Y_t I_t (1-\mu)}\left\{\frac{F}{2}\left[k\left[l_b^2 - (l_c + l_p)^2\right] + l_p^2 + 2l_p l_c\right] - M_p l_p\right\} \tag{29}$$

Equating Equation (29) and Equation (5), and then solving the obtained expression to find the moment M_p can be expressed as

$$M_p = \frac{F}{2l_p}\left[k\mu(l_b^2 - l_c^2) - (k-1)(l_p^2 + 2l_c l_p)\right] \tag{30}$$

Equation (31) can be used to calculate the constants derived before (Equation (20a) to Equation (20h). the applied voltage V_a can be found from substituting Equation (30) into Equation (25), thus

$$V_a = \frac{F\left[(k-1)(l_p^2 + 2l_c l_p) - k\mu(l_b^2 - l_c^2)\right]}{e_{31} l_p (t_b + t_p)}. \tag{31}$$

Equation (31) shows the effect of many important parameters on the magnitude of the applied voltage V_a ; these parameters are the length of the patch l_p, the length of the beam l_b, the stiffness parameter k and the flexibility parameter μ which their effects are not considered in the work of **Wang et al, 2002 and 2004**.

3.2 Design of Piezoelectric Patches

In order to ensure that the piezoelectric patch can effectively repair the cracked beam under the effect of the force F, it will be designed to withstand a larger force F_d which is related to the applied force F by the following relationship

$$\alpha_F = \frac{F_d}{F}, \tag{32}$$

where α_F is factor of safety has a value greater than one depending on the designer preferences. The maximum voltage V_d corresponding to the force F_d can be calculated from substituting Equation (32) into Equation (26), and then substituting the result into Equation (25), thus

$$V_d = -\frac{2\alpha_F F l_c}{e_{31}(t_b + t_p)}. \tag{33}$$

E is the maximum electrical field which can be applied on the piezoelectric patch. It is a property can be found in the datasheet of the used piezoelectric patch. Therefore, the desired applied voltage V_d can written as

$$V_d = E t_p. \tag{34}$$

Equating Equation (33) with Equation (34), and then solving the obtained expression to calculate the thickness of the piezoelectric patch t_p which is found to be

$$t_p = \frac{1}{2}\left(\sqrt{t_b^2 - \frac{8\alpha_F F l_c}{e_{31}E}} - t_b\right). \tag{35}$$

For completing the patch design, it is required to define a new factor α_V which will be called the voltage factor and defined as

$$\alpha_V = \frac{V_a}{V_{a_{max}}}. \tag{36}$$

The factor α_V is dimensionless factor has a value ranging from zero to one based on the desired applied voltage specified by the patch designer. From using Equation (27) and Equation (36), the applied voltage V_a can be expressed as

$$V_a = -\frac{2\alpha_V F l_c}{e_{31}(t_b + t_p)}. \tag{37}$$

The required length of the piezoelectric patch l_p can be calculated from equating Equation (31) with Equation (37), and then solving the obtained quadrant equation to calculate the length l_p which is found to be

$$l_p = \sqrt{\left(\frac{k\mu}{k-1}\right) l_b^2 + \left[\left(\frac{\alpha_V + k - 1}{k-1}\right)^2 - \left(\frac{k\mu}{k-1}\right)\right] l_c^2} - \left(\frac{\alpha_V + k - 1}{k-1}\right) l_c. \tag{38}$$

In the above analysis, the width of the piezoelectric patch is considered to be constant and equal to beam width.

4. FINITE ELEMENT MODEL

In this paper, a steel cantilever beam was chosen to be the case study. The beam has a rectangular cross sectional area of 10 mm width and 30 mm thickness. The length of the beam was chosen to be 1000 mm. The external force was applied at the tip of the beam of magnitude 100 N. The standard mechanical properties of steel material were used e.g. the value of modulus of elasticity was taken as $210\ \mathrm{GN/m^2}$. The piezoelectric is chosen to be hard lead zirconate titanate type EC-63 manufactured by EXELIS. **Table 1** shows the specifications of this material obtained from the datasheet. These specifications are required for running the finite elements model and simulating the analytical solution. The proposed cantilever beam was built using finite element model as shown in **Fig. 4**. The piezoelectric patch was considered to be perfectly bonded in the opposite side to the crack at the surface of the beam.

The crack flexibility parameter μ is identified from comparing the slopes obtained from the finite elements model and that obtained from the derived equation (Equation (29)). **Figure 5** shows the identified crack flexibility parameter μ at different crack depth and locations. The results shown in **Fig. 5** can be used with very good accuracy to a crack width b_c up to 5 mm.

In this finite elements model, the magnitude of the applied voltage on the piezoelectric patch is manipulated manually till the slope of the cracked beam becomes same as to that of the healthy beam.

5. RESULTS AND DISCUSSION

The proposed analytical model and the built finite element model were used to find the deflection profiles at different crack locations as shown in **Fig. 6**. Both models give almost the same deflection profiles. This is expected because the crack flexibility parameter μ used in the analytical model is identified from the finite elements model. Thus, the assumed relationship given in Equation (10c) is accurate and can be used at different crack depths and locations.

The patches shown in **Table 2** are designed using the proposed analytical model: Equation (35), Equation (37) and Equation (38). These patches are investigated using the derived analytical solution and the built finite elements model whereas very good agreements between results were obtained. It can be noticed in this table that increasing the factor of safety α_F has no significant effect on determining the patch length l_p or the applied voltage V_a. Furthermore, the decreasing of α_V cause to decrease the applied voltage V_a, but this decreasing causes increasing in the required patch length l_p.

The deflection profile of the repaired beam is almost the same when any of the designed patches shown in Table 2 is used. This because of that the effect of the piezoelectric patch is adjusted by changing the patch thickness t_p and length l_p with the applied voltage V_a all together as shown in **Table 2**. Therefore, **Fig. 7** includes only one curve representing the deflection profile of the repaired beam. This figure shows the deflection of the healthy beam and that of the repaired beam. It is clear that the designed piezoelectric patch repairs the beam effectively and causes a reduction of 6% in the maximum beam deflection.

Equation (35) stats that for certain beam dimensions and load condition, the piezoelectric patch thickness t_p depends on two parameters: the factor of safety α_F and the crack location l_c. **Figure 8** shows the effect of these parameters on the patch thickness t_p. This figure indicates two important things: increasing the factor of safety α_F cause to increase the patch thickness t_p and increasing the crack location l_c requires using a thicker patch.

It can be deduced from Equation (37) that for certain beam dimensions and load condition, the applied voltage V_a depends on the voltage factor α_V, crack location l_c and patch thickness t_p.

Increasing the voltage factor α_V and crack location l_c will increase the required applied voltage V_a, while increasing the patch thickness t_p will decrease the required applied voltage V_a.
Also for certain beam dimensions and load condition, Equation (38) shows that the length of the piezoelectric patch depends on four parameters: the voltage factor α_V, the crack location l_c, the crack flexibility parameter μ and piezoelectric patch thickness t_p. Actually, the crack flexibility parameter μ depends on the crack depth t_c and location l_c as shown in **Fig. 5**, while the piezoelectric patch thickness t_p depends on the factor of safety α_F and the crack location l_c as shown in **Fig. 8**.
Figure 9 shows the variation of the length of the piezoelectric patch l_p with the factor of safety α_F at different crack locations l_c. This figure indicates that increasing the factor of safety α_F causes to reduce the required length of the piezoelectric patch l_p. Also, it indicates that the patch length l_p is decreased when the crack location l_c is increased. This because of that the increasing in the crack location l_c is already encountered by increasing the patch thickness t_p as shown in **Fig. 8**.
Figure 10 shows the variation of the length of the piezoelectric patch l_p with the voltage factor α_V at different crack locations l_c. This figure shows that increasing the voltage factor α_V reduces the required length of the piezoelectric patch l_p. Also, it shows that crack location l_c has no significant effect at the patch length l_p at large voltage factors α_V.
Figure 11 shows the variation of the length of the piezoelectric patch l_p with the voltage factor α_V at different factors of safety α_F. At this situation, it can be deduced that increasing the factors of safety α_F has no advantage if the voltage factor α_V is large.
Figure 12 shows the variation of the length of the piezoelectric patch l_p with the voltage factor α_V at different crack depths t_c. In this figure, it is clear that increasing the crack depth t_c requires also increasing in the patch length l_p.

6. CONCULUSION

In this work, closed form solutions describing the deflection of a beam subjected to static load at two cases are introduced. The first case for a beam contains a crack, while the second case for a beam contains a crack and is repaired with a piezoelectric patch. Also, a novel model to design the required piezoelectric patch is introduced. Thus, the following points are remarked from the present work:

1. The factor of safety chosen by the patch designer is very important parameter which plays great rule in determining the patch thickness and hence the amount of applied voltage
2. The thickness of the piezoelectric patch is highly affected by the crack location whereas they are proportionally related
3. The additional flexibility of the beam due to crack depends mainly on the crack dimensions and location
4. The length of the piezoelectric patch depends on the thickness of the piezoelectric patch and the crack dimensions

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NOMENCLATURES

b_c	Width of the crack
b_p	Width of the piezoelectric patch
c_1 to c_4	Constants of the equation described the cracked beam deflection
d_{31}	Charge constant of the used piezoelectric material
E	Maximum applied field of the piezoelectric patch
e_{31}	Stress constant of the used piezoelectric material
F	Applied static force
F_d	Force used to design the piezoelectric patch
g_{31}	Voltage constant of the used piezoelectric material
I_b	Moment of inertia of the beam
I_p	Moment of inertia of the piezoelectric patch
I_t	Total moment of inertia of the beam with the piezoelectric patch
k	Ratio equal to $Y_t I_t / Y_b I_b$
l_b	Length of the beam
l_c	Length at which the crack appears (simply is called crack location)
l_p	Half length of the piezoelectric patch
$M_b(x)$	Total bending moment at x location
M_p	Bending moment induced by the piezoelectric patch
$M_{p_{max}}$	Maximum induced moment by the piezoelectric patch to counter the external load effect
r_1 to r_8	Constants of the equation described the repaired beam deflection
s_{11}^E and s_{12}^E	Elastic compliances of the used piezoelectric material
t_b	Thickness of the beam
t_c	Depth of the crack
t_p	Thickness of the piezoelectric patch
V_a	Applied voltage on the piezoelectric patch to achieve the beam repair

$V_{a_{max}}$	Maximum applied voltage on the piezoelectric patch to achieve the beam repair
V_d	Applied voltage corresponds to the force F_d
x and y	Used plane
Y_b	Modulus of elasticity of the beam
Y_p	Modulus of elasticity of the piezoelectric patch
y_c	Deflection of the cracked beam
y_h	Deflection of the healthy beam
Y_p	Modulus of elasticity of the piezoelectric patch
y_r	Deflection of the repaired beam
Y_t	Total Modulus of elasticity of the beam with the piezoelectric patch
α_F	Factor of safety
α_V	Voltage factor
θ_c	Slope of the cracked beam
θ_h	Slope of the healthy beam
θ_r	Slope of the repaired beam
μ	Crack flexibility parameter
σ_x	Stress in x direction

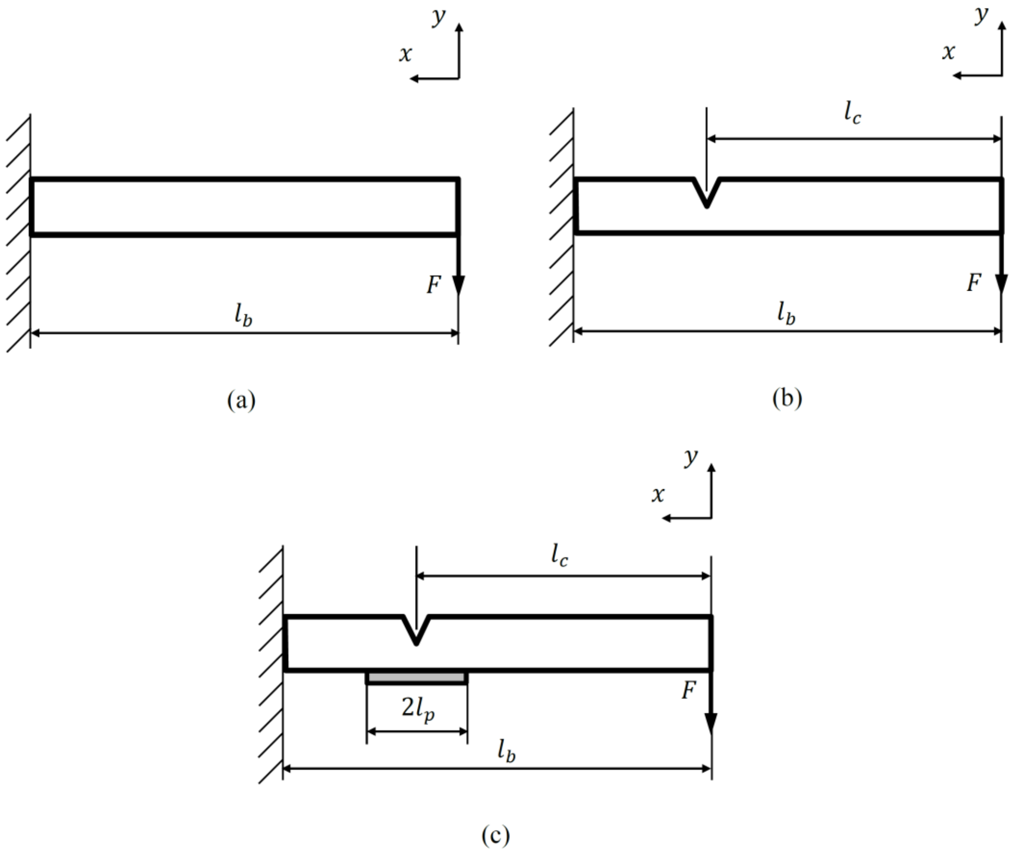

Figure 1. (a) Healthy cantilever beam, (b) Cantilever beam includes open type crack and (c) cracked cantilever beam repaired with piezoelectric patch

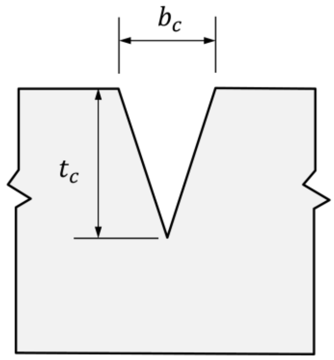

Figure 2. Part of the beam shows shape and dimensions of the crack

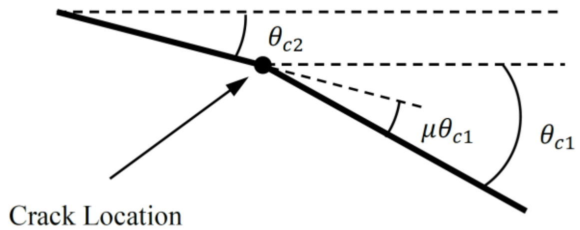

Figure 3. Slope of the beam at the crack location

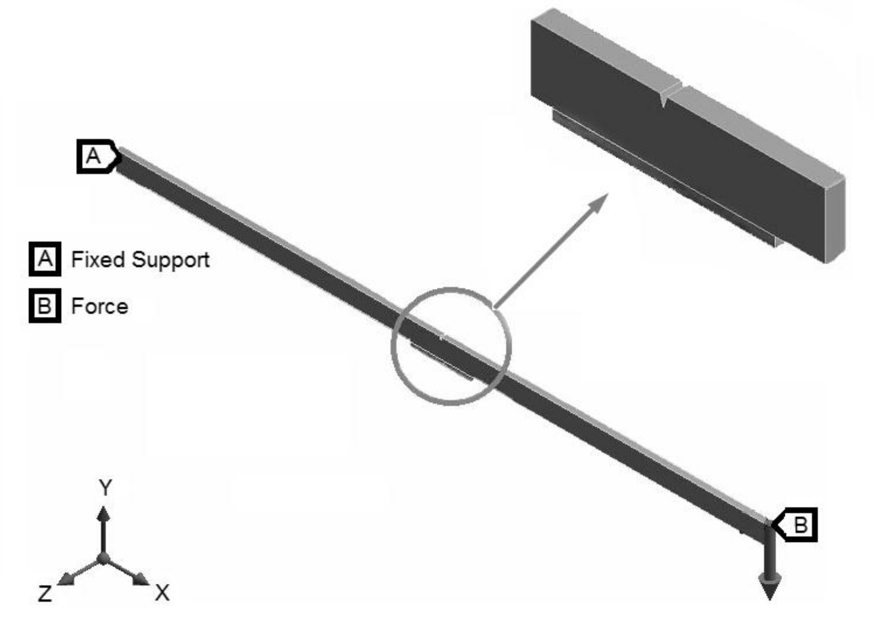

Figure 4. Implementation of the case study with finite element method

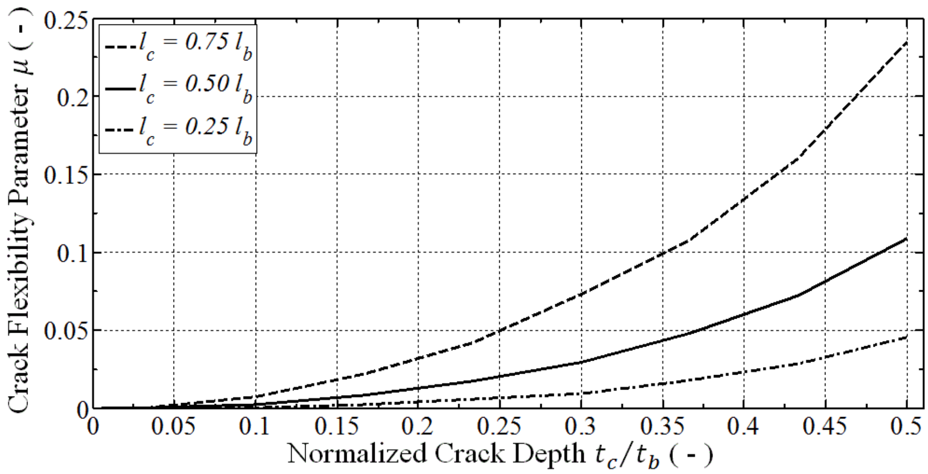

Figure 5. Identified crack flexibility parameter μ at different crack depths and locations

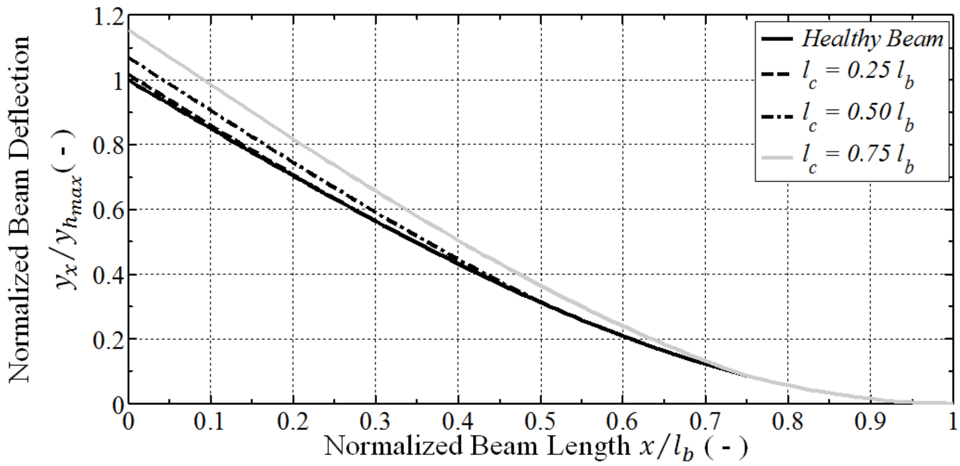

Figure 6. Deflections of the cracked beam at different crack locations (origin of x is located at the free end of the beam)

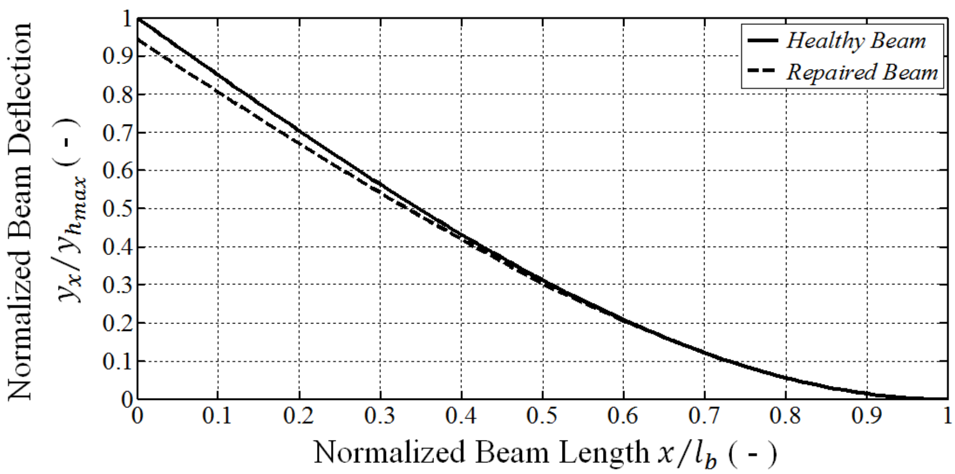

Figure 7. Deflections of the healthy and Repaired beams (origin of x is located at the free end of the beam)

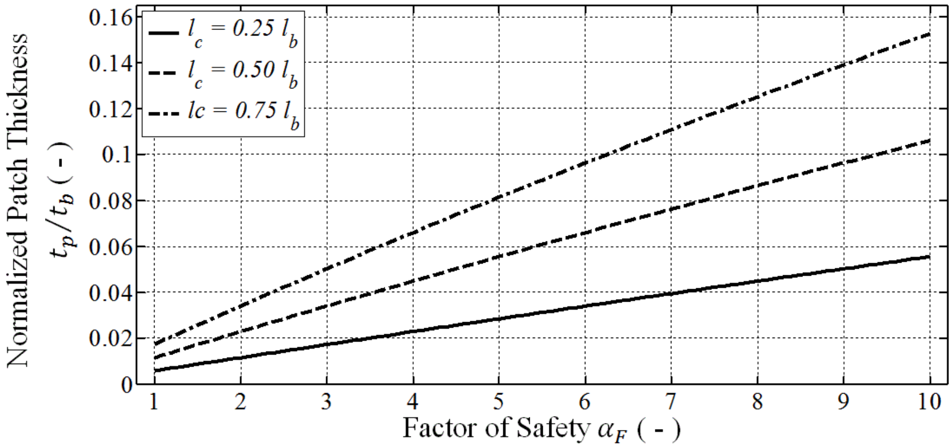

Figure 8. Variation of the piezoelectric patch thickness with the factor of safety at different crack locations.

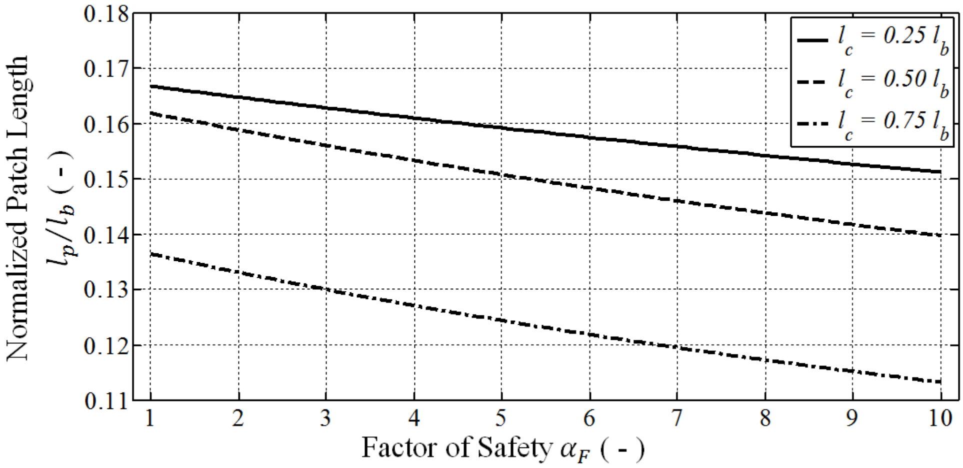

Figure 9. Variation of the piezoelectric patch length with the crack depth at different crack locations

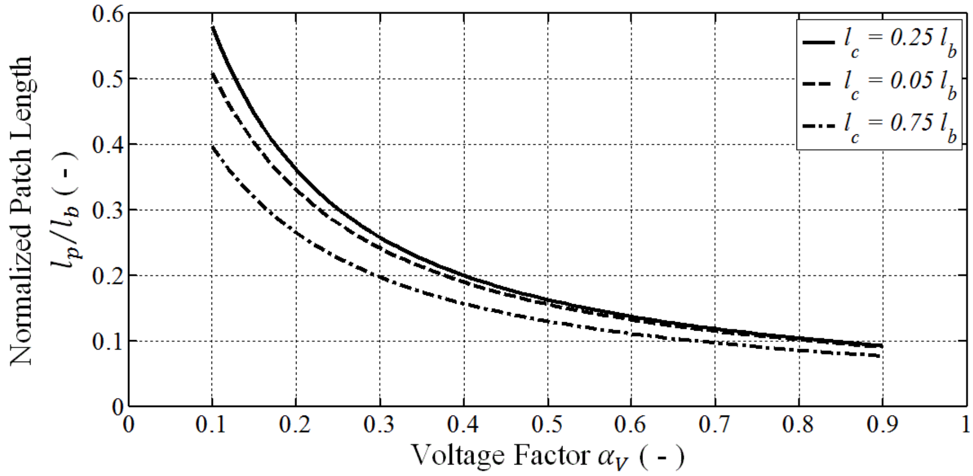

Figure 10. Variation of the length of the piezoelectric patch with the voltage factor at different crack locations

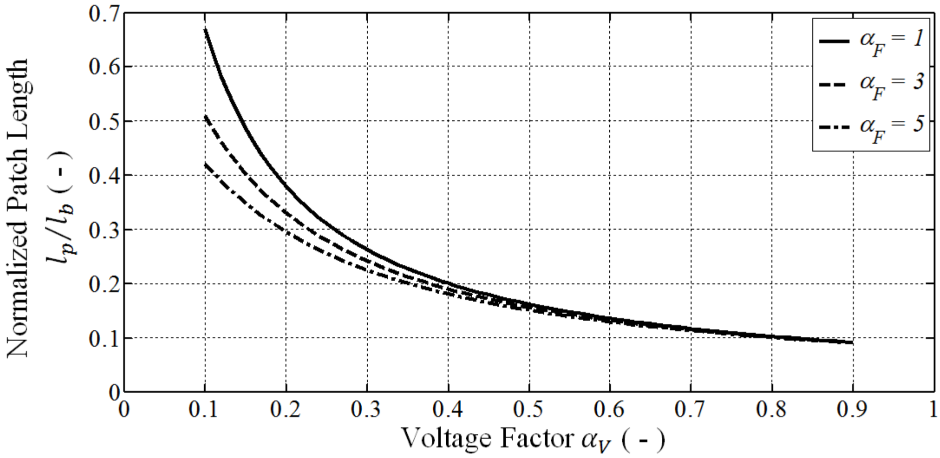

Figure 11. Variation of the length of the piezoelectric patch with the voltage factor at different factors of safety

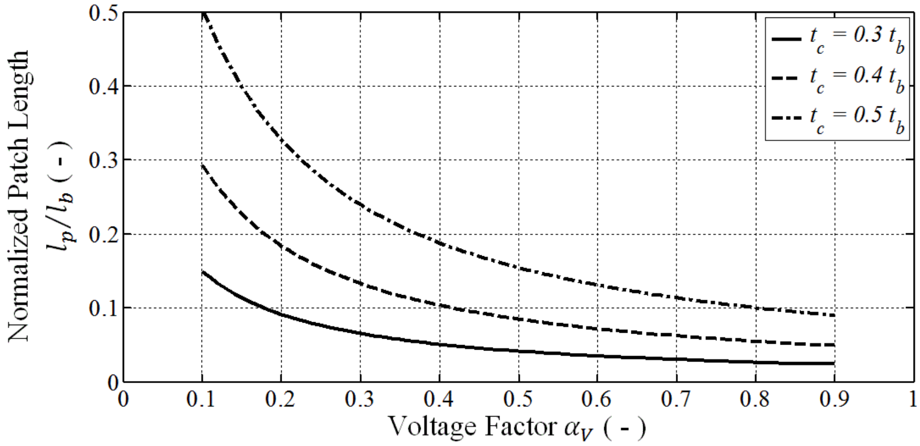

Figure 12. Variation of the length of the piezoelectric patch with the voltage factor at different depth of the crack

Table 1. Specifications of the used piezoelectric patch

Property	Magnitude	Unites
d_{31}	-120×10^{-12}	m/V
s_{11}^E	11.3×10^{-12}	m^2/N
s_{12}^E	-3.7×10^{-12}	m^2/N
E	394	V/mm

Table 2. Designed piezoelectric patches (based on the proposed model)

Designer Preferences		**Calculated Patch Dimensions (mm)**			
Factor of Safety α_F	Voltage Factor α_V	Thickness t_p	Length l_p	Width b_p	Calculated Required Voltage V_a (V)
1	1	0.35	82.40	10	208.69
3	1	1.00	82.20	10	204.16
5	1	1.70	82.00	10	200.00
5	0.75	1.70	106.30	10	150.00
5	0.5	1.70	150.70	10	100.00