IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Date of publication xxxx 00, 0000, date of current version xxxx 00, 0000.

Digital Object Identifier 10.1109/ACCESS.2017.DOI

A Novel Applications of Complex Intuitionistic Fuzzy Sets in Group Theory

MUHAMMAD GULZAR[1], M. HARIS MATEEN[2],DILSHAD ALGHAZZAWI[3],NASREEN KAUSAR[4]
[1]Department of Mathematics, Government College University Faisalabad, 38000, Pakistan (e-mail: 98kohly@gmail.com)
[2]Department of Mathematics, University of the Punjab, Lahore 54590, Pakistan (e-mail: haris.ier@pu.edu.pk)
[3]Department of Mathematics, King Abdulaziz University, Rabigh 21589, Saudi Arabia, (e-mail: dalghazzawi@kau.edu.sa)
[4]Department of Mathematics, University of Agriculture Faisalabad 38000, Pakistan, (kausar.nasreen57@gmail.com)

Corresponding author: Nasreen Kausar (e-mail:kausar.nasreen57@gmail.com).

This work was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR for technical and financial support.

ABSTRACT In this paper, we initiate the novel concept of complex intuitionistic fuzzy subgroups and prove that every complex intuitionistic fuzzy subgroup generates two intuitionistic fuzzy subgroups. We extend this ideology to define the concept of level subsets of complex intuitionistic fuzzy set and discuss its various fundamental algebraic characteristics. We also show that the level subset of the complex intuitionistic fuzzy subgroups is a subgroup. Furthermore, we investigate the homomorphic image and pre-image of complex intuitionistic fuzzy subgroup under group homomorphism. Moreover, we prove that the product of two complex intuitionistic fuzzy subgroups is also a complex intuitionistic fuzzy subgroup and develop some new results about direct product of complex intuitionistic fuzzy subgroups.

INDEX TERMS Complex intuitionistic fuzzy set, Complex intuitionistic fuzzy subgroup, Level subsets, Product of complex intuitionistic fuzzy sets, Product of complex intuitionistic fuzzy subgroups.

I. INTRODUCTION

In Mathematics, group theory is one of the most important part of algebra. Group theory has an effective framework to analyze an object that appears in symmetric form. It plays a vital role to classify the symmetries of molecules, atoms, regular polyhedral and crystal structure. This theory has become a standard and powerful tool to study universal genetic code and codon sequences behavior. Group theory is the most important branch of Mathematics that has been broadly used in algebraic geometry, theoretical physics and cryptography.

The fuzzy set theory offers a mathematical method to seize the ambiguity related to human cerebral process like intellectual and thoughtful. This theory also leads us to obtain for better way to resolve our daily life problems by proper procedure of decision making. Zahed [1] commenced the idea of fuzzy sets in 1965. Rosenfeld [2] developed a link between group theory and fuzzy sets and presented the theory of fuzzy subgroups. Atanassov [3] published his maiden well acknowledged article about intuitionistic fuzzy sets in 1986. Biswas [4] introduced the intuitionistic fuzzification algebraic structure and initiated the concept intuitionistic fuzzy subgroups. The more development about intuitionistic fuzzy subgroups may be viewed in [5], [6].

Decision-making involves the analysis and ranking of the finite set of alternatives in respective of how effective decision makers are even when all requirements are considered simultaneously. The rating values of each alternative in this process including both precise subjective information for the data and experts. But, generally, the knowledge they provide is considered to be crisp in nature. Owing to the day-by-day complexity of the system, but the real-life includes several problems with multi criteria decision-making where the detail is ambiguous, imprecise or uncertain in nature. To resolve this, the theory of fuzzy set or extended fuzzy sets such as intuitionist fuzzy set, interval-valued intuitionistic fuzzy sets [7] are the most popular, characterizing the criterion values in terms of membership degrees. Researchers found out from their applications that none of these models may account for partial ignorance of the data and variations during a given period of time executing them. In addition, uncertainty in our daily lives and the vagueness in the data arises at the same time as changes in the process (periodicity) of the data. The current theories are therefore inadequate to take account of the information and therefore there is a loss of information during process. To resolve this, Ramot *et al.* [8] initiated a

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034626, IEEE Access

complex fuzzy set in which the range of membership function is extended from real number to the complex number with the unit disc. As the complex fuzzy set takes only the degree of membership but does not weight on the non-membering portion of data entities, which also assume an equal part in the decision-making process for evaluating the system. In the real world, however, it is regularly difficult to express membership degree estimation by the exact value of a fuzzy set. In such situations, that may be easier to represent real-world vagueness and ambiguity use 2-dimensional information, rather than one. Therefore, an extension of the existing theories may be extremely useful for explaining uncertainties, since of their reluctant judgment in the complex problem of decision making. That's why, Alkouri and Salleh [9] extended the definition of complex fuzzy sets to complex intuitionistic fuzzy sets by incorporating complex degree of non-membership functions, and their basic properties were studied. Thereafter, a complex intuitionistic fuzzy set is a more general extension of the existing theories like fuzzy sets, intuitionistic fuzzy sets, complex fuzzy sets etc. Clearly the advantage of the complex intuitionistic fuzzy set is that it can contain substantially more data for the information to be expressed. Alkouri and Salleh initiated the concept of complex intuitionist fuzzy relation and developed fundamental operation of complex intuitionistic fuzzy sets in [10], [11]. Ersoy and Davvaz [12] explored the structure of intuitionistic fuzzy set in Γ-semihyper-ring. Broumi *et al.* [13] developed some new results of intuitionistic fuzzy soft sets. Davvaz and Sadrabadi [14] studied the sequences of join spaces and intuitionistic fuzzy set connected with the direct product of two hyper-groupoids with particular characterization. In 2014, Mandal and Ranadive [15] discussed rough semiprime intuitionistic fuzzy ideal and rough intuitionistic fuzzy ideal of intuitionistic fuzzy subring. The idea of intuitionistic fuzzy graph with categorical properties was investigated in [16]. Intuitionistic fuzzy set theory plays a significant role in vast range of medical fields. It works wonder in diagnosing and selecting suitable treatments in medicine. Beg and Rashid [17] extended the study of intuitionistic fuzzy theory to solve medical diagnosis decision making problems. Bakhadach *et al.* [18] explored some features of intuitionistic fuzzy prime ideal in 2016. Al-Husban and Salleh [19] presented the concept of complex fuzzy subgroups in 2016. Anandh and Giri [20] examined the notion of intuitionistic fuzzy subfield with respect to (T, S) norm in 2017. Makamba and Murali [21] discussed parabolic fuzzy subgroups in 2017. Yamin and Sharma [22] debated the intuitionistic fuzzy ring with operators in 2018. Lakshmi and Priyaa [23] discussed a novel approach on ambiguity using intuitionistic fuzzy based rule generation system that addresses the problem of deduction by including the degree of belief, disbelief and the hesitation margin in wireless sensor networks. Kellil [24] defined the product of fuzzy subrings in 2018. Solairaju and Mahalakshmi [25] investigated some characterization of hesitant intuitionistic fuzzy soft set and presented the idea hesitant intuitionistic fuzzy soft group. Alolaiyan and Abbas [26] established a stability of complete fuzzy hypergraphs and used it to find a safe and scientific way in CT scans. Latif *et al.* [27] defined the fundamental theorem of t-intuitionistic fuzzy isomorphism in 2019. Many interesting results about complex intuitionistic fuzzy graph and cellular network provider companies were presented in [28]. Gulistan *et al.* [29] expressed the notion (α, β) of complex fuzzy hyperideal. Aloaiyan *et al.* [30] described a novel framework of t-intuitionistic fuzzification of Lagrange's theorem. Garg and Rani [31-35] made remarkable effort to extend the notion of intuitionistic fuzzy sets in decision making problems. Shuaib *et al.* [36] explored some fundamental algebraic properties of η-intuitionistic fuzzy subgroups. Shuaib *et al.* [37] discussed some more characterization of ξ-intuitionistic fuzzy subgroups. The new development about Γ-interval valued fuzzification of Lagrange's theorem of Γ interval valued fuzzy subgroups was studied in [38]. Alcantud *et al.* [39] depicted the process of aggregating infinite sequences of intuitionistic fuzzy sets. Liu *et al.* [40] examined the existing technique of transformation between fuzzy sets and soft sets. Gulzar *et al.* [41] described a new class of t-intuitionistic fuzzy subgroups. In this class, they presented the t-intuitionistic fuzzy centralizer, t-intuitionistic fuzzy normalizer, t-intuitionistic fuzzy Abelian subgroups and t-intuitionistic fuzzy cyclic subgroups. Moreover, Gulzar *et al.* [42] launched the concept of Q-complex fuzzy subrings and discussed some fundamental algebraic properties of Q-complex fuzzy subrings.

The study of complex intuitionistic fuzzy subgroups is very important algebraic structure like intuitionistic fuzzy subgroups and intuitionistic fuzzy subrings. Moreover, it is powerful extension of intuitionistic fuzzy subgroups. In this way their studies help us to select the useful sample of the major structures. The phase term in the complex intuitionistic fuzzy sets can also be used to precisely epitomize the cycles appear in intuitionistic fuzzy algebraic structures. In the analysis of complex intuitionistic fuzzy algebraic theory, the structures comprise of a phase term and amplitude term. The phase term can be used to appropriately describe the cycles of the algebraic structures. In the dealing of intuitionistic fuzzy alternating groups, various cycles can be represented accurately and appropriately using the phase term if the intuitionistic fuzzy alternating groups are described in terms of complex intuitionistic fuzzy sets. This would accomplish it facile to identify various cycles and their corresponding membership functions and non-membership functions in a systematic manner. Whereas the amplitude term is equivalent to the membership function and non-membership function of classical intuitionistic fuzzy sets. The aspiration to draw this unique proficiency of the phase term existence in the complex intuitionistic fuzzy model in the study of fuzzy algebra served as the main motivation to commence and establish the theory of complex intuitionistic fuzzy subgroups in this article. Complete characterization of many group theory issues can be discussed by using this new complex intuitionistic fuzzy algebraic structure.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034626, IEEE Access

To achieve the objective 1, in this article, we have extended the intuitionistic fuzzy subgroups to the complex intuitionistic fuzzy subgroup in which membership and non-membership degrees are represented in unit disk. Also, we have discussed various fundamental results. Objective 2, we establish that every complex intuitionistic fuzzy subgroup produces intuitionistic fuzzy subgroup and Q-intuitionistic fuzzy subgroup. Moreover, we define level subset of complex intuitionistic fuzzy subset and show that level subset of complex intuitionistic fuzzy subgroup form a subgroup of group. We develop homomorphic image (inverse image) of complex intuitionistic fuzzy subgroup under classical group homomorphism. To achieve objective 3, we determine the precise signification of direct product of complex intuitionistic fuzzy subgroups and show that direct product of two complex intuitionistic fuzzy subgroups is a complex intuitionistic fuzzy subgroup. Some of the desirable characteristics of direct product of these newly define subgroups are investigated in detail work.

To do so, the rest of the paper is organized as follow: Section 2 contains some basic definitions of complex intuitionistic fuzzy sets (CIFSs) and intuitionistic fuzzy subgroups (IFSGs) and a related result which plays a significant role in our further discussion. In section 3 we introduce the novel idea of complex intuitionistic fuzzy subgroups (CIFSGs) and describe their fundamental properties. We also prove that every CIFSG generate two IFSGs. Furthermore, we define level subset of CIFSs and show that the level subsets of CIFSG are subgroups of group. The image and inverse image of CIFSG under natural group homomorphism are discussed in section 4. In section 5, we deal the notion direct product of CIFSGs and also investigate the algebraic properties of this phenomena. We also show that direct product of CIFSGs is CIFSG.

II. PRELIMINARIES

In this section, some concepts of CIFSs and CIFSGs are reviewed over the universe of discourse.

Definition 2.1: [3] An intuitionistic fuzzy set (IFS) A of universe of discourse P is of the form $A = \{(m, \eta_A(m), \hat{\eta}_A(m)) : m \in P\}$, where η_A and $\hat{\eta}_A$ give the membership and nonmembership values of m from unit interval, respectively such that $0 \leq \eta_A(m) + \hat{\eta}_A(m) \leq 1$, for any $m \in P$.

Definition 2.2: [4] An IFS A of a group G is called an IFSG of a G, if the following conditions hold:

1) $\eta_A(mn) \geq \min\{\eta_A(m), \eta_A(n)\}$,
2) $\eta_A(m^{-1}) \geq \eta_A(m)$,
3) $\hat{\eta}_A(mn) \leq \max\{\hat{\eta}_A(m), \hat{\eta}_A(n)\}$,
4) $\hat{\eta}_A(m^{-1}) \leq \hat{\eta}_A(m)$, for all $m, n \in H$.

Theorem 2.1: [4] Intersection of two IFSGs of group H is IFSG.

Definition 2.3: [9] A CIFS A of crisp non-empty set P is an object of the form $A = \{(m, \theta_A(m), \hat{\theta}_A(m)) : m \in P\}$, where the membership function $\theta_A(m) = \eta_A(m)e^{i\varphi_A(m)}$ and is defined as $\theta_A : P \to \{z \in C : |z| \leq 1\}$ and non-membership function $\hat{\theta}_A(m) = \hat{\eta}_A(m)e^{i\hat{\varphi}_A(m)}$ and is defined as $\hat{\theta}_A : P \to \{z \in C : |z| \leq 1\}$, where C is set of complex numbers. These membership and non-membership functions receive all degree of membership and non-membership from the unit disc on complex plane, respectively such that the sum of membership and non-membership values is also lies within unit disc of complex plane where $i = \sqrt{-1}$, $\eta_A(m)$, $\hat{\eta}_A(m)$, $\varphi_A(m)$, and $\hat{\varphi}_A(m)$ are real valued such that $0 \leq \eta_A(m) + \hat{\eta}_A(m) \leq 1$ and $0 \leq \varphi_A(m) + \hat{\varphi}_A(m) \leq 2\pi$. In this paper, for the sake of simplicity we shall use $\theta_A(m) = \eta_A(m)e^{i\varphi_A(m)}$, $\theta_B(m) = \eta_B(m)e^{i\varphi_B(m)}$ as membership function and $\hat{\theta}_A(m) = \hat{\eta}_A(m)e^{i\hat{\varphi}_A(m)}$, $\hat{\theta}_B(m) = \hat{\eta}_B(m)e^{i\hat{\varphi}_B(m)}$ as non-membership function of CIFSs A and B, respectively.

Definition 2.4: [10] Let A and B two CIFSs of set P. Then the intersection of CIFSs A and B is defined as:

$$A \cap B = \{(m, \theta_{A\cap B}(m), \hat{\theta}_{A\cap B}(m)) : m \in P\}.$$

Where

$$\begin{aligned}\theta_{A\cap B}(m) &= \eta_{A\cap B}(m)e^{i\varphi_{A\cap B}(m)} \\ &= \min\{\eta_A(m), \eta_B(m)\}e^{i\min\{\varphi_A(m),\varphi_B(m)\}}\end{aligned}$$

and

$$\begin{aligned}\hat{\theta}_{A\cap B}(m) &= \hat{\eta}_{A\cap B}(m)e^{i\hat{\varphi}_{A\cap B}(m)} \\ &= \max\{\hat{\eta}_A(m), \hat{\eta}_B(m)\}e^{i\max\{\hat{\varphi}_A(m),\hat{\varphi}_B(m)\}}.\end{aligned}$$

Definition 2.5: [10] Let A and B two CIFSs of set P. Then the union of CIFSs A and B is defined as:

$$A \cup B = \{(m, \theta_{A\cup B}(m), \hat{\theta}_{A\cup B}(m)) : m \in P\}.$$

Where

$$\begin{aligned}\theta_{A\cup B}(m) &= \eta_{A\cup B}(m)e^{i\varphi_{A\cup B}(m)} \\ &= \max\{\eta_A(m), \eta_B(m)\}e^{i\max\{\varphi_A(m),\varphi_B(m)\}},\end{aligned}$$

and

$$\begin{aligned}\hat{\theta}_{A\cup B}(m) &= \hat{\eta}_{A\cup B}(m)e^{i\hat{\varphi}_{A\cup B}(m)} \\ &= \min\{\hat{\eta}_A(m), \hat{\eta}_B(m)\}e^{i\min\{\hat{\varphi}_A(m),\hat{\varphi}_B(m)\}}.\end{aligned}$$

III. PROPERTIES OF COMPLEX INTUITIONISTIC FUZZY SUBGROUPS

This section is devoted the study of CIFSGs and level subsets of CIFSGs. We also found that a CIFSGs produces two IFSGs. We define level subset of CIFS and show that level subset of CIFSG form subgroup of group and explore some algebraic properties of this phenomena.

Definition 3.1: Let $A = \{< m, \psi_A(m), \hat{\psi}_A(m) >: m \in H\}$ be an intuitionistic fuzzy set (IFS). Then the π-intuitionistic fuzzy set (π-IFS) A_π is defined as $A_\pi = \{< m, \psi_{A_\pi}(m), \hat{\psi}_{A_\pi}(m) >: m \in H\}$, where the function $\psi_{A_\pi}(m) = 2\pi\psi_A(m)$ and $\hat{\psi}_{A_\pi}(m) = 2\pi\hat{\psi}_A(m)$ denote the degree of belongingness and non-belongingness of an element m of H, respectively and satisfy the following condition $0 \leq \psi_{A_\pi}(m) + \hat{\psi}_{A_\pi}(m) \leq 2\pi$.

Definition 3.2: A π-IFS A_π of group H is called π-intuitionistic fuzzy subgroup(π-IFSG) of H, $\forall\ m,\ n \in H$ if

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034626, IEEE Access

1) $\psi_{A_\pi}(mn) \geq \min\{\psi_{A_\pi} m), \psi_{A_\pi}(n)\}$,
2) $\psi_{A_\pi}(m^{-1}) \geq \psi_{A_\pi}(m)$,
3) $\hat{\psi}_{A_\pi}(mn) \leq \max\{\hat{\psi}_{A_\pi} m), \hat{\psi}_{A_\pi}(n)\}$,
4) $\hat{\psi}_{A_\pi}(m^{-1}) \leq \hat{\psi}_{A_\pi}(m)$.

Theorem 3.1: [5] A π-IFS A_π of group H is a π-IFSG of H if and only if A is IFSG of H.

Proof: The proof of this theorem is straightforward.

Definition 3.3: Let A and B be two CIFSs of H. Then

1) A CIFS A is homogeneous CIFS, if for all $m, x \in H$, we have
   (a) $\eta_A(m) \leq \eta_A(x)$ if and only if $\varphi_A(m) \leq \varphi_A(x)$,
   (b) $\hat{\eta}_A(m) \geq \hat{\eta}_A(x)$ if and only if $\hat{\varphi}_A(m) \geq \hat{\varphi}_A(x)$.
2) A CIFS A is homogeneous CIFS with B, if for all $m \in H$, we have
   (a) $\eta_A(m) \leq \eta_B(m)$ if and only if $\varphi_A(m) \leq \varphi_B(m)$,
   (b) $\hat{\eta}_A(m) \geq \hat{\eta}_B(m)$ if and only if $\hat{\varphi}_A(m) \geq \hat{\varphi}_B(m)$.

Throughout this paper we shall use CIFS as homogeneous CIFS.

Definition 3.4: A CIFS $A = \{(m, \theta_A(m), \hat{\theta}_A(m)) : m \in H\}$ of group H is called a CIFSG, $\forall\, m, n \in H$ if

1) $\theta_A(mn) \geq \min\{\theta_A(m), \theta_A(n)\}$,
2) $\theta_A(m^{-1}) \geq \theta_A(m)$,
3) $\hat{\theta}_A(mn) \leq \max\{\hat{\theta}_A(m), \hat{\theta}_A(n)\}$,
4) $\hat{\theta}_A(m^{-1}) \leq \hat{\theta}_A(m)$, for all $m, n \in H$.

Also, we define the definition CIFSG as follow:

1) $\eta_A(mn)e^{i\varphi_A(mn)} \geq \min\{\eta_A(m), \eta_A(n)\} \times e^{i\min\{\varphi_A(m),\varphi_A(n)\}}$,
2) $\eta_A(m^{-1})e^{i\varphi_A(m^{-1})} \geq \eta_A(m)e^{i\varphi_A(m)}$,
3) $\hat{\eta}_A(mn)e^{i\hat{\varphi}_A(mn)} \leq \max\{\hat{\eta}_A(m), \hat{\eta}_A(n)\}\times e^{i\min\{\hat{\varphi}_A(m),\hat{\varphi}_A(n)\}}$
4) $\hat{\eta}_A(m^{-1})e^{i\hat{\varphi}_A(m^{-1})} \leq \hat{\eta}_A(m)e^{i\hat{\varphi}_A(m)}$, for all $m, n \in H$.

Theorem 3.2: Let A be a CIFS of group H. Then A is a CIFSG of H if and only if

1) The fuzzy set $\overline{A} = \{(m, \eta_A(m), \hat{\eta}_A(m)) : m \in H, \eta_A(m),\ \hat{\eta}_A(m) \in [0, 1]\}$ is a IFSG.
2) The π-fuzzy set $\underline{A} = \{(m, \varphi_A(m), \hat{\varphi}_A(m)) : m \in H, \varphi_A(m), \hat{\varphi}_A(m) \in [0, 2\pi]\}$ is a π-IFSG.

Proof: Suppose that A is a CIFSG and $m, n \in H$. Then we have,

$$\begin{aligned} & \eta_A(mn)e^{i\varphi_A(mn)} \\ = \ & \theta_A(mn) \\ \geq \ & \min\{\theta_A(m), \theta_A(n)\} \\ = \ & \min\{\eta_A(m)e^{i\varphi_A(m)}, \eta_A(n)e^{i\varphi_A(n)}\} \\ = \ & \min\{\eta_A(m), \eta_A(n)\}e^{i\min\{\varphi_A(m),\varphi_A(n)\}}. \end{aligned}$$

As A is homogeneous, so

$$\begin{aligned} \eta_A(mn) &\geq \min\{\eta_A(m), \eta_A(n)\} \\ \text{and } \varphi_A(mn) &\geq \min\{\varphi_A(m), \varphi_A(n)\}. \end{aligned}$$

Moreover,

$$\begin{aligned} \eta_A(m^{-1})e^{i\varphi_A(m^{-1})} &= \theta_A(m^{-1}) \\ &\geq \theta_A(m) \\ &= \eta_A(m)e^{i\varphi_A(m).} \end{aligned}$$

This implies that

$$\eta_A(m^{-1}) \geq \eta_A(m) \text{ and } \varphi_A(m^{-1}) \geq \varphi_A(m).$$

Now

$$\begin{aligned} & \hat{\eta}_A(mn)e^{i\hat{\varphi}_A(mn)} \\ = \ & \hat{\theta}_A(mn) \\ \leq \ & \max\{\hat{\theta}_A(m), \hat{\theta}_A(n)\} \\ = \ & \max\{\hat{\eta}_A(m)e^{i\hat{\varphi}_A(m)}, \hat{\eta}_A(n)e^{i\hat{\varphi}_A(n)}\} \\ = \ & \max\{\hat{\eta}_A(m), \hat{\eta}_A(n)\}e^{i\max\{\hat{\varphi}_A(m),\hat{\varphi}_A(n)\}}. \end{aligned}$$

As A is homogeneous, so

$$\begin{aligned} \hat{\eta}_A(mn) &\leq \max\{\hat{\eta}_A(m), \hat{\eta}_A(n)\} \\ \text{and } \hat{\varphi}_A(mn) &\leq \max\{\hat{\varphi}_A(m), \hat{\varphi}_A(n)\}. \end{aligned}$$

Moreover,

$$\begin{aligned} \hat{\eta}_A(m^{-1})e^{i\hat{\varphi}_A(m^{-1})} &= \hat{\theta}_A(m^{-1}) \\ &\leq \hat{\theta}_A(m) \\ &= \hat{\eta}_A(m)e^{i\hat{\varphi}_A(m)} \end{aligned}$$

$$\hat{\eta}_A(m^{-1}) \leq \hat{\eta}_A(m) \text{ and } \hat{\varphi}_A(m^{-1}) \leq \hat{\varphi}_A(m).$$

Consequently, $\overline{A}$ is IFSG and $\underline{A}$ is π-IFSG.
Conversely, suppose that $\overline{A}$ and $\underline{A}$ is IFSG and π-IFSG, respectively. Then, we have

$$\begin{aligned} \eta_A(mn) &\geq \min\{\eta_A(m), \eta_A(n)\} \\ \eta_A(m^{-1}) &\geq \eta_A(m) \\ \hat{\eta}_A(mn) &\leq \max\{\hat{\eta}_A(m), \hat{\eta}_A(n)\} \\ \hat{\eta}_A(m^{-1}) &\leq \hat{\eta}_A(m) \end{aligned}$$

$$\begin{aligned} \varphi_A(mn) &\geq \min\{\varphi_A(m), \varphi_A(n)\} \\ \varphi_A(m^{-1}) &\geq \varphi_A(m) \\ \hat{\varphi}_A(mn) &\leq \max\{\hat{\varphi}_A(m), \hat{\varphi}_A(n)\} \\ \hat{\varphi}_A(m^{-1}) &\leq \hat{\varphi}_A(m). \end{aligned}$$

For this, we consider

$$\begin{aligned} & \theta_A(mn) \\ = \ & \eta_A(mn)e^{i\varphi_A(mn)} \\ \geq \ & \min\{\eta_A(m), \eta_A(n)\}e^{i\min\{\varphi_A(m),\varphi_A(n)\}} \\ = \ & \min\{\eta_A(m)e^{i\varphi_A(m)}, \eta_A(n)e^{i\varphi_A(n)}\} \\ = \ & \min\{\theta_A(m), \theta_A(n)\}. \end{aligned}$$

Also, we have

$$\begin{aligned} \theta_A(m^{-1}) &= \eta_A(m^{-1})e^{i\varphi_A(m^{-1})} \\ &\geq \eta_A(m)e^{i\varphi_A(m)} \\ &= \theta_A(m). \end{aligned}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034626, IEEE Access

Consider,

$$\begin{aligned} & \hat{\theta}_A(mn) \\ = & \ \hat{\eta}_A(mn)e^{i\hat{\varphi}_A(mn)} \\ \leq & \ \max\{\hat{\eta}_A(m), \hat{\eta}_A(n)\}e^{i\max\{\hat{\varphi}_A(m),\hat{\varphi}_A(n)\}} \\ = & \ \max\{\hat{\eta}_A(m)e^{i\hat{\varphi}_A(m)}, \hat{\eta}_A(n)e^{i\hat{\varphi}_A(n)}\} \\ = & \ \max\{\hat{\theta}_A(m), \hat{\theta}_A(n)\}. \end{aligned}$$

Further,

$$\begin{aligned} \hat{\theta}_A(m^{-1}) &= \hat{\eta}_A(m^{-1})e^{i\hat{\varphi}_A(m^{-1})} \\ &\leq \hat{\eta}_A(m)e^{i\hat{\varphi}_A(m)} \\ &= \hat{\theta}_A(m). \end{aligned}$$

Hence A is CIFSG.

The following result illustrates that intersection of two CIFSGs is CIFSG.

Theorem 3.3: Intersection of two CIFSGs of group H is CIFSG.

Proof: Let A and B be two CIFSGs of group H, for all $m, x \in H$. From Theorem 3.2, We have

1) The IFS $\{(m, \eta_A(m), \hat{\eta}_A(m)), m \in H\}$ and $\{(m, \eta_B(m), \hat{\eta}_B(m)), m \in H\}$ are IFSGs of a group H. In the view of Theorem 2.1, we get $\{(m, \eta_{A\cap B}(m), \hat{\eta}_{A\cap B}(m)), m \in H\}$ is IFSG of a group H.

2) The $\pi-$IFS $\{(m, \varphi_A(m), \hat{\varphi}_A(m)), m \in H\}$ and $\{(m, \varphi_B(m), \hat{\varphi}_B(m)), m \in H\}$ are π-IFSG of a group H. From Theorem 2.1 and Theorem 3.1, we obtain $\{(m, \varphi_{A\cap B}(m), \hat{\varphi}_{A\cap B}(m)), m \in H\}$ is $\pi-$IFSG of H.
   Consider

$$\begin{aligned} & \theta_{A\cap B}(mx) \\ = & \ \eta_{A\cap B}(mx)e^{i\varphi_{A\cap B}(mx)} \\ \geq & \ \min\{\eta_{A\cap B}(m), \eta_{A\cap B}(x)\}e^{\min\{i\varphi_{A\cap B}(m),\varphi_{A\cap B}(x)\}} \\ = & \ \min\{\eta_{A\cap B}(m)e^{i\varphi_{A\cap B}(m)}, \varphi_{A\cap B}(x)e^{i\varphi_{A\cap B}(x)}\} \\ = & \ \min\{\theta_{A\cap B}(m), \theta_{A\cap B}(x)\} \end{aligned}$$

Moreover,

$$\begin{aligned} \theta_{A\cap B}(m^{-1}) &= \eta_{A\cap B}(m^{-1})e^{i\varphi_{A\cap B}(m^{-1})} \\ &\geq \{\eta_{A\cap B}(m)\}e^{\{i\varphi_{A\cap B}(m)\}} \\ \theta_{A\cap B}(m^{-1}) &\geq \theta_{A\cap B}(m). \end{aligned}$$

Suppose that,

$$\begin{aligned} & \hat{\theta}_{A\cap B}(mx) \\ = & \ \hat{\eta}_{A\cap B}(mx)e^{i\hat{\varphi}_{A\cap B}(mx)} \\ \leq & \ \max\{\hat{\eta}_{A\cap B}(m), \hat{\eta}_{A\cap B}(x)\}e^{\max\{i\hat{\varphi}_{A\cap B}(m),\ \varphi_{A\cap B}(x)\}} \\ = & \ \max\{\hat{\eta}_{A\cap B}(m)e^{i\hat{\varphi}_{A\cap B}(m)}, \hat{\varphi}_{A\cap B}(x)e^{i\hat{\varphi}_{A\cap B}(x)}\} \\ = & \ \max\{\hat{\theta}_{A\cap B}(m), \hat{\theta}_{A\cap B}(x)\} \end{aligned}$$

Moreover,

$$\begin{aligned} \hat{\theta}_{A\cap B}(m^{-1}) &= \hat{\eta}_{A\cap B}(m^{-1})e^{i\hat{\varphi}_{A\cap B}(m^{-1})} \\ &\leq \{\hat{\eta}_{A\cap B}(m)\}e^{\{i\hat{\varphi}_{A\cap B}(m)\}} \\ \hat{\theta}_{A\cap B}(m^{-1}) &\leq \hat{\theta}_{A\cap B}(m). \end{aligned}$$

Thus conclude the proof.

Remark 3.1: The union of two CIFSGs of group H may not be CIFSG of group H.

The following example illustrates that the union of two CIFSGs of group may not be CIFSG of group.

Example 3.1: Let $H = Z$ be the group of integers under addition. Suppose A and B are two CIFSG of group H and defined as

$$\theta_A(m) = \begin{cases} 0.2e^{\frac{i\pi}{2}} & \text{if} m \in 3Z \\ 0 & \text{otherwise.} \end{cases}$$

$$\hat{\theta}_A(m) = \begin{cases} 0.1e^{\frac{i\pi}{9}} & \text{if } x \in 3Z \\ 0.5e^{\frac{i\pi}{3}} & \text{otherwise.} \end{cases}$$

$$\theta_B(m) = \begin{cases} 0.1e^{\frac{i\pi}{3}} & \text{if } m \in 2Z \\ 0.01e^{\frac{i\pi}{8}} & \text{otherwise.} \end{cases}$$

$$\hat{\theta}_B(m) = \begin{cases} 0.3e^{\frac{i\pi}{8}} & \text{if } x \in 2Z \\ 0.4e^{\frac{i\pi}{4}} & \text{otherwise.} \end{cases}$$

It can be easily verified that A and B are two CIFSGs of group H. From Definition 2.6 $A \cup B = \{(m, \theta_{A\cup B}, \hat{\theta}_{A\cup B})\}$. Therefore,

$$\theta_{A\cup B}(m) = \begin{cases} 0.2e^{\frac{i\pi}{2}} & \text{if } m \ \in 3Z \\ 0.1e^{\frac{i\pi}{3}} & \text{if } m \ \in 2Z - 3Z \\ 0.01e^{\frac{i\pi}{8}} & \text{otherwise.} \end{cases}$$

$$\hat{\theta}_{A\cup B}(m) = \begin{cases} 0.1e^{\frac{i\pi}{10}} & \text{if } m \ \in 3Z \\ 0.3e^{\frac{i\pi}{8}} & \text{if } m \ \in 2Z - 3Z \\ 0.4e^{\frac{i\pi}{4}} & \text{otherwise.} \end{cases}$$

Take $m = 9$ and $x = 4$. Then $\theta_{A\cup B}(9) = 0.2e^{\frac{i\pi}{2}}$ and $\theta_{A\cup B}(4) = 0.1e^{\frac{i\pi}{3}}$, then $\theta_{A\cup B}(9-4) = \theta_{A\cup B}(5) = 0.01e^{\frac{i\pi}{8}}$ and $\min\{\theta_A(9), \hat{\theta}_B(4)\} = \min\{0.2e^{\frac{i\pi}{2}}, 0.1e^{\frac{i\pi}{3}}\} = 0.1e^{\frac{i\pi}{3}}$. Clearly, $\theta_{A\cup B}(9-4) < \min\{\theta_A(9), \hat{\theta}_B(4)\}$. This condition does not hold. Consequently, $A \cup B$ is not CIFSG of H.

Definition 3.5: Let $A = \{(m, \theta_A(m), \hat{\theta}_A(m)) : m \in H\}$ be a CIFS of H, for any $r,\ \hat{r} \in [0,1]$, and $t,\ \hat{t} \in [0, 2\pi]$. The level subset of CIFS is defined by

$$\begin{aligned} A_{(r,t)}^{(\hat{r},\hat{t})} &= \{m \in H : \eta_A(m) \geq r, \\ \varphi_A(m) &\geq t, \hat{\eta}_A(m) \leq \hat{r}, \hat{\varphi}_A(m) \leq \hat{t}\}. \end{aligned}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034626, IEEE Access

For $\hat{t} = 0 = t$, we obtain, the level subset $A_r^{\hat{r}} = \{m \in H : \eta_A(m) \geq r, \hat{\eta}_A(m) \leq \hat{r}\}$ and for $\hat{t} = 0 = t$, we obtain, the level subset $A_t^{\hat{t}} = \{m \in H : \varphi_A(m) \geq t, \hat{\varphi}_A(m) \leq \hat{t}\}$.

Theorem 3.4: Let A be CIFSG of group H. Then $A_{(r,t)}^{(\hat{r},\hat{t})}$ is a subgroup of group H, for all $r, \hat{r} \in [0,1]$,and $t, \hat{t} \in [0, 2\pi]$, where $\eta_A(e) \geq r, \ \ \varphi_A(e) \geq t$, $\hat{\eta}_A(e) \leq \hat{r}, \ \ \hat{\varphi}_A(e) \leq \hat{t}$, also e is identity element of H.

Proof: Note that $A_{(r,t)}^{(\hat{r},\hat{t})}$ is nonempty, as $e \in A_{(r,t)}^{(\hat{r},\hat{t})}$. Let $m, n \in A_{(r,t)}^{(\hat{r},\hat{t})}$ be any two elements. Then

$$\begin{aligned} \eta_A(m) &\geq r,\ \varphi_A(m) \geq t, \hat{\eta}_A(m) \leq \hat{r},\ \hat{\varphi}_A(m) \leq t, \\ \text{and } \eta_A(n) &\geq r, \varphi_A(n) \geq t, \hat{\eta}_A(n) \leq \hat{r}, \hat{\varphi}_A(n) \leq \hat{t}. \end{aligned}$$

Now we suppose that,

$$\begin{aligned} & \eta_A(mn)e^{i\varphi_A(mn)} \\ =\ & \theta_A(mn) \\ \geq\ & \min\{\theta_A(m), \theta_A(n)\} \\ =\ & \min\{\eta_A(m)e^{i\varphi_A(m)}, \eta_A(n)e^{i\varphi_A(n)}\} \\ =\ & \min\{\eta_A(m), \eta_A(n)\}e^{i\min\{\varphi_A(m),\varphi_A(n)\}}. \end{aligned}$$

As A is homogeneous, so

$$\begin{aligned} \eta_A(mn) &\geq \min\{\eta_A(m), \eta_A(n)\} = \min\{r, r\} = r, \\ \varphi_A(mn) &\geq \min\{\varphi_A(m), \varphi_A(n)\} = \min\{t, t\} = t. \end{aligned}$$

Further,

$$\begin{aligned} & \hat{\eta}_A(mn)e^{i\hat{\varphi}_A(mn)} \\ =\ & \hat{\theta}_A(mn) \\ \leq\ & \max\{\hat{\theta}_A(m), \hat{\theta}_A(n)\} \\ =\ & \max\{\hat{\eta}_A(m)e^{i\hat{\varphi}_A(m)}, \hat{\eta}_A(n)e^{i\hat{\varphi}_A(n)}\} \\ =\ & \max\{\hat{\eta}_A(m), \hat{\eta}_A(n)\}e^{i\max\{\hat{\varphi}_A(m),\hat{\varphi}_A(n)\}}. \end{aligned}$$

By homogeneity, so

$$\begin{aligned} \hat{\eta}_A(mn) &\leq \max\{\hat{\eta}_A(m), \hat{\eta}_A(n)\} = \max\{\hat{r}, \hat{r}\} = \hat{r}, \\ \hat{\varphi}_A(mn) &\leq \max\{\hat{\varphi}_A(m), \hat{\varphi}_A(n)\} = \max\{\hat{t}, \hat{t}\} = \hat{t}. \end{aligned}$$

This implies that $mn \in A_{(r,t)}^{(\hat{r},\hat{t})}$.

Also, $\eta_A(m^{-1})e^{i\varphi_A(m^{-1})} = \theta_A(m^{-1}) \geq \theta_A(m) = \eta_A(m)e^{i\varphi_A(m)}$

$\eta_A(m^{-1}) \geq \eta_A(m) \geq r$ and $\varphi_A(m^{-1}) \geq \varphi_A(m) \geq t$ (by homogeneity)

Moreover, we have

$$\begin{aligned} \hat{\eta}_A(m^{-1})e^{i\hat{\varphi}_A(m^{-1})} &= \hat{\theta}_A(m^{-1}) \\ &\leq \hat{\theta}_A(m) \\ &= \hat{\eta}_A(m)e^{i\hat{\varphi}_A(m)} \end{aligned}$$

$\hat{\eta}_A(m^{-1}) \leq \hat{\eta}_A(m) \leq \hat{r}$ and, $\hat{\varphi}_A(m^{-1}) \leq \hat{\varphi}_A(m) \leq \hat{t}$
$\Rightarrow m^{-1} \in A_{(r,t)}^{(\hat{r},\hat{t})}$. Hence, $A_{(r,t)}^{(\hat{r},\hat{t})}$ is subgroup.

Theorem 3.5: Let $A_{(r,t)}^{(\hat{r},\hat{t})}$ be a subgroup of group H, then A is CIFSG of H, for all $r \in [0, 1]$, and $t \in [0, 2\pi]$, where $\eta_A(e) \geq r, \varphi_A(e) \geq t,\ \hat{\eta}_A(e) \leq \hat{r}, \hat{\varphi}_A(e) \leq \hat{t}$, also e is identity element of H.

Proof: Assume that $\min\{\eta_A(m), \eta_A(n)\} = r$, $\min\{\varphi_A(m), \varphi_A(n)\} = t$, and $\max\{\hat{\eta}_A(m), \hat{\eta}_A(n)\} = \hat{r}$, $\max\{\hat{\varphi}_A(m), \hat{\varphi}_A(n)\} = \hat{t}$. Then we have $\eta_A(m) \geq r$, $\hat{\eta}_A(m) \leq \hat{r}$, $\varphi_A(m) \geq t$, $\hat{\varphi}_A(m) \leq \hat{t}$ and $\eta_A(n) \geq r$, $\hat{\eta}_A(n) \leq \hat{r}$, $\varphi_A(n) \geq t$, $\hat{\varphi}_A(n) \leq \hat{t}$. This implies that $m \in A_{(r,t)}^{(\hat{r},\hat{t})}$ and $n \in A_{(r,t)}^{(\hat{r},\hat{t})}$. As $A_{(r,t)}^{(\hat{r},\hat{t})}$ is subgroup, so $mn \in A_{(r,t)}^{(\hat{r},\hat{t})}$. Then we have

$$\eta_A(mn) \geq r \text{ and } \varphi_A(mn) \geq t$$
$$\hat{\eta}_A(mn) \leq \hat{r} \text{ and } \hat{\varphi}_A(mn) \leq \hat{t}$$

Implies that

$$\begin{aligned} \eta_A(mn) &\geq \min\{\eta_A(m), \eta_A(n)\} \\ \text{and } \varphi_A(mn) &\geq \min\{\varphi_A(m), \varphi_A(n)\}. \end{aligned}$$

$$\begin{aligned} \hat{\eta}_A(mn) &\leq \max\{\hat{\eta}_A(m), \hat{\eta}_A(n)\} \\ \text{and } \hat{\varphi}_A(mn) &\leq \max\{\hat{\varphi}_A(m), \hat{\varphi}_A(n)\}. \end{aligned}$$

Thus,

$$\begin{aligned} & \theta_A(mn) \\ =\ & \eta_A(mn)e^{i\eta_A(mn)} \\ \geq\ & \min\{\eta_A(m), \eta_A(n)\}e^{i\min\{\varphi_A(m),\varphi_A(n)\}} \\ =\ & \min\{\eta_A(m)e^{i\varphi_A(m)}, \eta_A(n)e^{i\varphi_A(n)}\} \\ \theta_A(mn) \geq\ & \min\{\theta_A(m), \theta_A(n)\}. \end{aligned}$$

$$\begin{aligned} & \hat{\theta}_A(mn) \\ =\ & \hat{\eta}_A(mn)e^{i\hat{\eta}_A(mn)} \\ \leq\ & \max\{\hat{\eta}_A(m), \hat{\eta}_A(n)\}e^{i\max\{\hat{\varphi}_A(m),\hat{\varphi}_A(n)\}} \\ =\ & \max\{\hat{\eta}_A(m)e^{i\hat{\varphi}_A(m)}, \hat{\eta}_A(n)e^{i\hat{\varphi}_A(n)}\} \\ \hat{\theta}_A(mn) \leq\ & \max\{\hat{\theta}_A(m), \hat{\theta}_A(n)\} \end{aligned}$$

Further, let $m \in H$ be any element. Let $\eta_A(m) = r$, $\varphi_A(m) = t$, $\hat{\eta}_A(m) = \hat{r}$, and $\hat{\varphi}_A(m) = \hat{t}$.

Then, $\eta_A(m) \geq r$, and $\varphi_A(m) \geq t$, and $\hat{\eta}_A(m) \leq \hat{r}$, $\hat{\varphi}_A(m) \leq \hat{t}$ is true. Implies that $m \in A_{(r,t)}^{(\hat{r},\hat{t})}$. As $A_{(r,t)}^{(\hat{r},\hat{t})}$ is subgroup. So, $m^{-1} \in A_{(r,t)}^{(\hat{r},\hat{t})} \Rightarrow \eta_A(m^{-1}) \geq r$, $\varphi_A(m^{-1}) \geq t, (and), \hat{\eta}_A(m^{-1}) \leq \hat{r},\ \hat{\varphi}_A(m^{-1}) \leq \hat{t}, \Rightarrow \eta_A(m^{-1}) \geq \eta_A(m)$, $\varphi_A(m^{-1}) \geq \varphi_A(m)$, and $\hat{\eta}_A(m^{-1}) \leq \hat{\eta}_A(m)$, $\hat{\varphi}_A(m^{-1}) \leq \hat{\varphi}_A(m)$. Consider that,

$$\begin{aligned} \theta_A(m^{-1}) &= \eta_A(m^{-1})e^{i\varphi_A(m^{-1})} \\ &\geq \eta_A(m)e^{i\varphi_A(m)} \\ &= \theta_A(m). \end{aligned}$$

$$\begin{aligned} \hat{\theta}_A(m^{-1}) &= \hat{\eta}_A(m^{-1})e^{i\hat{\varphi}_A(m^{-1})} \\ &\leq \hat{\eta}_A(m)e^{i\hat{\varphi}_A(m)} \\ &= \hat{\theta}_A(m). \end{aligned}$$

Corollary 3.1: Let A be a CIFSG of H, then the level subsets $A_r^{\hat{r}}$ and $A_t^{\hat{t}}$ is a subgroups of group H, for all $\hat{r}, r \in [0, 1]$, and $\hat{t},\ t \in [0, 2\pi]$, Where $\eta_A(e) \geq r, \ \ \varphi_A(e) \geq t$ and $\hat{\eta}_A(e) \leq \hat{r}, \ \ \hat{\varphi}_A(e) \leq \hat{t}$, e is identity element of H.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034626, IEEE Access

IV. HOMOMORPHISM OF COMPLEX INTUITIONISTIC FUZZY SUBGROUPS

In this section, we define the homomorphic image and pre-image of CIFSG. We discuss some results of CIFSG under group homomorphism.

Definition 4.1: Let $f : H \to G$ be a homomorphism from group H to group G. Let A and B be two CIFSG of groups H and G, respectively, for all $x \in H$, $m \in G$. The set $f(A)(m) = \{(m, f(\theta_A)(m), f(\hat{\theta}_A)(m))\}$ is image of A, where

$$f(\theta_A)(m) = \begin{cases} sup\{\theta_A(x), & \text{if } f(x) = m\},\ f^{-1}(m) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

$$f(\hat{\theta}_A)(m) = \begin{cases} inf\{\hat{\theta}_A(x), & \text{if } f(x) = m\},\ f^{-1}(m) \neq \emptyset \\ 1, & \text{otherwise.} \end{cases}$$

. The set $f^{-1}(B)(x) = \{(x, f^{-1}(\theta_B)(x), f^{-1}(\hat{\theta}_B)(x))\}$ is called pre image of B, where

$$f^{-1}(\theta_B)(x) = (\theta_B)(f(x))$$
$$f^{-1}(\hat{\theta}_B)(x) = (\hat{\theta}_B)(f(x)), \forall x \in H$$

Theorem 4.1: [5] Let $f : H \to G$ be homomorphism from group H to group G. Let A be IFSG of H and B be IFSG of G. Then $f(A)$ is IFSG of G and $f^{-1}(B)$ is IFSG of H.

Lemma 4.1: Let $f : H \to G$ be a homomorphism from group H to group G. Let A and B be two CIFSG. Then

1) $f(\theta_A)(m) = f(\eta_A)(m)e^{if(\varphi_A)(m)}$, for all $m \in G$,
2) $f(\hat{\theta}_A)(m) = f(\hat{\eta}_A)(m)e^{if(\hat{\varphi}_A)(m)}$, for all $m \in G$,
3) $f^{-1}(\theta_B)(x) = f^{-1}(\eta_B)(x)e^{if^{-1}(\varphi_B)(x)}$, for all $x \in H$,
4) $f^{-1}(\hat{\theta}_B)(x) = f^{-1}(\hat{\eta}_B)(x)e^{if^{-1}(\hat{\varphi}_B)(x)}$, for all $x \in H$.

Proof:

1) Suppose that

$$\begin{aligned} & f(\theta_A)(m) \\ = & \max\{\theta_A(x),\ \text{if } f(x) = m\} \\ = & \max\{\eta_A(x)e^{if(\varphi_A)(x)},\ \text{if } f(x) = m\} \\ = & \max\{\eta_A(x), \text{if } f(x) = m\}e^{i\max\{(\varphi_A)(x),\ \text{if } f(x)=m\}} \\ = & f(\eta_A)(m)e^{if(\varphi_A)(m)}. \end{aligned}$$

Hence,

$$f(\theta_A)(m) = f(\eta_A)(m)e^{if(\varphi_A)(m)}.$$

2) Assume that

$$\begin{aligned} & f(\hat{\theta}_A)(m) \\ = & \min\{\hat{\theta}_A(x),\ \text{if } f(x) = m\} \\ = & \min\{\hat{\eta}_A(x)e^{if(\hat{\varphi}_A)(x)},\ \text{if } f(x) = m\} \\ = & \min\{\hat{\eta}_A(x), \text{if } f(x) = m\}e^{i\min\{(\hat{\varphi}_A)(x),\ \text{if } f(x)=m\}} \\ = & f(\hat{\eta}_A)(m)e^{if(\hat{\varphi}_A)(m)}. \end{aligned}$$

3) Hence,

$$f(\hat{\theta}_A)(m) = f(\hat{\eta}_A)(m)e^{if(\hat{\varphi}_A)(m)}.$$

4) Consider,

$$\begin{aligned} & f^{-1}(\theta_B)(x) = \theta_B(f(x)) \\ = & \eta_B(f(x))e^{i\varphi_B(f(x))} \\ = & f^{-1}(\eta_B)(x)e^{if^{-1}(\varphi_B)(x)} \end{aligned}$$

5) Consequently,

$$f^{-1}(\theta_B)(x) = f^{-1}(\eta_B)(x)e^{if^{-1}(\varphi_B)(x)}.$$

6) Consider,

$$\begin{aligned} & f^{-1}(\hat{\theta}_B)(x) = \hat{\theta}_B(f(x)) \\ = & \hat{\eta}_B(f(x))e^{i\hat{\varphi}_B(f(x))} \\ = & f^{-1}(\hat{\eta}_B)(x)e^{if^{-1}(\hat{\varphi}_B)(x)}. \end{aligned}$$

7) Consequently,

$$f^{-1}(\hat{\theta}_B)(x) = f^{-1}(\hat{\eta}_B)(x)e^{if^{-1}(\hat{\varphi}_B)(x)}.$$

The following result specifies that the homomorphic image of CIFSG is always CIFSG.

Theorem 4.2: Let $f : H \to G$ be a group homomorphism from H to G. Let A be CIFSG of H. Then $f(A)$ is CIFSG of G.

Proof: Obviously, $\overline{A} = \{(x, \eta_A(x), \hat{\eta}_A(x)) : x \in H, \eta_A(x),\ \hat{\eta}_A(x) \in [0,1]\}$ and $\underline{A} = \{(x, \varphi_A(x), \hat{\varphi}_A(x)) : x \in H, \varphi_A(m),\ \hat{\varphi}_A(m) \in [0, 2\pi]\}$ are IFSG and π-IFSG, respectively. From Theorem 3.1 and Theorem 4.1 the homomorphic image of $\overline{A} = \{(x, \eta_A(x), \hat{\eta}_A(x)) : x \in H, \eta_A(x),\ \hat{\eta}_A(x) \in [0,1]\}$ and $\underline{A} = \{(x, \varphi_A(x), \hat{\varphi}_A(x)) : x \in H, \varphi_A(x),\ \hat{\varphi}_A(x) \in [0, 2\pi]\}$ are IFSG and π-IFSG, respectively. For all $m, n \in G$. Then we have

$$\begin{aligned} f(\eta_A)((mn) & \geq \min\{f(\eta_A)(m), f(\eta_A)(n)\} \\ f(\eta_A)(m^{-1}) & \geq f(\eta_A)(m) \\ f(\hat{\eta}_A)((mn) & \leq \max\{f(\hat{\eta}_A)(m), f(\hat{\eta}_A)(n)\} \\ f(\hat{\eta}_A)(m^{-1}) & \leq f(\hat{\eta}_A)(m) \\ f(\varphi_A)(mn) & \geq \min\{f(\varphi_A)(m), f(\varphi_A)(n)\} \\ f(\varphi_A)(m^{-1}) & \geq f(\varphi_A)(m) \\ f(\hat{\varphi}_A)(mn) & \leq \max\{f(\hat{\varphi}_A)(m), f(\hat{\varphi}_A)(n)\} \\ f(\hat{\varphi}_A)(m^{-1}) & \leq f(\hat{\varphi}_A)(m) \end{aligned}$$

In the view of Lemma 4.1 [1], we know that,

$$\begin{aligned} & f(\theta_A)(mn) \\ = & f(\eta_A)(mn))e^{if(\varphi_A)(mn)},\ m, n \in G \\ \geq & \min\{f(\eta_A)(m), f(\eta_A)(n)\} \times \\ & e^{i\min\{f(\varphi_A)(m), f(\varphi_A)(n)\}} \\ \geq & \min\{f(\eta_A)(m)e^{if(\varphi_A)(m)}, \\ & f(\eta_A)(n)e^{if(\varphi_A)(n)}\} \\ = & \min\{f(\theta_A)(m), f(\theta_A)(n)\} \end{aligned}$$

Consequently, $f(\theta_A)(mn) \geq min\{f(\theta_A)(m), f(\theta_A(n))\}$

Moreover,

$$\begin{aligned} & f(\theta_A)(m^{-1}) \\ = \ & f(\eta_A)(m^{-1})e^{if(\varphi_A)(m^{-1})}, \forall m \in G \\ \geq \ & f(\eta_A)(m)e^{if(\varphi_A)(m)} \\ = \ & f(\theta_A)(m) \end{aligned}$$

Thus,

$$f(\theta_A)(m^{-1}) \geq f(\theta_A)(m).$$

From Lemma 4.1[2], we know that,

$$\begin{aligned} & f(\hat{\theta}_A)(mn) \\ = \ & f(\hat{\eta}_A)(mn))e^{if(\hat{\varphi}_A)(mn)}, \ m,n \in G \\ \leq \ & \max\{f(\hat{\eta}_A)(m), f(\hat{\eta}_A)(n)\} \times \\ & e^{i\max\{f(\hat{\varphi}_A)(m), f(\hat{\varphi}_A)(n)\}} \\ \leq \ & \max\{f(\hat{\eta}_A)(m)e^{if(\hat{\varphi}_A)(m)}, \\ & f(\hat{\eta}_A)(n)e^{if(\hat{\varphi}_A)(n)}\} \\ = \ & \max\{f(\hat{\theta}_A)(m), f(\hat{\theta}_A)(n)\} \end{aligned}$$

Consequently, $f(\hat{\theta}_A)(mn) \leq max\{f(\hat{\theta}_A)(m), f(\hat{\theta}_A(n))\}$
Moreover,

$$\begin{aligned} & f(\hat{\theta}_A)(m^{-1}) \\ = \ & f(\hat{\eta}_A)(m^{-1})e^{if(\hat{\varphi}_A)(m^{-1})}, \ \forall m \in G \\ \leq \ & f(\hat{\eta}_A)(m)e^{if(\hat{\varphi}_A)(m)} \\ = \ & f(\hat{\theta}_A)(m) \end{aligned}$$

Thus,

$$f(\hat{\theta}_A)(m^{-1}) \leq f(\hat{\theta}_A)(m).$$

This establishes the proof.

The following result indicates that the inverse homomorphic image of CIFSG is CIFSG.

Theorem 4.3: Let $f : H \to G$ be a group homomorphism from H to G. Let B be a complex fuzzy subgroup of G. Then $f^{-1}(B)$ is CIFSG of H.

Proof: Note that, $\overline{B} = \{(m, \eta_B(m), \hat{\eta}_B(m)) : m \in H, \eta_B(m), \ \hat{\eta}_B(m) \in [0,1]\}$ and $\underline{B} = \{(m, \varphi_B(m), \hat{\varphi}_B(m)) : m \in H, \varphi_B(m), \ \hat{\varphi}_B(m) \in [0, 2\pi]\}$ are IFSG and π-IFSG, respectively. Then from Theorem 3.1 and Theorem 4.1 the inverse image of $\overline{B} = \{(m, \eta_B(m), \hat{\eta}_B(m)) : m \in H, \eta_B(m), \ \hat{\eta}_B(m) \in [0,1]\}$ and $\underline{B} = \{(m, \varphi_B(m), \hat{\varphi}_B(m)) : m \in H, \varphi_B(m), \ \hat{\varphi}_B(m) \in [0, 2\pi]\}$ are IFSG and π-IFSG, respectively, for all $x, y \in H$. Then we have,

$$\begin{aligned} f^{-1}(\eta_B)(xy) & \geq \min\{f^{-1}(\eta_B)(x), f^{-1}(\eta_B)(y)\} \\ f^{-1}(\eta_B)(x^{-1}) & \geq f^{-1}(\eta_B)(x) \\ f^{-1}(\hat{\eta}_B)(xy) & \leq \max\{f^{-1}(\hat{\eta}_B)(x), f^{-1}(\hat{\eta}_B)(y)\} \\ f^{-1}(\hat{\eta}_B)(x^{-1}) & \leq f^{-1}(\hat{\eta}_B)(x) \\ f^{-1}(\varphi_B)(xy)) & \geq \min\{f^{-1}(\varphi_B)(x), f^{-1}(\varphi_B)(y)\} \\ f^{-1}(\varphi_B)(x^{-1}) & \geq f^{-1}(\varphi_B)(x) \\ f^{-1}(\hat{\varphi}_B)(xy)) & \leq \max\{f^{-1}(\hat{\varphi}_B)(x), f^{-1}(\hat{\varphi}_B)(y)\} \\ f^{-1}(\hat{\varphi}_B)(x^{-1}) & \leq f^{-1}(\hat{\varphi}_B)(x) \end{aligned}$$

In the view of Lemma 4.1[3], we know

$$\begin{aligned} & f^{-1}(\theta_B)(xy) \\ = \ & f^{-1}(\eta_B)(xy) \times e^{if^{-1}(\varphi_B)(xy)}, \forall x, y \in H \\ \geq \ & \min\{f^{-1}(\eta_B)(x), f^{-1}(\eta_B)(y)\} \times \\ & e^{i\min\{f^{-1}(\varphi_B)(x), f^{-1}(\varphi_B)(y)\}} \\ \geq \ & \min\{f^{-1}(\eta_B)(x)e^{if^{-1}(\varphi_B)(x)}, \\ & f^{-1}(\eta_B)(y)e^{if^{-1}(\varphi_B)(y)}\} \\ = \ & \min\{f^{-1}(\theta_B)(x), f^{-1}(\theta_B)(y)\} \end{aligned}$$

Therefore, $f^{-1}(\theta_B)(xy)) \geq \min\{f^{-1}(\theta_B)(x), f^{-1}(\theta_B)(y)\}$
Further,

$$\begin{aligned} & f^{-1}(\theta_B)(x^{-1}) \\ = \ & f^{-1}(\eta_B)(x^{-1}) \times e^{if^{-1}(\varphi_B)(x^{-1})}, \forall x \in H \\ \geq \ & f^{-1}(\eta_B)(x)e^{if^{-1}(\eta_B)(x)} \\ \geq \ & f^{-1}(\theta_B)(x) \end{aligned}$$

As a result, we got

$$f^{-1}(\theta_B)(x^{-1}) \geq f^{-1}(\theta_B)(x).$$

By using Lemma 4.1[4], we have,

$$\begin{aligned} & f^{-1}(\hat{\theta}_B)(xy) \\ = \ & f^{-1}(\hat{\eta}_B)(xy)e^{if^{-1}(\hat{\varphi}_B)(xy)}, \forall x, y \in H \\ \leq \ & \max\{f^{-1}(\hat{\eta}_B)(x), f^{-1}(\hat{\eta}_B)(y)\} \times \\ & e^{i\max\{f^{-1}(\hat{\varphi}_B)(x), f^{-1}(\hat{\varphi}_B)(y)\}} \\ \leq \ & \max\{f^{-1}(\hat{\eta}_B)(x)e^{if^{-1}(\hat{\varphi}_B)(x)}, \\ & f^{-1}(\hat{\eta}_B)(y)e^{if^{-1}(\hat{\varphi}_B)(y)}\} \\ = \ & \max\{f^{-1}(\hat{\theta}_B)(x), f^{-1}(\hat{\theta}_B)(y)\}. \end{aligned}$$

Therefore,
$f^{-1}(\hat{\theta}_B)(xy)) \leq \max\{f^{-1}(\hat{\theta}_B)(x), f^{-1}(\hat{\theta}_B)(y)\}$
Further,

$$\begin{aligned} & f^{-1}(\hat{\theta}_B)(x^{-1}) \\ = \ & f^{-1}(\hat{\eta}_B)(x^{-1})e^{if^{-1}(\hat{\varphi}_B)(x^{-1})}, \ \forall x \in H \\ \leq \ & f^{-1}(\hat{\eta}_B)(x)e^{if^{-1}(\hat{\eta}_B)(x)} \\ \leq \ & f^{-1}(\hat{\theta}_B)(x). \end{aligned}$$

As a result, we got

$$f^{-1}(\hat{\theta}_B)(x^{-1}) \leq f^{-1}(\hat{\theta}_B)(x).$$

This concluded the proof.

V. PROPERTIES OF THE DIRECT PRODUCT OF COMPLEX INTUITIONISTIC FUZZY SUBGROUPS

This section presents the novel framework of direct product of CIFSGs. We use the concept of CIFS to define direct product of CIFS. We prove that direct product of two CIFSGs is CIFSG and investigate their properties.

Definition 5.1: Let A and B be any two π-IFSs of sets H_1 and H_2, respectively. The Cartesian product of π-IFS A and

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034626, IEEE Access

B is defined as $(A_\pi \times B)_\pi(m,n) = \{((m,n), \psi_{A_\pi\times B_\pi}(m,n), \hat{\psi}_{A_\pi\times B_\pi}(m,n))\}, \forall\, m\in H_1,\ n\in H_2$.

Remark 5.1: Let A and B be two π-IFSGs of H_1 and H_2, respectively. Then $A_\pi \times B_\pi$ is π-IFSG of $H_1\times H_2$.

Remark 5.2: A π-IFS $A_\pi \times B_\pi$ of group $H_1\times H_2$ is a π-IFSG of $H_1\times H_2$ if and only if $A\times B$ is IFSG of $H_1\times H_2$

Definition 5.2: Let A and B two CIFSs of sets P and Q. The Cartesian product of CIFSs A and B is defined by a function

$$\begin{aligned}
A\times B &= \{((m,n), \theta_{A\times B}(m,n), \hat{\theta}_{A\times B}(m,n))\},\\
\theta_{A\times B}(m,n) &= \eta_{A\times B}(m,n)e^{i\varphi_{A\times B}(m,n)}\\
&= \min\{\eta_A(m), \eta_B(n)\}e^{i\min\{\varphi_A(m),\varphi_B(n)\}},\\
\hat{\theta}_{A\times B}(m,n) &= \hat{\eta}_{A\times B}(m,n)e^{i\hat{\varphi}_{A\times B}(m,n)}\\
&= \max\{\hat{\eta}_A(m), \hat{\eta}_B(n)\}e^{i\max\{\hat{\varphi}_A(m),\hat{\varphi}_B(n)\}}.
\end{aligned}$$

For the sake of simplicity, throughout this paper we shall use $\theta_{A\times B}(m,n) = \eta_{A\times B}(m,n)e^{i\varphi_{A\times B}(m,n)}$ and $\hat{\theta}_{A\times B}(m,n) = \hat{\eta}_{A\times B}(m,n)e^{i\hat{\varphi}_{A\times B}(m,n)}$ for the membership and non-membership functions, respectively, of cartesian product of CIFS $A\times B$.

The following theorem lead us to note that the cartesian product of two CIFSGs is CIFSG.

Theorem 5.1: Let A and B be two CIFSGs of H_1 and H_2, respectively. Then $A\times B$ is CIFSG of $H_1\times H_2$.

Proof: Let $m,x\in H_1$ and $n,y\in H_2$ be elements. Then $(m,n),(x,y)\in H_1\times H_2$. Consider

$$\begin{aligned}
&\theta_{A\times B}((m,n)(x,y))\\
=\ &\theta_{A\times B}(mx,ny)\\
=\ &\eta_{A\times B}(mx,ny)e^{i\varphi_{A\times B}(mx,ny)}\\
=\ &\min\{\eta_A(mx),\eta_B(ny)\}\times e^{i\min\{\varphi_A(mx),\varphi_B(ny)\}}\\
=\ &\min\{\eta_A(mx)e^{i\varphi_A(mx)},\eta_B(ny)e^{i\varphi_A(ny)}\}\\
=\ &\min\{\theta_A(mx),\theta_B(ny)\}\\
\geq\ &\min\{\min\{\theta_A(m),\theta_A(x)\},\min\{\theta_B(n),\theta_B(y)\}\}\\
=\ &\min\{\min\{\theta_A(m),\theta_B(n)\},\min\{\theta_A(x),\theta_B(y)\}\}\\
\geq\ &\min\{\theta_{A\times B}(m,n),\theta_{A\times B}(x,y)\}
\end{aligned}$$

$$\theta_{A\times B}((m,n)(x,y)) \geq \min\{\theta_{A\times B}(m,n),\theta_{A\times B}(x,y)\}.$$

Further,

$$\begin{aligned}
&\theta_{A\times B}(m^{-1},n^{-1})\\
=\ &\eta_{A\times B}(m^{-1},n^{-1})e^{i\varphi_{A\times B}(m^{-1},n^{-1})}\\
=\ &\min\{\eta_A(m^{-1}),\eta_B(n^{-1})\}e^{i\min\{\varphi_A(m^{-1}),\varphi_B(n^{-1})\}}\\
=\ &\min\{\eta_A(m^{-1})e^{i\varphi_A(m^{-1})},\eta_B(n^{-1})e^{i\varphi_A(n^{-1})}\}\\
=\ &\min\{\theta_A(m^{-1}),\theta_B(n^{-1})\}\\
\geq\ &\min\{\theta_A(m),\theta_B(n)\}.
\end{aligned}$$

Consequently,

$$\theta_{A\times B}(m^{-1},n^{-1}) \geq \theta_{A\times B}(m,n).$$

Assume that,

$$\begin{aligned}
&\hat{\theta}_{A\times B}((m,n)(x,y))\\
=\ &\hat{\theta}_{A\times B}(mx,ny)\\
=\ &\hat{\eta}_{A\times B}(mx,ny)e^{i\hat{\varphi}_{A\times B}(mx,ny)}\\
=\ &\max\{\hat{\eta}_A(mx),\hat{\eta}_B(ny)\}\times e^{i\max\{\hat{\varphi}_A(mx),\hat{\varphi}_B(ny)\}}\\
=\ &\max\{\hat{\eta}_A(mx)e^{i\hat{\varphi}_A(mx)},\hat{\eta}_B(ny)e^{i\hat{\varphi}_A(ny)}\}\\
=\ &\max\{\hat{\theta}_A(mx),\theta_B(ny)\}\\
\leq\ &\max\{\max\{\hat{\theta}_A(m),\hat{\theta}_A(x)\},\max\{\hat{\theta}_B(n),\hat{\theta}_B(y)\}\}\\
=\ &\max\{\max\{\hat{\theta}_A(m),\hat{\theta}_B(n)\},\max\{\hat{\theta}_A(x),\theta_B(y)\}\}\\
\leq\ &\max\{\hat{\theta}_{A\times B}(m,n),\hat{\theta}_{A\times B}(x,y)\}
\end{aligned}$$

Thus

$$\hat{\theta}_{A\times B}((m,n)(x,y)) \leq \max\{\hat{\theta}_{A\times B}(m,n),\hat{\theta}_{A\times B}(x,y)\}.$$

Further,

$$\begin{aligned}
&\hat{\theta}_{A\times B}(m^{-1},n^{-1})\\
=\ &\hat{\eta}_{A\times B}(m^{-1},n^{-1})e^{i\hat{\varphi}_{A\times B}(m^{-1},n^{-1})}\\
=\ &\max\{\eta_A(m^{-1}),\hat{\eta}_B(n^{-1})\}\ e^{i\max\{\hat{\varphi}_A(m^{-1}),\hat{\varphi}_B(n^{-1})\}}\\
=\ &\max\{\hat{\eta}_A(m^{-1})e^{i\hat{\varphi}_A(m^{-1})},\eta_B(n^{-1})e^{i\hat{\varphi}_A(n^{-1})}\}\\
=\ &\max\{\hat{\theta}_A(m^{-1}),\hat{\theta}_B(n^{-1})\}\\
\leq\ &\max\{\hat{\theta}_A(m),\hat{\theta}_B(n)\}\ .
\end{aligned}$$

Consequently,

$$\hat{\theta}_{A\times B}(m^{-1},n^{-1}) \leq \hat{\theta}_{A\times B}(m,n).$$

Thus conclude the proof.

Corollary 5.1: Let $A_1, A_2\ldots\ A_n$ be CIFSGs of $H_1, H_2,\ldots\ H_n$, respectively. Then $A_1\times A_2\times\ldots\times A_n$ is CIFSG of $H_1\times H_2\times\ldots\times H_n$.

Remark 5.3: Let A and B be two CIFSs of H_1 and H_2, respectively and $A_1\times A_2$ be CIFSG of $H_1\times H_2$. Then it is not compulsory both A_1 and A_2 should be CIFSG of H_1 and H_2, respectively.

The following example illustrates that the above stated remark.

Example 5.1: Let $H_1=\{1,-1\}$ and $H_2=\{e,a,a^2,a^3\}$ be two groups and
$H_1\times H_2 = \{(1,e),(1,a),(1,a^2),(1,a^3),$
$(-1,e),(-1,a),(-1,a^2),(-1,a^3)\}$. Then two CIFS A_1 and B_2 is defined by

$$\begin{aligned}
\theta_{A_1} &= \{(1,0.2e^{i\frac{\pi}{12}}),(-1,0.1e^{i\frac{\pi}{15}})\},\\
\hat{\theta}_{A_1} &= \{(1,.55e^{i\frac{\pi}{6}}),(-1,0.7e^{i\frac{\pi}{3}})\},\\
\theta_{A_2} &= \{(e,0.3e^{i\frac{\pi}{3}}),(a,4.5e^{i\frac{\pi}{2}}),(a^2,.33e^{i\frac{\pi}{3}}),(a^3,0.4e^{i\pi})\},\\
\hat{\theta}_{A_2} &= \{(e,0.24e^{i\frac{\pi}{8}}),(a,.2e^{i\frac{\pi}{10}}),(a^2,.1e^{i\frac{\pi}{9}}),(a^3,0.23e^{i\frac{\pi}{6}})\},
\end{aligned}$$

$$\theta_{A_1\times A_2}(x) = \begin{cases} 0.2e^{i\frac{\pi}{12}}, \\ \forall\ x\in\{(1,e),(1,a),(1,a^2),(1,a^3)\} \\ 0.1e^{i\frac{\pi}{15}}, \\ \forall\ x\in\{(-1,e),(-1,a),(-1,a^2),(-1,a^3)\} \end{cases}$$

$$\hat{\theta}_{A_1\times A_2}(x)=\begin{cases}0.1e^{i\frac{\pi}{6}},\\ \forall\, x\in\{(1,e),(1,a),(1,a^2),(1,a^3)\}\\ 0.7e^{i\frac{\pi}{3}},\\ \forall\, x\in\{(-1,e),(-1,a),(-1,a^2),(-1,a^3)\}\end{cases}$$

Here, $A_1\times A_1$ is CIFSG of $H_1\times H_2$ and A_1 is CIFSG of H_1. But A_2 is not a CIFSG of H_2 because $A_2^{(0.23,\frac{\pi}{6})}=\{a,a^3\}$ is not a subgroup.

Remark 5.4: Let $A\times B$ be a CIFSG of group $H_1\times H_2$. Then $\eta_{A\times B}(e,e^{'})\geq\eta_{A\times B}(m,n)$, $\varphi_{A\times B}(e,e^{'})\geq\varphi_{A\times B}(m,n)$, $\hat{\eta}_{A\times B}(e,e^{'})\leq\hat{\eta}_{A\times B}(m,n)$ and $\hat{\varphi}_{A\times B}(e,e^{'})\leq\hat{\varphi}_{A\times B}(m,n), \forall m\in K_1,\ n\in K_2$. Where e and $e^{'}$ are identities of H_1 and H_2, respectively.

Theorem 5.2: Let A and B be two CFS of groups H_1 and H_2 =, respectively. If $A\times B$ is a CIFSG of $H_1\times H_2$, then at least one of the following assertions must be hold

1) $\eta_A(e)\leq\eta_B(n)$, $\varphi_A(e)\leq\varphi_B(n)$ and $\hat{\eta}_A(e)\geq\hat{\eta}_B(n)$, $\hat{\varphi}_A(e)\geq\hat{\varphi}_B(n)$ $\forall n\in H_2$.
2) $\eta_B(e^{'})\leq\eta_A(m)$, $\varphi_B(e^{'})\leq\varphi_A(m)$, $\hat{\eta}_B(e^{'})\geq\eta_A(m)$, $\hat{\varphi}_B(e^{'})\geq\hat{\varphi}_A(m)$ $\forall m\in H_1$.

Where e and $e^{'}$ are identities of H_1 and H_2, respectively.

Proof: Let $A\times B$ be a CIFSG of $H_1\times H_2$. On contrary, suppose that the statements [1] and [2] do not hold. Then there exist $m\in H_1$ and $n\in H_2$ such that

1) $\eta_A(e)\leq\eta_B(n)$,$\varphi_A(e)\leq\varphi_B(n)$ and $\hat{\eta}_A(e)\geq\hat{\eta}_B(n)$,$\hat{\varphi}_A(e)\geq\hat{\varphi}_B(n)$ $\forall\, n\in H_2$.
2) $\eta_B(e^{'})\leq\eta_A(m)$,$\varphi_B(e^{'})\leq\varphi_A(m)$ $\hat{\eta}_B(e^{'})\geq\eta_A(m)$,$\hat{\varphi}_B(e^{'})\geq\hat{\varphi}_A(m)$ $\forall\, m\in H_1$.

Consider,

$$\theta_{A\times B}(m,n)=\min\{\eta_A(m),\eta_B(n)\}\ e^{i\min\{\varphi_A(m),\varphi_B(n)\}}$$

$$\geq\min\{\eta_A(e),\eta_B(e^{'})\}e^{imin\{\varphi_A(e),\varphi_B(e^{'})\}}=\theta_{A\times B}(e,e^{'}).$$

and

$$\hat{\theta}_{A\times B}(m,n)=\max\{\hat{\eta}_A(m),\hat{\eta}_B(n)\}\ e^{i\max\{\hat{\varphi}_A(m),\hat{\varphi}_B(n)\}}$$

$$\leq\max\{\hat{\eta}_A(e),\hat{\eta}_B(e^{'})\}e^{imax\{\hat{\varphi}_A(e),\hat{\varphi}_B(e^{'})\}}=\hat{\theta}_{A\times B}(e,e^{'}).$$

But $A\times B$ is CIFSG. Hence, at least one of the following statements must be hold.

1) $\eta_A(e)\leq\eta_B(n)$,$\varphi_A(e)\leq\varphi_B(n)$ and $\hat{\eta}_A(e)\geq\hat{\eta}_B(n)$,$\hat{\varphi}_A(e)\geq\hat{\varphi}_B(n)$ $\forall n\in H_2$
2) $\eta_B(e^{'})\leq\eta_A(m)$,$\varphi_B(e^{'})\leq\varphi_A(m)$ $\hat{\eta}_B(e^{'})\geq\eta_A(m)$,$\hat{\varphi}_B(e^{'})\geq\hat{\varphi}_A(m)$ $\forall m\in H_1$

Theorem 5.3: Let A and B CIFSs of H_1 and H_2 and $\eta_B(e^{'})\geq\eta_A(m)$,$\varphi_B(e^{'})\geq\varphi_A(m)$, $\hat{\eta}_B(e^{'})\leq\hat{\eta}_A(m)$,$\hat{\varphi}_B(e^{'})\leq\hat{\varphi}_A(m)$ $\forall\, m\in H_1$, $e^{'}$ is identity of H_2. If $A\times B$ is CIFSG of $H_1\times H_2$, then A is CIFSG of H_1.

Proof: Let $(m,e^{'}),(x,e^{'})$ be elements of $H_1\times H_2$. By given condition $\eta_B(e^{'})\geq\eta_A(m)$ and $\varphi_B(e^{'})\geq\varphi_A(m)$, for all $m,x\in H_1$ and $e^{'}\in H_2$. Consider,

$$\begin{aligned}&\theta_A(mx)\\ =\ &\eta_A(mx)e^{i\varphi_A(mx)}\\ =\ &\min\{\eta_A(mx)e^{i\varphi_A(mx)},\eta_B(e^{'}e^{'})e^{i\varphi_B(e^{'}e^{'})}\}\\ =\ &\eta_{A\times B}((m,e^{'})(x,e^{'}))e^{i\varphi_{A\times B}((m,e^{'})(x,e^{'}))}\\ \geq\ &\min\{\eta_{A\times B}(m,e^{'}),\eta_{A\times B}(x,e^{'})\}\times\\ &e^{i\min\{\varphi_{A\times B}(m,e^{'}),\varphi_{A\times B}(x,e^{'})\}}\\ =\ &\min\{\min\{\eta_A(m),\eta_B(e^{'})\},\min\{\eta_A(x),\eta_B(e^{'})\}\}\\ &e^{i\min\{\min\{\hat{\varphi}_A(m),\varphi_B(e^{'})\}\},\min\{\hat{\varphi}_A(x),\varphi_B(e^{'})\}\ \}}\\ =\ &\min\{\theta_A(m),\theta_A(x)\}\end{aligned}$$

Thus, $\theta_A(mx)\geq\min\{\theta_A(m),\theta_A(x)\}$
Further,

$$\begin{aligned}&\hat{\theta}_A(mx)\\ =\ &\hat{\eta}_A(mx)e^{i\hat{\varphi}_A(mx}\\ =\ &\{\max\{\hat{\eta}_A(mx)e^{i\hat{\varphi}_A(mx)},\hat{\eta}_B(e^{'}e^{'})e^{i\hat{\varphi}_B(e^{'}e^{'})}\}\}\\ =\ &\{\hat{\eta}_{A\times B}((m,e^{'})(x,e^{'}))\}e^{i\{\hat{\varphi}_{A\times B}((m,e^{'})(x,e^{'}))\}}\\ \leq\ &\max\{\hat{\eta}_{A\times B}(m,e^{'}),\hat{\eta}_{A\times B}(x,e^{'})\}\times\\ &e^{i\max\{\hat{\varphi}_{A\times B}(m,e^{'}),\varphi_{A\times B}(x,e^{'})\}}\\ =\ &\max\{\max\{\hat{\eta}_A(m),\hat{\eta}_B(e^{'})\},\max\{\hat{\eta}_A(x),\hat{\eta}_B(e^{'})\}\}\times\\ &e^{i\max\{\max\{\hat{\varphi}_A(m),\varphi_B(e^{'})\}\},\max\{\hat{\varphi}_A(x),\varphi_B(e^{'})\}\ \}}\\ =\ &\max\{\hat{\theta}_A(m),\hat{\theta}_A(x)\}\end{aligned}$$

Thus, $\hat{\theta}_A(mx)\leq\max\{\hat{\theta}_A(m),\hat{\theta}_A(x)\}$.
Also, assume that

$$\begin{aligned}&\theta_A(m^{-1})\\ =\ &\eta_A(m^{-1})e^{i\varphi_A(m^{-1})}\\ =\ &\min\{\eta_A(m^{-1})e^{i\varphi_A(m^{-1})},\eta_B((e^{'})^{-1})e^{i\varphi_B((e^{'})^{-1})}\}\\ =\ &\min\{\eta_A(m^{-1}),\eta_B((e^{'})^{-1})\}e^{i\min\{\varphi_A(m^{-1}),\varphi_B((e^{'})^{-1})\}}\\ =\ &\eta_{A\times B}(m^{-1},(e^{'})^{-1})e^{i\varphi_{A\times B}(m^{-1},(e^{'})^{-1})}\\ \geq\ &\eta_{A\times B}(m,e^{'})e^{i\varphi_{A\times B}(m,e^{'})}\\ =\ &\min\{\eta_A(m),\eta_B(e^{'})\}e^{i\min\{\varphi_A(m),\varphi_B(e^{'})\}}\\ =\ &\min\{\eta_A(m),\eta_A(m)\}e^{i\min\{\varphi_A(m),\varphi_A(m)\}}\\ =\ &\eta_A(m)e^{i\varphi_A(m)}=\theta_A(m)\end{aligned}$$

Consequently, we have

$$\theta_A(m^{-1})\geq\theta_A(m).$$

Moreover,

$$
\begin{aligned}
& \hat{\theta}_A(m^{-1}) \\
= \; & \hat{\eta}_A(m^{-1})e^{i\hat{\varphi}_A(m^{-1})} \\
= \; & \max\{\hat{\eta}_A(m^{-1})e^{i\hat{\varphi}_A(m^{-1})}, \hat{\eta}_B((e')^{-1})e^{i\hat{\varphi}_B((e')^{-1})}\} \\
= \; & \max\{\hat{\eta}_A(m^{-1}), \hat{\eta}_B((e')^{-1})\}e^{i\max\{\hat{\varphi}_A(m^{-1}),\hat{\varphi}_B((e')^{-1})\}} \\
= \; & \hat{\hat{\eta}}_{A\times B}(m^{-1},(e')^{-1})e^{i\hat{\varphi}_{A\times B}(m^{-1},(e')^{-1})} \\
\leq \; & \hat{\hat{\eta}}_{A\times B}(m,e')e^{i\hat{\varphi}_{A\times B}(m,e')} \\
= \; & \max\{\hat{\eta}_A(m), \hat{\eta}_B(e')\}e^{i\max\{\hat{\varphi}_A(m),\hat{\varphi}_B(e')\}} \\
= \; & \max\{\hat{\eta}_A(m), \hat{\eta}_A(m)\}e^{i\max\{\hat{\varphi}_A(m),\hat{\varphi}_A(m)\}} \\
= \; & \hat{\eta}_A(m)e^{i\hat{\varphi}_A(m)} = \hat{\theta}_A(m)
\end{aligned}
$$

As a result, we have

$$\hat{\theta}_A(m^{-1}) \leq \hat{\theta}_A(m). \; Hence, \; proved \; our \; claim.$$

Theorem 5.4: Let A and B two CIFSs of H_1 and H_2 such that $\eta_A(e) \geq \eta_B(n)$ and $\varphi_A(e) \geq \varphi_B(n)$, $\forall n \in H_2$ and e is identity of H_1. If $A \times B$ is CIFSG of $H_1 \times H_2$, then B is a CIFSG of H_2.

*Proof:*The proof of this theorem similar to Theorem 5.7.

Corollary 5.2: Let A and B two CIFSs of H_1 and H_2, respectively. If $A \times B$ is CIFSG of $H_1 \times H_2$, then A is a CIFSG of H_1 or B is a CIFSG of H_2.

VI. CONCLUSION

In this article, we have defined the notion of π-IFSG, CIFSG, level subset and intersection of CIFS. We have proved that every CIFSG generate two IFSGs and have investigated key feature of this fact. We have demonstrated that level subset of CIFSG form subgroup of group and have discussed some algebraic properties of level subset. Moreover, we have proved that homomorphic image (pre image) of CIFSG is CIFSG. We have also defined direct product of CIFS and have examined that the direct product of two CIFSGs is CIFSG and have developed many important results about this phenomena. In the future, we will extend the commenced approach to the different algebraic structure and then apply to the various field of group theory and ring theory.

REFERENCES

[1] L.A. Zadeh, "Fuzzy sets," Information and control, vol. 8, no. 3, pp. 338 - 353, 1965.
[2] A. Rosenfeld, "Fuzzy groups," Journal of Mathematical Analysis and Applications , vol. 35 no. 3, pp. 512-517, Sep.1971.
[3] K. T. Atanassov, "Intuitionistic fuzzy sets," Fuzzy Sets and Systems, vol. 20, 87-96, Aug. 1986.
[4] R. Biswas, "Intuitionistic fuzzy subgroups," Mathematical Forum, vol. 10, pp. 37-46, 1989.
[5] P.K Sharma, "Homomorphism of Intuitionistic fuzzy groups," International Mathematical Forum, vol. 6, no. 64, pp. 3169-3178, 2011.
[6] P. K. Sharma, "Intuitionistic fuzzy groups," International Journal of Data Warehousing and Mining, vol. 1, no. 1, pp. 86-94, 2011.
[7] K. Atanassov, G. Gargov, "Interval-valued intuitionistic fuzzy sets," Fuzzy Sets and System, vol. 31, pp. 343–349, 1989.
[8] D. Ramot, R. Milo, M. Friedman, A. Kandel, "Complex fuzzy sets," IEEE Transaction on Fuzzy Systems, vol. 10, no. 2, 450-461, Apr. 2002.
[9] A. Alkouri and A.R. Salleh, "Complex Atanassov's intuitionistic fuzzy sets," International Conference on Fundamental and Applied Sciences, AIP Conference Proceedings, vol. 1482, no. 1, pp. 464-470, 2012.
[10] A. Alkouri, A.R. Salleh, "Some operations on Complex Atanassov's intuitionistic fuzzy sets," AIP Conf. Proc, 1571, pp. 987-993, Dec. 2013.
[11] A. Alkouri, A.R. Salleh, "Complex Atanassov's intuitionistic fuzzy relation," Abstract and Applied Analysis, Article ID 287382, pp 18, Jun. 2013.
[12] Ersoy and Davvaz, "Structure of intuitionistic fuzzy sets in Γ-semihyperrings," Abstract and Applied Analysis, pp. 9, Feb. 2013, ID. 560698.
[13] S. Broumi, F. Smarandache, M. Dhar and P. Majumdar, "New results of intuitionistic fuzzy soft set," International Journal of Information Engineering and Electronic Business, vol. 2, pp. 47-52, Apr. 2014.
[14] B. Davvaz, H. Sadrabadi, "On Atanassov's intuitionistic fuzzy grade of the direct product of two hypergroupoids," Kuwait Journal of Science, vol. 41, no. 3, pp.47-61, 2014.
[15] P. Mandal, A. S. Ranadive, "The rough intuitionistic fuzzy ideals of intuitionistic fuzzy subrings in a commutative ring," Fuzzy Information and Engineering, vol. 6, no. 3, pp. 279-297, Sep. 2014.
[16] H. Rashmanlou, R. A. Borzooei, "Intuitionistic fuzzy graphs with categorical properties," Fuzzy Information and Engineering, vol. 7, no. 3 pp. 317-334, Sep. 2015.
[17] I. Beg, T. Rashid, "A system for medical diagnosis based on intuitionistic fuzzy relation," Notes on Intuitionistic Fuzzy Sets, vol. 21, no. 3 pp. 80-89, 2015.
[18] I. Bakhadach, S. Melliani, M. Oukessou, and L. S. Chadli, "Intuitionistic fuzzy ideal and intuitionistic fuzzy prime ideal in a ring," Notes on Intuitionistic Fuzzy Sets, vol. 22, no. 2 pp. 59-63, 2016.
[19] A. Al-Husban, A.R Salleh. "Complex fuzzy group based on complex fuzzy space," Global Journal of Pure and Applied Mathematics, vol. 12, no. 2 pp. 1433-50, 2016.
[20] B. Anandh, R. Giri, "Intuitionistic fuzzy subfields of a field with respect to (T, S)-norm," Advanced in Fuzzy Mathematics, vol. 12, no. 4, pp. 1007-1015, 2017.
[21] B.B. Makamba, Murali, "A class of fuzzy subgroups of finite reflection groups," Journal of Intelligent and Fuzzy Systems, vol. 33, no. 2, pp. 979-983, 2017.
[22] M. Yamin and P. K. Sharma, "Intuitionistic fuzzy ring with operators," International Journal of Mathematics and Computer Science, vol 6, no, 2, pp 1860-1866, 2018.
[23] N. G. Lakshmi, D. S. Priya, "Evolving intuitionistic fuzzy rule based security attack detection in wireless sensor," International Journal of Pure and Applied Mathematics vol. 119, no. 10 , pp. 349-365, 2018.
[24] R. Kellil, "Sum and product of Fuzzy ideals of a ring," International Journal of Mathematics and Computer Science, 13, pp.187-205, 2018.
[25] A. Solairaju, S. Mahalakshmi, "Hesitant intuitionistic fuzzy soft groups," International Journal of Pure and Applied Mathematics, vol. 118, no. 10, pp. 223-232, 2018.
[26] H. Aloaiyan, M. Abbas, "An application of stability of fuzzy hypergraphs in medical field," Journal of Computational and theoretical Nanoscience, vol. 15, no.4, pp. 1247-1254, Apr 2018.
[27] L. Latif, U. Shuaib, H. Alolaiyan and A. Razaq, "On Fundamental Theorems of t-Intuitionistic Fuzzy Isomorphism of t-Intuitionistic Fuzzy Subgroups," IEEE Access, vol. 6, pp. 74547-74556, Nov. 2018.

[28] N. Yaqoob, M. Gulistan, S. Kadry, H. A. Wahab, "Complex intuitionistic fuzzy graph with Application in cellular network provider companies," Mathematics, vol. 7, no. 1, pp. 35, Jan. 2019.

[29] M. Gulistan, N. Yaqoob, S. Nawaz, M. Azhar, "A study of (α, β)-complex fuzzy hyperideals in non-associative hyperrings," Journal of Intelligent and Fuzzy Systems, vol. 36, no. 6, pp. 6025-6036, 2019.

[30] H. Aloaiyan, U. Shuaib, Laila Latif, A. Razaq, " t-Intuitionistic fuzzification of Lagrange's theorem of t-intuitionistic fuzzy subgroup," IEEE Access , vol. 7, pp. 158419-158426, Oct. 2019.

[31] H. Garg, D. Rani, "A robust correlation coefficient measure of complex intuitionistic fuzzy sets and their application in decision-making," Applied Intelligence, vol. 49, no.2, pp. 496 - 512, Feb. 2019.

[32] H. Garg, D. Rani, "Generalized geometric aggregation operators based on T-norm operations for complex Intuitionistic fuzzy sets and their application in decision-making," Cognitive Computation, vol. 12, pp. 679–698,Dec. 2020.

[33] H. Garg, D. Rani, "New generalised Bonferroni mean aggregation operators of complex intuitionistic fuzzy information based on Archimedean t-norm and t-conorm," Journal of Experimental and Theoretical Artificial Intelligence, vol. 32, no. 1, pp. 81-109, 2019.

[34] H. Garg, D. Rani, "Some results on information measures for complex intuitionistic fuzzy sets," International Journal of Intelligent Systems, vol. 34, no. 10, pp. 2319-2363, Oct. 2019.

[35] H. Garg, D. Rani, "Novel aggregation operators and ranking method for complex intuitionistic fuzzy sets and their application to decision-making process," Artificial Intelligence Review, vol. 53, pp. 3595-3620, Oct. 2020.

[36] U. Shuaib, H. Alolaiyan, A. Razaq , S. Dilbar, F. Tahir, "On some algebraic aspects of η-intuitionistic fuzzy subgroups," Journal of Taibah University for Science, vol. 14, no.1, pp. 463-9, 2020.

[37] U. Shuaib, M. Amin, S. Dilbar, F. Tahir, "On Algebraic Attributes of ξ-Intuitionistic Fuzzy Subgroups," International Journal of Mathematics and Computer Science, vol. 15, no.1, pp. 395-411, 2019.

[38] U. Shuaib, A. Razaq, H. Alolaiyan, M. S. Saif and A. Rafiq, "On Γ-Interval Valued Fuzzification of Lagrange's Theorem of Γ-Interval Valued Fuzzy Subgroups," *IEEE Access*, vol. 8, pp. 96261- 96270, May. 2020.

[39] J. C. R. Alcantud, A. Z. Khameneh, A. Kilicman, "Aggregation of infinite chains of intuitionistic fuzzy sets and their application to choices with temporal intuitionistic fuzzy information," *Information Sciences*, vol. 514, pp. 106-117, Apr. 2020.

[40] Z. Liu, J. C. R. Alcantud, K. Qin, L. Xiong, "The Soft sets and fuzzy sets based neural networks and application," *IEEE Access*, vol. 8, pp. 41615-41625, Feb. 2020.

[41] M. Gulzar, D. Alghazzawi, M. H. Mateen and N. Kausar, "A Certain Class of t-Intuitionistic Fuzzy Subgroups," *IEEE Access*, vol. 8, pp. 163260-163268, Aug. 2020.

[42] M. Gulzar, D. Alghazzawi, M. Haris Mateen, M. Premkumar, "On some characterization of Q -complex fuzzy sub-rings." *Journal of Mathematics and Computer Science*, vol 22, no.3, pp. 295-305, 2021.

MUHAMMAD GULZAR was born in Gojra Pakistan. He received the M.Phil degree in field of fuzzy group theory from Government College University, Faisalabad, Pakistan, in 2018. His area of research is applications of fuzzy sets in group theory, ring theory, and various intuitionistic fuzzy algebraic varieties.

MR. HARIS MATEEN received his degrees of M.Phil from University of the Punjab, Pakistan. Presently, he is pursuing his PhD degree from Mathematics Department, University of the Punjab . His area of research is Discrete Mathematics, fuzzy algebraic structure.

PLACE PHOTO HERE

DILSHAD ALGHAZZAWI received the Ph.D. degree in Mathematical Sciences from Artois University, France, Lens, in 2018. From 2012 to 2019, she was a Lecturer, King Abdulaziz University, Rabigh Campus. Since 2019, she has been an Assistant Professor in Department of Mathematical Sciences, King Abdul Aziz University Rabigh Campus. Her research interests include theory of rings, non-commutative rings and their Applications.

PLACE PHOTO HERE

NASREEN KAUSAR received the Ph.D. degree in Mathematics from Quaid-i-Azam university Islamabad, Pakistan. She is currently a Assistant Professor of Mathematics in University of Agriculture Faisalabad, Pakistan. Her research interests are associative and commutative (non-associative and non-commutative algebraic structure and its applications on fuzzy structures.

• • •