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A Novel Approach to Estimate Solution of Volterra Integral Equations

Research Article

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Abstract. Approximating numerical solution of integral equations is considered to be very important as such equations have number of applications in various fields. In this paper we introduce novel approach to estimate numerical solutions of Volterra integral equations. In the proposed technique, Chebyshev polynomial is employed to approximate solution for an unknown function in the Volterra integral equation. It is observed that the proposed technique is highly suitable for such problems and have very encouraging results. We compare accuracy and efficiency of the method with existing techniques.

Keywords. Volterra integral equations; Chebyshev Polynomials; Numerical solution

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1. Introduction

Integral equations arise in many scientific and engineering problems. A large class of initial and boundary value problems can be converted to Volterra integral equations. Linear and nonlinear Volterra integral equations contribute in various scientific approaches such as spread of epidemics, population dynamics and semi-conductor devices.

The Volterra integral equations are divided into two groups referred to as the first and the second kind. Volterra integral equation named after Italian mathematician and physicist Vito Volterra, known for his contributions to mathematical biology and integral equations. Abel (Italian mathematician) first produced integral equation in connection with the famous tautochrone problem in 1825 [10, 22, 24].

In recent years, there has been a growing interest in the Volterra integral equations arising in various fields of physics and engineering [11] such as Dirichlet problems and potential theory, electrostatics, diffusion problems, the partial transport problems of astrophysics and reactor theory, contact problems [1] and heat transfer problems. In applied mathematics, theory of integral equations is an important subject. In engineering and science, two dimensional integral equations are an important tool for modeling many problems.

The applications of such equations can be seen in different fields such as electrodynamics and electromagnetic, heat and mass transfer, elasticity and dynamic contact, acoustic, chemical and electrochemical process, fluid mechanics, population, molecular physics, medicine and in many other fields. The numerical solution of these equations have been used in collocation and Nyström method which are very important approaches.

To solve Volterra integral equations using various polynomials [17], many researchers have developed some valid numerical methods. In approximation techniques, Bernstein polynomials are used by Maleknejad *et al*. [13] and Mandal and Bhattacharya [15]. Taylor Expansion method and Block-Pulse functions are used by Shahsavaran [20]. These are also used by Rawashdeh and Bellour [5] and Wang [23] with computer algebra. For the solution of first order nonlinear differential equations and the solution of 2nd order linear differential equations Bernstein polynomials used by Bhatti and Bracken [6]. Shirin and Islam have used these polynomials for the solution of second kind Fredholm integral equations [21]. An augmented Galerkin technique has been described by Babolian and Delves [4] for the numerical solution of Fredholm integral equations of first kind. A piecewise interpolation, studied for the numerical solution of first kind Fredholm integral equations by Hanna *et al*. [9]. Computational method for the solution of first kind integral equations has gone through by Lewis [12].

Recently, Dastjerdi and Ghaini [7] proposed a new method based on Chebyshev polynomials and moving least squares method for numerical solution of second kind of Volterra-Fredholm integral equations [7]. The main advantage of this method is that it does not require integration to proceed for interpolation or for mesh. Zakied avazzadeh and Mohammad Heydari [3] introduced an efficient method for solving two dimensional Volterra and Fredholm integral equations of second kind. Their method is also developed on Chebyshev polynomials and approximate numerical solution of Volterra and Fredholm integral equations.

Babolian and Delves [4, 8] described an augmented Galerkin method for the numerical solution of Fredholm integral equations of first kind, which is simple to use and which has the considerable advantage of providing a cheaply computed numerical criterion for the existence of a solution of integral equations, particularly Fredholm integral equations.

Piessen [16] presented a survey for Chebyshev polynomials for the use in the solution of integral equations, in the computation and the inversion of integral transformation. In his paper, his main focus on the problems which show singularity. Rahman [19] solved numerically second kind of Volterra integral equations by the Galerkin method. For this purpose he derived an efficient and simple matrix formulation using Hermite polynomials as trial functions.

Hanna *et al.* [9] used piecewise interpolation method for the first-type numerical inversion of Fredholm integral equations. The main advantage of this technique was the choice of collocation points and grid. In 2012, Rahman and Islam [18] solved linear Volterra integral equations with the help of Legendre piecewise polynomials. In their work they considered problems related to both first and second kind integral equations with weakly singular as well as regular kernals. Adibi and Assari [2] used a computational method to solve the first kind of Fredholm integral equations. To solve a system of algebraic equations they utilized Chebyshev wavelets method as basis in Galerkin method to constructed on the unit interval.

We are interested to study the available approaches in deep and develop new approaches so that numerical solution of integral equations could be improved. We introduced efficient approaches, constructed on Chebyshev polynomials. The formulation is derived to solve nonlinear and linear Volterra integral equations of first and second kind having both regular and weakly singular kernels. Several examples of Volterra integral equations of different kinds are considered to verify the proposed technique. The results of these examples show the rapid convergence of numerical solution along with high degree of accuracy and error reduction. The proposed techniques are very convenient to apply to find exact and approximated numerical solutions of Volterra integral equations of different kind.

2. Integral Equations

In this section, we discuss some well known integral equations by defining them and providing examples.

Definition 1. An integral equation is an equation in which the unknown function appears under an integral sign. A standard integral equation could be of the form

$$g(x)h(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)h(t)dt,$$

where $\alpha(x)$ and $\beta(x)$ are the limits of integration, λ is parameter, and $K(x,t)$ is a function of two variables x and t, called the kernel or the nucleus of the integral equation, $g(x)$ and $f(x)$ are known functions. The function $h(x)$ is to be determined appears under the integral sign, and it it could appears inside and outside the integral sign as well. It is to be noted that the limits of integration $\alpha(x)$ and $\beta(x)$ may be both variables, constants, or mixed.

2.1 Homogeneity of Integral Equations

In this section, we discuss homogeneity of integral equations.

Definition 2. An integral equation of the form

$$g(x)h(x)=f(x)+\lambda\int_{\alpha(x)}^{\beta(x)}K(x,t)h^n(t)dt$$

is called homogeneous, if the known function $f(x)$ is zero, otherwise it is called inhomogeneous.

The integral equation given below

$$u(x)=\sin x+\int_0^x xtu(t)dt$$

is called inhomogeneous integral equation, because of the function $f(x)=\sin(x)$.

Similarly, the integral equation given below

$$u(x)=\int_0^x (2+x-t)u(t)dt$$

is called homogeneous integral equation, due to that fact that the function $f(x)=0$.

2.2 Linearity of Integral Equations

In this section, we explain linearity of integral equations.

Definition 3. An integral equation is called linear, if the exponent of the unknown function under the integral sign is one.

$$h(x)=f(x)+\lambda\int_{\alpha(x)}^{\beta(x)}K(x,t)h^n(t)dt \tag{2.1}$$

Therefore, if $n=1$ in the integral equation (2.1) then it is linear. On the other hand, If the exponent $n\neq 1$, or if the equation contains nonlinear functions such as e^u, $\sinh(u)$, $\cos(u)$, $\ln(1+u)$, then it is called nonlinear integral equation.

$$u(x)=1-\int_0^x (x-t)u(t)dt, \tag{2.2}$$

$$v(x)=1+\int_0^1 (1+x-t)v^4(t)dt. \tag{2.3}$$

Equation (2.2) is linear integral equation, while equation (2.3) is non-linear integral equation.

2.3 Some Important Types of Integral Equations

There are many types of integral equations. These types mostly depend on the limits of integration and nucleus of the integral equations.

2.4 Fredholm Integral Equations

Definition 4. For Fredholm integral equations, the limits of integration are fixed. Moreover, the unknown function $u(x)$ may appear only inside integral sign. The integral equation (2.4) is

$$f(x)=\int_a^b K(x,t)u(t)dt \tag{2.4}$$

called Fredholm integral equation of the first kind.

$$u(x)=f(x)+\lambda\int_a^b k(x,t)u(t)dt. \tag{2.5}$$

However, the integral equation (2.5) is called Fredholm integral equations of second kind.

Note that, in Fredholm integral equation of second kind, unknown function $u(x)$ appears inside and outside the integral sign.

Example 1.

$$\frac{1}{2}x^2 - \frac{2}{3}x + \frac{1}{4} = \int_0^1 (x-t)u(t)dt$$

is called Fredholm Integral Equation of first kind.

Example 2.

$$u(x) = \frac{3}{2}x - \frac{1}{3} - \int_0^1 (x-t)u(t)dt$$

is called Fredholm integral equation of second kind.

2.5 Volterra Integral Equations

Definition 5. In Volterra integral equations, at least one of the limits of integration is a variable. For the first kind Volterra integral equation, the unknown function $u(x)$ appears only inside integral sign in the form

$$f(x) = \int_0^x K(x,t)u(t)dt.$$

However, in Volterra integral equations of the second kind, the unknown function $u(x)$ appears inside and outside the integral sign. The second kind may represented in the form

$$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt.$$

Examples of the Volterra integral equations of the first and second kind are

Example 3.

$$x = \int_0^x (1+x-t)u(t)dt$$

and

$$\frac{1}{6}x^3 = \int_0^x (x-t)u(t)dt$$

are called Volterra integral equations of first kind.

Example 4.

$$u(x) = 1 - \int_0^x u(t)dt$$

and

$$u(x) = x + \frac{1}{6}x^3 - \int_0^x (x-t)u(t)dt$$

are called Volterra integral equations of second kind.

2.6 Volterra-Fredholm Integral Equations

Definition 6. The Volterra-Fredholm integral equations [14, 24] arise from parabolic boundary value problems, from the mathematical modeling of the spatio-temporal development of an epidemic and from various physical and biological models. The Volterra-Fredholm integral equations appear in the literature appears in two forms, namely

$$u(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)u(t)dt + \lambda_2 \int_a^b K_2(x,t)u(t)dt$$

and

$$u(x) = f(x) + \lambda \int_0^x \int_a^b K(x,t)u(t)dtd$$

where $f(x)$ and $K(x,t)$ are analytic functions. Note that, first equation consists disjoint Volterra and Fredholm integrals. Whereas second equation consists of mixed Volterra and Fredholm integrals. Moreover, the unknown functions $u(x)$ appears inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown function appears only inside the integral sign, the resulting equations are of first kind.

Example 5.

$$u(x) = 6x + 3x^2 + 2 - \int_0^x xu(t)dt - \int_0^1 tu(t)dt$$

and

$$u(x) = x - \frac{1}{3}x^3 + \int_0^x tu(t)dt + \int_{-1}^1 t^2u(t)dt$$

are called disjoint Volterra-Fredholm Integral Equations of second kind.

Example 6.

$$u(x) = x + \frac{17}{2}x^2 - \int_0^x \int_0^1 (x-t)u(t)drdt$$

and

$$u(x) = 4 + 14x - 2x^2 + \int_0^x \int_{-1}^1 (x^3 - t^3)u(t)dtdr$$

are called mixed Volterra-Fredholm Integral Equations of second kind.

2.7 Singular Volterra Integral Equations

Definition 7. Volterra integral equations of the first kind

$$f(x) = \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt$$

or of the second kind

$$u(x) = f(x) + \int_{g(x)}^{h(x)} K(x,t)u(t)dt$$

is called singular if one of the limits of integration $g(x)$, $h(x)$ or both are infinite.

Moreover, the above two equations are called singular if the kernel $K(x,t)$ becomes unbounded at one or more points in the interval of integration.

The equations of the form

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt, \quad 0 < \alpha < 1$$

is called generalized Abel's integral equation. The integral equation of the second kind given below

$$u(x) = f(x) + \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt, \quad 0 < \alpha < 1$$

is called weakly singular integral equations. For $\alpha = \frac{1}{2}$, the integral equation

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$$

is called the Abel's singular integral equation. Notice that the kernel in each equation becomes infinity if $t = x$.

Example 7.

$$\sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$$

is called Abel's integral equation.

Example 8.

$$x^3 = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u(t)dt$$

is called generalized Abel's integral equation.

Example 9.

$$u(x) = 1 + \sqrt{x} + \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u(t)dt$$

is called weakly singular integral equation.

2.8 Chebyshev Polynomial

Pafnuty Chebyshev was born in 1821 in Russia. His early research was devoted to number theory. He defended his doctoral thesis "*Teoria sravneny*" (*Theory of congruences*) in 1849. In 1850, he became extraordinary and in 1860 full professor of Mathematics at Petersburg University.

Definition 8. The general form of the Chebyshev polynomials of nth degree is defined by

$$T_n(x) = \sum_{m=0}^{\frac{n}{2}} (-1)^m \frac{n!}{(2m)!(n-2m)!} (1-x^2)^m x^{n-2m}. \tag{2.6}$$

Performing computation on MAPLE, the first few Chebyshev polynomials for different values of n in equation (2.6) are given below

$$T_0(x) = 1,$$

$$T_1(x) = x,$$

$$T_2(x) = 2x^2 - 1,$$

$$T_3(x) = 4x^3 - 3x,$$

$$T_4(x) = 8x^4 - 8x^2 + 1,$$

$T_5(x) = 16x^5 - 20x^3 + 5x,$

$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1.$

3. Novel Approach to Approximate Numerical Solution of Volterra Integral Equations

In this section, we introduce novel approach to approximate numerical solution of Volterra integral equations. This approach is based on Chebyshev and Laguerre Polynomial. The steps involved in the approach are organized as algorithm. The algorithm is further implemented through programming and using Maple.

We performed experiments by considering several examples of different kind of Volterra integral equations. Numerical solutions are observed to estimate the degree of accuracy and reducing errors. It is noted by making comparison with existing approaches that our algorithms provide better results than existing techniques.

Next, we illustrate the novel approach developed on the basis of Chebyshev polynomial.

4. Novel Approach

Novel approach is designed over Chebyshev polynomial and applied to estimate numerical solution of Volterra integral equations. To find the numerical solution of Volterra Integral equation we use the following steps.

Procedure:

(S-1) As a first step, we compute Chebyshev Polynomial (4.1), (already defined in previous discussion).

$$T_n(x) = \sum_{m=0}^{\frac{n}{2}} (-1)^m \frac{n!}{(2m)!(n-2m)!}(1-x^2)^m x^{n-2m} \tag{4.1}$$

for different values of n, for example, the first value will be $T_0 = 1$.

(S-2) In the second step, take the integral equation of second kind as given below.

$$I(x) = f(x) + \lambda \int_a^b k(x,t)I(t)dt \tag{4.2}$$

where $f(x)$ and $k(x,t)$ are known functions and λ is a constant.

(S-3) In this step, we use approximate solution of equation (4.2) to formulate the equation (4.2) in the form so that it could be used for required analysis. The approximate solution of equation (4.2) is given below (4.3).

$$I(x) = \sum_{i=0}^{n} \alpha_i T_i(x) \tag{4.3}$$

where α_i, $i = 0,1,2,\dots,n$ are unknown constants.

Using equation (4.3) in equation (4.2),

$$\sum_{i=0}^{n} \alpha_i T_i(x) = f(x) + \lambda \int_a^b k(x,t) \sum_{i=0}^{n} \alpha_i T_i(t)dt \tag{4.4}$$

(S-4) Calculate the equation (4.4) at $x = x_i$, $i = 0,1,2,\dots,n$ and form the system of n equations. Further, $x_i = 10^{-n} + \frac{i+1}{n+1}$, $i = 0,1,2,\dots,n$. and we can assume $x_0 = a$ if it is necessary.

(S-5) Solve the system of equations for α_i, $i = 0,1,2,\dots,n$ and substitute values of α_i, $i = 0,1,2,\dots,n$ in equation (4.3).

Next, we implement the Novel approach and illustrate it by providing examples.

Implementation:

We implement the procedure by taking different types of Volterra integral equations. We evaluated numerically the Volterra integral equations by performing experiments using Maple.

4.1 Implementing Novel Approach over Linear Volterra Integral Equations

We apply Novel approach initially over linear Volterra integral equations and tabulated the results by taking different numerical values.

Example 10. Consider the Volterra integral equation of second kind

$$I(x) = x + \int_0^x (t-x)I(t)dt. \tag{4.5}$$

Using equation (4.1), we get the Chebyshev polynomials of degree $n = 3$

$$T_0(x) = 1,\ T_1(x) = x,\ T_2(x) = 2x^2 - 1,\ T_3(x) = 4x^3 - 3x.$$

Now using equation (4.3), we get the following result

$$\hat{I}(x) = a_0 + a_1 x + 2a_2 x^2 - a_2 + 4a_3 x^3 - 3a_3 x.$$

Using approximate solution in equation (4.5),

$$a_0 + a_1 x + 2a_2 x^2 - a_2 + 4a_3 x^3 - 3a_3 x + \ldots - a_2 x^2 = 0.$$

Using (S-4) and (S-5), we get the following values of a_i

$$a_0 = -0.0206245747,\ \ a_1 = 0.9144778434,\ \ a_2 = -0.01874288007,\ \ a_3 = -0.03350889415.$$

Consequently, we have the following approximate solution

$$\widetilde{I}(x) = -0.001881694508 + 1.015004526x - 0.03748576013x^2 - 0.1340355766x^3.$$

The exact solution is

$$I(x) = \sin(x).$$

The curves representing numerical solutions corresponding to different values are presenting in Figure 1.

Taking different values of n, the numerical solutions are tabulated in Table 1.

It is observed that if we increase the degree of Chebyshev polynomials, the error decrease.

5. Conclusion

An efficient and simple approach based on the Chebyshev polynomial is developed to solve first and second kind Volterra integral equations. The efficiency increases as we increase the degree of equation. Numerical values tabulated in tables and graphical results show the efficiency of the approach. The convergence of proposed technique is reliable.

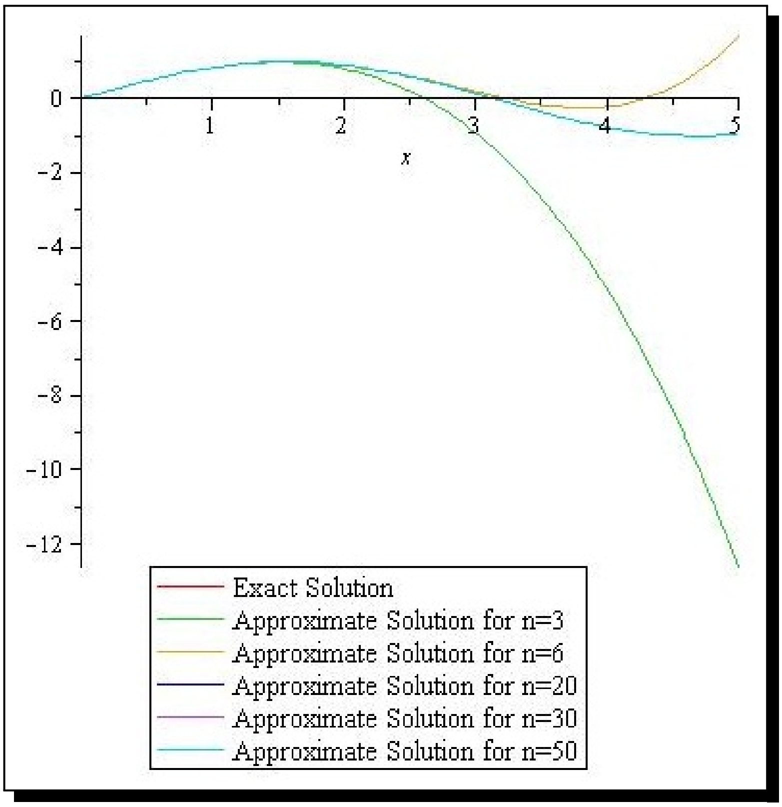

Figure 1

Table 1

x	$n=3$	$n=6$	$n=20$	$n=30$	$n=50$
.0	1.88×10^{-3}	1.06×10^{-6}	1.49×10^{-28}	5.13×10^{-47}	7.19×10^{-88}
.1	7.23×10^{-4}	8.57×10^{-8}	0.00×10^{0}	0.00×10^{0}	1.00×10^{-11}
.2	1.21×10^{-4}	2.68×10^{-8}	0.00×10^{0}	0.00×10^{0}	1.00×10^{-10}
.3	1.06×10^{-4}	9.70×10^{-9}	0.00×10^{0}	0.00×10^{0}	0.00×10^{0}
.4	1.25×10^{-4}	1.41×10^{-8}	0.00×10^{0}	0.00×10^{0}	1.00×10^{-10}
.5	6.91×10^{-5}	2.28×10^{-8}	0.00×10^{0}	0.00×10^{0}	1.00×10^{-10}
.6	3.19×10^{-5}	2.28×10^{-8}	0.00×10^{0}	0.00×10^{0}	0.00×10^{0}
.7	6.15×10^{-5}	2.69×10^{-8}	0.00×10^{0}	0.00×10^{0}	0.00×10^{0}
.8	1.48×10^{-4}	3.57×10^{-8}	0.00×10^{0}	0.00×10^{0}	0.00×10^{0}

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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