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A NOVEL HIGH ORDER SPACE-TIME SPECTRAL METHOD FOR THE TIME FRACTIONAL FOKKER–PLANCK EQUATION*

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Abstract. The fractional Fokker–Planck equation is an important physical model for simulating anomalous diffusions with external forces. Because of the nonlocal property of the fractional derivative an interesting problem is to explore high accuracy numerical methods for fractional differential equations. In this paper, a space-time spectral method is presented for the numerical solution of the time fractional Fokker–Planck initial-boundary value problem. The proposed method employs the Jacobi polynomials for the temporal discretization and Fourier-like basis functions for the spatial discretization. Due to the diagonalizable trait of the Fourier-like basis functions, this leads to a reduced representation of the inner product in the Galerkin analysis. We prove that the time fractional Fokker–Planck equation attains the same approximation order as the time fractional diffusion equation developed in [X. Li and C. Xu, *SIAM J. Numer. Anal.*, 47 (2009), pp. 2108–2131] by using the present method. That indicates an exponential decay may be achieved if the exact solution is sufficiently smooth. Finally, some numerical results are given to demonstrate the high order accuracy and efficiency of the new numerical scheme. The results show that the errors of the numerical solutions obtained by the space-time spectral method decay exponentially.

Key words. Riemann–Liouville fractional derivative, Caputo fractional derivative, spectral method, time fractional Fokker–Planck equation

AMS subject classifications. 26A33, 65M06, 65N12, 65M70

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1. Introduction.

The following Fokker–Planck equation has been commonly utilized to describe the Brownian motion of particles:

$$\partial_t C(x,t) = -\partial_x[P(x)C(x,t)] + K\partial_{xx}C(x,t). \tag{1.1}$$

Here $C(x,t)$ denotes the concentration field, K is the diffusion coefficient, and $P(x)$ is the drift coefficient. The notations ∂_t and ∂_x denote the first order derivatives with respect to t and x, respectively, and ∂_{xx} the second order derivative with respect to x. In many studies of diffusion processes where the diffusion takes place in a highly non-homogeneous medium, the traditional Fokker–Planck equation in (1.1) may not be adequate [2, 3]. The nonhomogeneities of the medium may alter the laws of the Markov diffusion in a fundamental way. In particular, the corresponding probability density of the concentration field may have a heavier tail than the Gaussian density. This phenomenon is called anomalous diffusion [30].

Replacing the time and/or space derivative of the traditional Fokker–Planck by a fractional derivative leads to a variety of space-time fractional Fokker–Planck equations (STFFPE). These STFFPEs have been useful for modeling many relevant physical processes [2, 3, 4, 18]. Some STFFPEs have been solved using analytical

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methods [19, 26, 35, 40], but it must be stressed that an analytical solution to the STFFPE can be obtained only in some simple cases such as constant coefficients or linearity. Therefore, numerical approaches are needed for solving these equations.

In this paper, we present a numerical method for solving the time fractional Fokker–Planck initial-boundary value problem

$$\begin{cases} \partial_t u = {}_0D_t^{1-\alpha}\left\{\partial_x[p(x)u(x,t)] + K_\alpha \partial_{xx} u(x,t) + f(x,t)\right\}, (x,t) \in (a,b)\times(0,T], \\ u(x,0) = u_0(x), x \in (a,b), \\ u(a,t) = u_a(t),\ u(b,t) = u_b(t), t \in (0,T], \end{cases} \tag{1.2}$$

where the unknown function $u(x,t)$ denotes the probability density function to find the test particle at a certain position x at a given time t. $K_\alpha > 0$ is a diffusion constant, $p(x)$ is nonpositive and decreases monotonically in the interval $[a,b]$, and $f(x,t)$ is the source term.

The fractional Fokker–Planck equation (1.2) was derived by Metzler, Barkai, and Klafter [28, 29], who describe an anomalous diffusion of a system close to thermal equilibrium under the influence of an external force field. Here, the derivative ${}_0D_t^{1-\alpha}(0<\alpha<1)$ denotes the left Riemann–Liouville fractional derivative defined by

$${}_0D_t^{\gamma} f(t) = \frac{d^m}{dt^m} \circ {}_0I_t^{m-\gamma} f(t)\ \forall m-1 \le \gamma < m,$$

in which ${}_0I_t^{\gamma} f(t)$ is the left Riemann–Liouville fractional integral defined by

$${}_0I_t^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s) ds.$$

Correspondingly, the right Riemann–Liouville fractional derivative is defined by

$${}_tD_T^{\gamma} f(t) = \frac{d^m}{d(-t)^m} \circ {}_tI_T^{m-\gamma} f(t)\ \forall\, m-1 \le \gamma < m,$$

where the right Riemann–Liouville fractional integral ${}_tI_T^{\alpha} f(t)$ is defined by

$${}_tI_T^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_t^T (s-t)^{\gamma-1} f(s) ds.$$

Problem (1.2) can be rewritten in the form

$${}_0D_t^{\alpha}\left[u(x,t) - u_0(x)\right] = \partial_x[p(x)u(x,t)] + K_\alpha \partial_{xx} u(x,t) + f(x,t) \tag{1.3}$$

with the same boundary conditions as (1.2) and zero initial condition. In fact, noting that if $g(t) \in L^\infty[0,t]$, then it follows that $\int_0^t (t-s)^{\alpha-1} g(s) ds|_{t=0} = 0$. By integrating the first equation of (1.2) over $[0,t]$, we obtain

$$\begin{aligned} u(x,t) &= u(x,0) + \int_0^t {}_0D_s^{1-\alpha}\left\{\partial_x[p(x)u(x,s)] + K_\alpha \partial_{xx} u(x,s) + f(x,s)\right\} ds \\ &= u(x,0) + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\left\{\partial_x[p(x)u(x,s)] + K_\alpha \partial_{xx} u(x,s) + f(x,s)\right\} ds. \end{aligned}$$

Hence, $u(x,t) - u(x,0) = {}_0I_t^{\alpha}\left\{\partial_x[p(x)u(x,s)] + K_\alpha \partial_{xx} u(x,s) + f(x,s)\right\}$. This gives (1.3) by the fact that ${}_0D_t^{\alpha} \circ {}_0I_t^{\alpha} f(t) = f(t)$.

Equation (1.2) is therefore equivalent to

$$\begin{cases} {}_0^C D_t^{\alpha} u(x,t) = \partial_x[p(x)u(x,t)] + K_\alpha \partial_{xx} u(x,t) + f(x,t), \\ u(x,0) = u_0(x), \\ u(a,t) = u_a(t), u(b,t) = u_b(t), \end{cases} \tag{1.4}$$

where

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds $$

is the Caputo derivative. It is often more convenient to use the formulation (1.3) than (1.4). Furthermore, we rewrite (1.3) as

$$ {}_0D_t^\alpha u(x,t) = \partial_x [p(x)u(x,t)] + K_\alpha \partial_{xx} u(x,t) + \tilde{f}(x,t), \tag{1.5} $$

in which $\tilde{f}(x,t) = f(x,t) + {}_0D_t^\alpha u_0(x)$.

Several methods have been developed for solving the fractional diffusion equation numerically. Langlands and Henry studied a time fractional diffusion equation based on an L^1 scheme [21]. The authors proposed an implicit difference scheme that is unconditionally stable. Based on the Grünwald–Letnikov formula, an explicit difference scheme has been presented for the time fractional diffusion equation [42, 43].

Numerical solution techniques for space fractional diffusion equations have also been studied by some authors. It is interesting to note that the explicit and implicit difference schemes based on the Grünwald–Letnikov formula are all unconditionally unstable for space fractional diffusion and advection-diffusion equations [27, 39]. However, the authors proved that a shifted Grünwald–Letnikov formula can produce a stable difference scheme [27, 39].

The finite difference method (FDM) is the most common method for solving fractional differential equation [5, 12, 13, 14, 16]. Recent works are found in [1, 8, 36]. However, high order schemes are seldom derived using the FDM. In general, an extrapolation method is applied in order to obtain high accuracy [13, 39].

Numerical solution of the fractional Fokker–Planck equation has been studied by a number of authors. The authors of [25] studied the space fractional Fokker–Planck equation by the FDM. A finite difference scheme is also obtained in [9, 10, 41] for time fractional and time-space fractional Fokker–Planck equations. For the case of a time fractional equation, an FDM based on the Grünwald–Letnikov definition can obtain an accuracy of order $(2-\alpha)$ for the temporal discretization, where $\alpha \in (0,1)$. Although the implementation of FDM approaches relatively straightforward, a challenge is that the convergence is algebraic and the accuracy is limited. Moreover, the FDM approaches suffer from heavy costs of computing because the fractional derivative is nonlocal. This fact would suggest that a global scheme, such as the spectral method, is suitable for the discretization of the fractional derivative.

It is well known that spectral methods are superior to FDMs in many instances for partial differential equations [7, 6, 17, 20]. However, the extension of the spectral method to a fractional differential equation is not trivial. Since the work of Erin and Roop [15], which established the variational theory for fractional differential equations, the spectral/finite element methods have been applied to fractional diffusion equations [11, 24, 46, 47, 48]. A high order finite element method for the space and time fractional Fokker–Planck equation was proposed by Deng [11]. The author proved the convergence order is $O(\tau^{2-\alpha} + \tau^{-1}h^l)$, where τ and h are the time and space step size, respectively. Lin and Xu [24] proposed a finite difference/spectral approximation for the time fractional diffusion equation, and the convergence order of $O(\tau^{2-\alpha} + \tau^{-1}N^{-m})$ was proved, where N and m are respectively the polynomial degree and regularity in the space variable of the exact solution. Obviously the convergence rates in these works are not optimal due to the presence of the factor τ^{-1}. In the paper [46] the authors proposed a finite difference/element approach that has

convergence order $O(\tau^{2-\alpha} + h^{r+1})$ for the sufficiently smooth exact solution, where r is the degree of the polynomial in space. Nevertheless the accuracy limitation in time caused a lower global convergence rate. On the other hand, the Galerkin spectral method, which can be implemented in both time and space directions, has the potential to achieve high order accuracy in both time and space. This motivates us to construct high order numerical approaches based on the space-time spectral method.

Actually, unlike the integer counterparts, the fractional derivative of any order possesses a variational formulation. This fact and the properties of the Jacobi polynomials [31] suggest one may use a spectral method for the discretization of the time fractional derivative. However, there are other implementations. In [22], Li, Zeng, and Liu proposed the spectral approximation to the fractional integral and derivative based on the Legendre, Chebyshev, and Jacobi polynomials, and some effective algorithms were also presented. Recently, a new spectral theory for fractional Sturm–Liouville problems has been developed in [44]. The spectral discretization in both time and space can generate an exponentially accurate numerical scheme in terms of the basis functions of Jacobi polynomials or Jacobi polyfractonomials [44]. In [23] the authors employed the space-time spectral method for solving a fractional diffusion equation and spectral accuracy was obtained in both space and time. A time-space fractional spectral collocation method was also examined in [45] and an exponentially accurate scheme was presented. However, high order methods for solving the fractional differential equations are still under development.

This paper is the first work we propose on high order methods for solving fractional differential equations. We will examine the numerical solution of the time fractional Fokker–Planck equation (1.2) by making use of the space-time spectral approach. Our method is based on the variational construction of the time and space fractional derivatives. Different from the work of [23], our scheme discretizes the space variable using Fourier-like basis functions. Moreover, we consider a more general initial-boundary value problem. The Fourier-like basis functions, which were first proposed by Shen and Wang [38], possess some properties similar to the Fourier basis functions and lead to a diagonal stiff matrix. Due to the diagonalizable trait, the time-space spectral formulation may be written as a relatively simple linear system. A proof of the theoretical approximation order is presented in this paper. Exponential decay can be seen from the numerical examples. Although the spectral accuracy may be obtained theoretically, the convergence order is lower due to the effect of the quadrature evaluation. We will investigate the numerical convergence order and compare the approximation orders of space-time spectral methods for the time fractional diffusion and convection-diffusion equations in section 6. The numerical examples show that the space-time spectral approach possesses high order accuracy and efficiency.

Our method is superior to the scheme developed in [23]. In fact, the computational magnitude of the present method is $O(M^2N)$ for solving the time fractional diffusion equation, while the method of [23] resulted in a magnitude of $O(M^2N^2)$. Another advantage of our scheme is that the method may be extended to high dimensional time fractional diffusion equations, which will be developed in our future work.

The paper is arranged as follows. In section 2 some basic concepts of fractional derivatives and some properties of the Jacobi polynomials are presented. In section 3 the variational formulation of the time fractional Fokker–Planck equation (1.2) is proposed. In section 4, the space-time spectral method is proposed and an error

estimate is derived. We study the implementation of the spectral method in section 5, and a new scheme for the spatial discretization is presented. Some numerical examples are provided in section 6, which illustrate the higher accuracy and efficiency of the space-time spectral method. Finally, some conclusions and remarks are given in section 7.

2. Fractional derivative space and Jacobi polynomials. Set $I = (0, T], \Lambda = (a, b), \Omega = I \times \Lambda$. Denote $(\cdot, \cdot)_D$ and $\|\cdot\|_{0,D}$ as the inner product and the norm in $L^2(D)$, respectively. Let ${}_0C^\infty(I)$ denote the space of smooth functions with compact support in I. Recall that the notation $A \lesssim B$ means that there exists a generic positive constant c s.t. $A \leq cB$. The following section introduces several definitions of fractional derivative space used in the present paper.

DEFINITION 2.1 (see [23, 33]). *For any $\gamma > 0$ define the left-fractional derivative space*

$$J_L^\gamma(I) = \left\{u \in L^2(I) | {}_0D_t^\gamma u \in L^2(I)\right\}$$

and

$$J_{L,0}^\gamma(I) = \left\{u \in J_L^\gamma(I) | \exists u_k \in {}_0C^\infty(I),\ \textit{for which}\ \|u_k - u\|_{J_L^\gamma} \to 0\ \textit{as}\ k \to \infty\right\},$$

where the norm $\|\cdot\|_{J_L^\gamma}$ is defined by

$$\|u\|_{J_L^\gamma(I)} = \left(\|{}_0D_t^\gamma u\|_{0,I}^2 + \|u\|_{0,I}^2\right)^{\frac{1}{2}}.$$

DEFINITION 2.2 (see [23, 33]). *For any $\gamma > 0$ define the right-fractional derivative space*

$$J_R^\gamma(I) = \left\{u \in L^2(I) | {}_tD_T^\gamma u \in L^2(I)\right\}.$$

and

$$J_{R,0}^\gamma(I) = \left\{u \in J_R^\gamma(I) | \exists u_k \in {}_0C^\infty(I),\ \textit{for which}\ \|u_k - u\|_{J_R^\gamma} \to 0\ \textit{as}\ k \to \infty\right\},$$

where the norm $\|\cdot\|_{J_R^\gamma}$ is defined by

$$\|u\|_{J_R^\gamma(I)} = \left(\|{}_tD_T^\gamma u\|_{0,I}^2 + \|u\|_{0,I}^2\right)^{\frac{1}{2}}.$$

DEFINITION 2.3 (see [23, 33]). *For any $\gamma > 0$ and $\gamma \neq n - \frac{1}{2}, n \in \mathbb{N}$, define the symmetric-fractional derivative space by*

$$J_S^\gamma(I) = \left\{u \in L^2(I) | {}_0D_t^\gamma u \in L^2(I)\ \textit{and}\ {}_tD_T^\gamma u \in L^2(I)\right\}$$

and

$$J_{S,0}^\gamma(I) = \left\{u \in J_S^\gamma(I) | \exists u_k \in {}_0C^\infty(I),\ \textit{for which}\ \|u_k - u\|_{J_S^\gamma} \to 0\ \textit{as}\ k \to \infty\right\},$$

where the $\|\cdot\|_{J_S^\gamma(I)}$ is defined by

$$\|u\|_{J_S^\gamma(I)} = \left(|({}_0D_t^\gamma u, {}_tD_T^\gamma u)_I| + \|u\|_{0,I}^2\right)^{\frac{1}{2}}.$$

DEFINITION 2.4 (see [23, 33]). *For any $\gamma > 0$ and $\gamma \neq n - \frac{1}{2}, n \in \mathbb{N}$, define the fractional Sobolev space*

$$H^\gamma(I) = \left\{u \in L^2(I) | \, |\omega|^\gamma \mathcal{F}(\tilde{u}) \in L^2(\mathbb{R})\right\}$$

endowed with the norm

$$\|u\|_{\gamma,I} = \| \, |\omega|^\gamma \mathcal{F}(\tilde{u})\|_{0,\mathbb{R}},$$

in which $\tilde{u}$ is the extension of zero of u outside I, and

$$ {}_0H^{\gamma}(I) = \left\{ u \in H^{\gamma}(I) | \exists u_k \in {}_0C^{\infty}(I), \text{ for which } \|u_k - u\|_{\gamma,I} \to 0 \text{ as } k \to \infty \right\}. $$

LEMMA 2.1. *If $\gamma \neq n - \frac{1}{2}, n \in \mathbb{N}$, then the fractional derivative spaces $J^{\gamma}_{L,0}(I)$, $J^{\gamma}_{R,0}(I)$, $J^{\gamma}_{S,0}(I), {}_0H^{\gamma}(I)$ are equal with equivalent norm.*

Proof. The equivalence can be verified by the same method as obtained in [23]. ∎

Additionally, the following property is needed in this paper.

LEMMA 2.2 (adjoint property). *If $u \in J^{\gamma}_L(I), v \in J^{\gamma}_R(I)$, then*

$$ ({}_0D^{\gamma}_t u(t), v)_I = (u, {}_tD^{\gamma}_T v)_I. $$

Furthermore, if $u, v \in {}_0H^{\frac{\alpha}{2}}(I)$, then

$$ ({}_0D^{\alpha}_t u, v)_I = ({}_0D^{\frac{\alpha}{2}}_t u, {}_tD^{\frac{\alpha}{2}}_T v)_I. $$

Proof. The first equality is the modification of Corollary 2 of Theorem 2.4 in [34]. Indeed, if $u \in J^{\gamma}_L(I)$, then there exists a unique $\varphi_1(t) \in L^2(I)$ satisfying Abel's equation ${}_0I^{\gamma}_t \varphi_1(t) = u(t)$. Similarly, there exists a unique $\varphi_2(t) \in L^2(I)$ such that $v(t) = {}_tI^{\gamma}_T \varphi_2(t)$. Thus, the first equality follows from the well-known adjoint property of the fractional integral.

Notice that

$$ {}_0D^{\alpha}_t u = {}_0D^{\frac{\alpha}{2}}_t {}_0D^{\frac{\alpha}{2}}_t u $$

if $u \in {}_0H^{\frac{\alpha}{2}}(I)$. Hence, the second equality follows immediately from the first equality. ∎

Next, let us recall some properties of the Jacobi orthogonal polynomials. The Jacobi polynomials $J^{\alpha,\beta}_n(x)(n = 0, 1, \ldots)$ are orthogonal on $(-1, 1)$ associated to the weight function $\omega^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}(\alpha, \beta > -1)$ and satisfy the following three-term recurrence relation [37]:

$$ \begin{cases} J^{\alpha,\beta}_0(x) = 1, \\ J^{\alpha,\beta}_1(x) = \frac{\alpha+\beta+2}{2}x + \frac{\alpha-\beta}{2}, \\ J^{\alpha,\beta}_{n+1}(x) = (A^{\alpha,\beta}_n x - B^{\alpha,\beta}_n) J^{\alpha,\beta}_n(x) - C^{\alpha,\beta}_n J^{\alpha,\beta}_{n-1}(x), n = 1, 2, \ldots, \end{cases} $$

in which

$$ \begin{aligned} A^{\alpha,\beta}_n &= \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}, \\ B^{\alpha,\beta}_n &= \frac{(\beta^2-\alpha^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}, \\ C^{\alpha,\beta}_n &= \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}. \end{aligned} $$

The Legendre orthogonal polynomials are a special case of the Jacobi polynomials with the choice $\alpha = \beta = 0$. Let $Q^{\alpha,\beta}_m(x) = J^{\alpha,\beta}_m(x)(1-x)^{\alpha}(1+x)^{\beta}$. The following properties hold (see [31]).

PROPOSITION 2.3. *For any $\alpha \in (0,1)$, the left Riemann–Liouville fractional integral of $Q_m^{0,-\alpha}(x)$ and the right Riemann–Liouville fractional integral of $Q_m^{-\alpha,0}(x)$ are respectively*

$$ {}_{-1}I_t^{\alpha} Q_m^{0,-\alpha}(t) = \frac{\Gamma(m+1-\alpha)}{\Gamma(m+1)} J_m^{-\alpha,0}(t) $$

and

$$ {}_{t}I_1^{\alpha} Q_m^{-\alpha,0}(t) = \frac{\Gamma(m+1-\alpha)}{\Gamma(m+1)} J_m^{0,-\alpha}(t). $$

PROPOSITION 2.4. *For any $\alpha \in (0,1)$, the left Riemann–Liouville fractional derivative of $J_m^{-\alpha,0}(x)$ and the right Riemann–Liouville fractional derivative of $Q_m^{0,-\alpha}(x)$ are respectively*

$$ {}_{-1}D_t^{\alpha} J_m^{-\alpha,0}(t) = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} Q_m^{0,-\alpha}(t) $$

and

$$ {}_{t}D_1^{\alpha} J_m^{0,-\alpha}(t) = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)} Q_m^{-\alpha,0}(t). $$

The proofs of these propositions can be found in Podlubny's book [31]. Notice that Proposition 2.4 can be derived directly from Proposition 2.3 by taking a fractional derivative on both sides of the two equations in Proposition 2.3. The following is the fractional derivative of the shifted Jacobi polynomials.

PROPOSITION 2.5. *For $0 < t < T$, $0 < \alpha < 1$,*

$$ {}_{0}I_t^{\alpha} Q_m^{0,-\alpha}\left(\frac{2t}{T}-1\right) = \frac{T^{\alpha}\Gamma(m+1-\alpha)}{2^{\alpha}\Gamma(m+1)} J_m^{-\alpha,0}\left(\frac{2t}{T}-1\right), $$

$$ {}_{t}I_T^{\alpha} Q_m^{-\alpha,0}\left(\frac{2t}{T}-1\right) = \frac{T^{\alpha}\Gamma(m+1-\alpha)}{2^{\alpha}\Gamma(m+1)} J_m^{0,-\alpha}\left(\frac{2t}{T}-1\right). $$

PROPOSITION 2.6. *For $0 < t < T$, $0 < \alpha < 1$,*

$$ {}_{0}D_t^{\alpha} J_m^{-\alpha,0}\left(\frac{2t}{T}-1\right) = \frac{\Gamma(m+1)t^{-\alpha}}{\Gamma(m+1-\alpha)} J_m^{0,-\alpha}\left(\frac{2t}{T}-1\right), $$

$$ {}_{t}D_T^{\alpha} J_m^{0,-\alpha}\left(\frac{2t}{T}-1\right) = \frac{\Gamma(m+1)(T-t)^{-\alpha}}{\Gamma(m+1-\alpha)} J_m^{-\alpha,0}\left(\frac{2t}{T}-1\right). $$

Notice that Proposition 2.6 may be derived in a straightforward manner from Proposiston 2.5. In order to make the paper self-contained, we present a proof of Proposition 2.5 in Appendix A.

3. Variational formulation. In this section, we investigate the variational formulation of the time fractional Fokker–Planck equation (1.2). In order to perform the spectral analysis conveniently, we make the transformation

$$ v(x,t) = u(x,t) + \frac{x-b}{b-a} u_a(t) + \frac{a-x}{b-a} u_b(t) \tag{3.1} $$

so as to derive the homogeneous boundary conditions. Denote $\xi(x,t) = \frac{x-b}{b-a}u_a(t) + \frac{a-x}{b-a}u_b(t)$. In terms of (3.1), $v(x,0) = u_0(x) + \xi(x,0)$. Thus, similar to (1.5), (1.2) is written as

$$ {}_0D_t^\alpha[v(x,t) - \xi(x,t) - u_0(x)] = \partial_x[p(x)v(x,t)] + K_\alpha \partial_{xx} v(x,t) - \partial_x(p(x)\xi(x,t)) + f(x,t), $$

that is,

$$ \begin{aligned} {}_0D_t^\alpha v(x,t) = {} & \partial_x[p(x)v(x,t)] + K_\alpha \partial_{xx} v(x,t) - \partial_x(p(x)\xi(x,t)) \\ & + {}_0D_t^\alpha \xi(x,t) + {}_0D_t^\alpha u_0(x) + f(x,t), \end{aligned} \tag{3.2} $$

together with the homogeneous initial-boundary conditions

$$ v(x,0) = 0, v(a,t) = v(b,t) = 0. $$

Hence, in what follows all the analysis is based on homogeneous boundary conditions unless otherwise stated. In order to simplify the notation we write (3.2) in the form

$$ {}_0D_t^\alpha u(x,t) = \partial_x[p(x)u(x,t)] + K_\alpha \partial_{xx} u(x,t) + \tilde{f}(x,t), \tag{3.3} $$

where $\tilde{f}(x,t) = f(x,t) - \partial_x(p(x)\xi(x,t)) + {}_0D_t^\alpha \xi(x,t) + {}_0D_t^\alpha u_0(x)$.

Let us introduce some function spaces on $\Omega = I \times \Lambda$. Assume that X is a Banach space on Λ with norm $\|\cdot\|_X$; the space $L^q(I;X)$ is defined as

$$ L^q(I;X) := \left\{ v : I \to X \,\middle|\, \int_I \|v(t)\|_X^q \, dt < \infty \right\} $$

for $1 \le q < \infty$, endowed with the norm

$$ \|v\|_{L^q(I;X)} := \left(\int_I \|v(t)\|_X^q \right)^{1/q}. $$

When $X = L^2(\Lambda)$, $L^2(I;X) = L^2(\Omega)$. Define the space $H^\gamma(I;X)$ as

$$ \begin{aligned} H^\gamma(I;X) &:= \left\{ v \in L^2(I;X) \middle|\ \|v(\cdot,t)\|_X \in H^\gamma(I) \right\}, \\ {}_0H^\gamma(I;X) &:= \left\{ v \in L^2(I;X) \middle|\ \|v(\cdot,t)\|_X \in {}_0H^\gamma(I) \right\} \end{aligned} $$

for $\gamma \ge 0$, endowed with the norm

$$ \|v\|_{H^\gamma(I;X)} := \| \, \|v(\cdot,t)\|_X \|_{\gamma,I}. $$

Set $V_\gamma = H^\gamma(I;L^2(\Lambda)) \cap L^2(I;H_0^1(\Lambda))$. Define the norm $\|\cdot\|_{V_\gamma}$ as

$$ \|u\|_{V_\gamma} = \|u\|_{\gamma,0} + \|u\|_{0,1}, $$

where

$$ \|u\|_{\gamma,0} = \|u\|_{H^\gamma(I;L^2(\Lambda))} \text{ and } \|u\|_{0,1} = \|u\|_{L^2(I;H_0^1(\Lambda))}. $$

It can be verified that the space V_γ is a Banach space with the norm $\|\cdot\|_{V_\gamma}$.

Now let us consider the weak formulation of problem (1.2). Multiplying (3.3) by a test function $v \in V_{\frac{\alpha}{2}}$ and integrating, by Lemma 2.2 we obtain

$$({}_0D_t^{\frac{\alpha}{2}}u, {}_tD_T^{\frac{\alpha}{2}}v)_\Omega + K_\alpha(\partial_x u, \partial_x v)_\Omega - (\partial_x(pu), v)_\Omega = (\tilde{f}, v)_\Omega.$$

Therefore, we derive the variational formulation of problem (1.2) as

$$\mathcal{B}(u, v) = \mathcal{F}(v) \quad \forall v \in V_{\frac{\alpha}{2}}, \tag{3.4}$$

where

$$\mathcal{B}(u, v) = ({}_0D_t^{\frac{\alpha}{2}}u, {}_tD_T^{\frac{\alpha}{2}}v)_\Omega + K_\alpha(\partial_x u, \partial_x v)_\Omega - (\partial_x(pu), v)_\Omega, \quad \mathcal{F}(v) = (\tilde{f}, v)_\Omega.$$

LEMMA 3.1. *For $0 < \alpha < 1$, the bilinear operator $\mathcal{B}(u, v)$ is coercive and continuous, that is, for any $u, v \in V_{\frac{\alpha}{2}}$ there exist $\kappa_1 > 0, \kappa_2 > 0$ such that*

$$\mathcal{B}(u, u) \geq \kappa_1 \|u\|^2_{V_{\frac{\alpha}{2}}}, \quad |\mathcal{B}(u, v)| \leq \kappa_2 \|u\|_{V_{\frac{\alpha}{2}}} \|v\|_{V_{\frac{\alpha}{2}}}.$$

Proof. Note that (see [33])

$$\begin{aligned}\mathcal{B}(u, u) &\geq \cos\frac{\pi\alpha}{2} \int_\Lambda \left\| {}_0D_t^{\frac{\alpha}{2}}u \right\|^2_{0,I} dx + K_\alpha \int_I \|\partial_x u\|^2_{0,\Lambda} dt - (\partial_x p, u^2)_{0,\Omega} \\ &\geq \frac{1}{2}\min\left\{C_1\cos\frac{\pi\alpha}{2}, C_2 K_\alpha\right\} \|u\|^2_{V_{\frac{\alpha}{2}}}\end{aligned}$$

by the assumption of $p(x)$ and Lemma 2.1, where C_1, C_2 are positive constants. Therefore, the coercivity of $\mathcal{B}$ is derived.

$$|\mathcal{B}(u, v)| \leq \left\| {}_0D_t^{\frac{\alpha}{2}}u \right\|_{0,\Omega} \left\| {}_0D_t^{\frac{\alpha}{2}}v \right\|_{0,\Omega} + K_\alpha \|\partial_x u\|_{0,\Omega} \|\partial_x v\|_{0,\Omega} + \|\partial_x(pu)\|_{0,\Omega} \|v\|_{0,\Omega}. \tag{3.5}$$

Since $\|\partial_x(pu)\|_{0,\Omega} \leq \|p\|_{1,\Lambda} \|u\|_{0,1}$, from (3.5) we derive

$$\begin{aligned}|\mathcal{B}(u, v)| &\leq \left\| {}_0D_t^{\frac{\alpha}{2}}u \right\|_{0,\Omega} \left\| {}_0D_t^{\frac{\alpha}{2}}v \right\|_{0,\Omega} + \max\{K_\alpha, \|p\|_{1,\Lambda}\} \|u\|_{0,1} \|v\|_{0,1} \\ &\leq \max\{1, K_\alpha, \|p\|_{1,\Lambda}\} \|u\|_{V_{\frac{\alpha}{2}}} \|v\|_{V_{\frac{\alpha}{2}}}.\end{aligned}$$

So, the continuity of $\mathcal{B}$ holds. ∎

We immediately obtain the following well-posedness theorem of the variational problem (3.4) through the previous lemma.

THEOREM 3.2. *For all $0 < \alpha < 1$ and $f \in L^2(\Omega)$, problem (3.4) is well-posed. Furthermore, there exists constant C only dependent of α, T, p, Ω, satisfying*

$$\|u\|_{V_{\frac{\alpha}{2}}} \leq C(\|f\|_{L^2(\Omega)} + \|u_a\|_{H^{\frac{\alpha}{2}}(I)} + \|u_b\|_{H^{\frac{\alpha}{2}}(I)} + \|u_0\|_{L^2(\Lambda)}). \tag{3.6}$$

Proof. Notice that

$$\begin{aligned}&\left(\tilde{f}, v\right)_\Omega \leq |(f, v)| + |(p(x)\xi, \partial_x v)| + \left|\left({}_0D_t^{\frac{\alpha}{2}}\xi, {}_tD_T^{\frac{\alpha}{2}}v\right)\right| + \left|\left({}_0D_t^{\frac{\alpha}{2}}u_0, {}_tD_T^{\frac{\alpha}{2}}v\right)\right| \\ &\lesssim \left(\|f\|_{L^2(\Omega)} + \|u_a\|_{H^{\frac{\alpha}{2}}(I)} + \|u_b\|_{H^{\frac{\alpha}{2}}(I)} + \|u_0\|_{L^2(\Lambda)}\right) \|v\|_{V_{\frac{\alpha}{2}}}.\end{aligned}$$

The result holds by the coercivity of $\mathcal{B}$. ∎

Remark 3.1. From the proof of Lemma 3.1, one can see that, for the well-posedness of problem (1.2), the assumption of $p(x)$ decreasing monotically is sufficient but not necessary.

4. Space-time spectral method. In this section, we shall consider the space-time spectral analysis on (3.3). The stability and convergence of the space-time spectral method are presented here.

For the homogeneous initial-boundary data, we assume hereafter that $u \in {}_0H^{\gamma}(I; L^2(\Lambda)) \cap L^2(I; H_0^1(\Lambda))$. Let $P_M(I)$ (resp., $P_N(\Lambda)$) denote the set of polynomials of degree M (resp., N) with respect to t (resp., x). Set

$$\mathring{P}_M(I) = \{p(t) \in P_M(I) | p(0) = 0\}, \quad \mathring{P}_N(\Lambda) = P_N(\Lambda) \cap H_0^1(\Lambda).$$

We denote by $\Pi_N^{0,1} : H_0^1(\Lambda) \to \mathring{P}_N(\Lambda)$ the orthogonal projection operator on $\mathring{P}_N(\Lambda)$ in $H_0^1(\Lambda)$ such that

$$\left((\Pi_N^{0,1} u - u)', v'\right)_{\Lambda} = 0 \ \forall v \in H_0^1(\Lambda).$$

Denote by $\Pi_M : L^2(I) \to \mathring{P}_M(I)$ the orthogonal projection operator on $\mathring{P}_M(I)$ in $L^2(I)$, and denote the projection operator $\Pi_M^1 : {}_0H^1(I) \to \mathring{P}_M(I)$ by

$$\Pi_M^1 u(t) = \int_0^t \Pi_{M-1} u'(\tau) d\tau \ \forall u \in {}_0H^1(I).$$

Let us denote $L = (M, N)$ and define $W_L = \mathring{P}_M(I) \otimes \mathring{P}_N(\Lambda)$. We construct the following spectral scheme for the variational problem (3.4): find $u_L \in W_L$ such that

$$\mathcal{B}(u_L, v_L) = \mathcal{F}(v_L) \ \forall v_L \in W_L, \tag{4.1}$$

where $\mathcal{F}(\cdot) = (f, \cdot)_{\Omega}$.

The discrete problem (4.1) is well-posed due to the fact that W_L is a subspace of $V_{\frac{\alpha}{2}}$, and a similar stability to (3.6) can be derived for u_L.

THEOREM 4.1 (stability). *For all $0 < \alpha < 1$, if $f \in L^2(\Omega)$, $p \in C^1(\Lambda)$, $u_0 \in L^2(\Lambda)$, and $u_a, u_b \in H^{\frac{\alpha}{2}}(I)$, then the spectral approximate solution of* (1.2) *is stable. Furthermore, the following estimate holds:*

$$\|u_L\|_{V_{\frac{\alpha}{2}}} \le C(\|f\|_{L^2(\Omega)} + \|u_a\|_{H^{\frac{\alpha}{2}}(I)} + \|u_b\|_{H^{\frac{\alpha}{2}}(I)} + \|u_0\|_{L^2(\Lambda)}),$$

where the constant C is independent of f, u_0, u_a, and u_b.

In order to estimate the error on the approximate solution u_L, let us recall some known approximation properties.

LEMMA 4.2 (approximation property).

(i) *For any $u \in H^m(\Lambda) \cap H_0^1(\Lambda)$, $m \ge 1, s < m$,*

$$\left\|\Pi_N^{1,0} u - u\right\|_{s,\Lambda} \le c_{\Lambda} N^{s-m} \|u\|_{m,\Lambda}; \tag{4.2}$$

(ii) *for any $u \in H^m(I) \cap {}_0H^1(I)$, $m \ge 1, s < m$,*

$$\left\|\Pi_M^1 u - u\right\|_{s,I} \le c_I M^{s-m} \|u\|_{m,I}; \tag{4.3}$$

where c_{Λ} and c_I are the constants independent of N and M, respectively.

Proof. The proof of (4.2) can be found in [37] and that of (4.3) in [23]. ∎

THEOREM 4.3 (convergence). *Let $0 < \alpha < 1, r > 1, s \geq 1$, and $f \in L^2(\Omega)$. Assume that u_L is the approximate solution of discrete problem* (4.1) *and u the exact solution of variational problem* (3.4) *satisfying $u \in {}_0H^{\frac{\alpha}{2}}(I; H^r(\Lambda)) \cap H^s(I; H_0^1(\Lambda))$. Then, the error $\|u - u_L\|_{V_{\frac{\alpha}{2}}}$ satisfies*

$$\begin{aligned}\|u - u_L\|_{V_{\frac{\alpha}{2}}} \lesssim{}& \Theta c_I M^{-s}(M^{\frac{\alpha}{2}} \|u\|_{s,0} + \|u\|_{s,1}) + \Theta c_\Lambda N^{-r}(N \|u\|_{0,r} + \|u\|_{\frac{\alpha}{2},r}) \\ &+ \Theta c_I c_\Lambda M^{\frac{\alpha}{2}-s} N^{-r} \|u\|_{s,r}, \end{aligned} \tag{4.4}$$

in which $\Theta = 1 + \kappa_2/\kappa_1$.

Proof. Set $\tilde{u}_L = \Pi_M^1 \Pi_N^{0,1} u$. By (3.4) and (4.1), we obtain

$$\mathcal{B}(u_L - \tilde{u}_L, v_L) = \mathcal{B}(u - \tilde{u}_L, v_L).$$

Taking $v_L = u_L - \tilde{u}_L$, ones obtains by Lemma 3.1

$$\kappa_1 \|u_L - \tilde{u}_L\|_{V_{\frac{\alpha}{2}}} \leq \kappa_2 \|u - \tilde{u}_L\|_{V_{\frac{\alpha}{2}}}.$$

Hence,

$$\|u - u_L\|_{V_{\frac{\alpha}{2}}} \leq \|u - \tilde{u}_L\|_{V_{\frac{\alpha}{2}}} + \|\tilde{u}_L - u_L\|_{V_{\frac{\alpha}{2}}} \leq (1 + \kappa_2/\kappa_1) \|u - \tilde{u}_L\|_{V_{\frac{\alpha}{2}}}. \tag{4.5}$$

The following allows us to estimate $\|u - \tilde{u}_L\|_{V_{\frac{\alpha}{2}}}$ using Lemma 4.2. First,

$$\begin{aligned}&\left\|u - \Pi_M^1 \Pi_N^{0,1} u\right\|_{\frac{\alpha}{2},0} \leq \left\|u - \Pi_M^1 u\right\|_{\frac{\alpha}{2},0} \\ &+ \left\|(\Pi_M^1 - I_d)(u - \Pi_N^{0,1} u)\right\|_{\frac{\alpha}{2},0} + \left\|u - \Pi_N^{0,1} u\right\|_{\frac{\alpha}{2},0} \\ &\lesssim c_I M^{\frac{\alpha}{2}-s} \|u\|_{s,0} + c_I c_\Lambda M^{\frac{\alpha}{2}-s} N^{-r} \|u\|_{s,r} + c_\Lambda N^{-r} \|u\|_{\frac{\alpha}{2},r}, \end{aligned} \tag{4.6}$$

where I_d denotes the identity operator.

On the other hand,

$$\begin{aligned}&\left\|u - \Pi_M^1 \Pi_N^{0,1} u\right\|_{0,1} \leq \left\|u - \Pi_N^{0,1} u\right\|_{0,1} + \left\|\Pi_N^{0,1} u - \Pi_N^{0,1} \Pi_M^1 u\right\|_{0,1} \\ &\lesssim c_\Lambda N^{1-r} \|u\|_{0,r} + c_I M^{-s} \|u\|_{s,1}. \end{aligned} \tag{4.7}$$

Here the second inequality employs the orthogonality of $\Pi_N^{0,1}$.

Substituting (4.6) and (4.7) into (4.5), the result (4.4) is obtained by the definition of norm $\|\cdot\|_{V_{\frac{\alpha}{2}}}$. ∎

Remark 4.1. From the above convergence theorem we know that the approximation orders of the space-time spectral method for solving the time fractional diffusion equation and convection-diffusion equation are the same (see [23]), which we can see from the numerical examples in section 6.

5. Implementation of the spectral method. In this section, we shall consider the implementation of the space-time spectral method. It is worth noting that the implementation of the space-time spectral method is challenging and requires special consideration, which we now discuss.

5.1. Fourier-like basis function and spatial discretization. First, we consider the basis function for space discretization. Denote by $L_k(x)$ the Legendre orthogonal polynomials and set

$$z_k(x) = \lambda_k(L_k(\hat{x}) - L_{k+2}(\hat{x})),\ \hat{x} = \frac{2}{b-a}\left(x - \frac{b+a}{2}\right),$$

for $k = 0, 1, \ldots, N-2$, $x \in (a, b)$, and λ_k to be determined. By the properties of the Legendre polynomials,

$$(z_k, z_l)_\Lambda = g_{kl} = \begin{cases} -\frac{b-a}{2k+1}\lambda_k\lambda_{k-2}, & l = k-2, \\ 0, & l = k-1, \\ \frac{(b-a)(4k+6)}{(2k+1)(2k+5)}\lambda_k^2, & l = k, \\ 0, & l = k+1, \\ -\frac{b-a}{2k+5}\lambda_k\lambda_{k+2}, & l = k+2, \\ 0, & |l-k| > 2, \end{cases} \tag{5.1}$$

and

$$(z_k'', z_l)_\Lambda = -\frac{4(2k+3)}{b-a}\lambda_k^2\delta_{kl}, \tag{5.2}$$

where δ_{kl} is the Kronecker symbol. Thus, $G = (g_{ij})_{(N-1)\times(N-1)}$ is a symmetric positive definite pentadiagonal matrix. The equality (5.1) can be verified easily. For the equality (5.2), we note the fact (see [37])

$$L_{n+1}'(x) - L_{n-1}'(x) = (2n+1)L_n(x).$$

The equality (5.2) follows immediately by the orthogonality of the Legendre polynomials.

Now, let us construct the Fourier-like basis function using the process of [38]. Choose $\lambda_k = \sqrt{b-a/4(2k+3)}$ so that $(z_k', z_l')_\Lambda = \delta_{kl}$. Let $\tau_i (0 \le i \le N-2)$ be the eigenvalues of G and $q_i = (q_{0i}, q_{1i}, \ldots, q_{N-2,i})^T$ the eigenvector corresponding to τ_i. Denote $\Pi = \mathrm{diag}(\tau_0, \tau_1, \ldots, \tau_{N-2})$, $Q = (q_0, q_1, \ldots, q_{N-2})$. We have $GQ = Q\Pi$, and $Q^TQ = I_{N-1}$, in which I_{N-1} is the identity matrix of $(N-1)\times(N-1)$. Taking

$$h_k(x) = \sum_{j=0}^{N-2} q_{jk} z_j(x), k = 0, 1, \ldots, N-2,$$

then $\mathring{P}_N(\Lambda) = \mathrm{span}\{h_0(x), h_1(x), \ldots, h_{N-2}(x)\}$, it can be proved that the following proposition holds.

PROPOSITION 5.1. *For the basis functions $h_k(x), x \in \Lambda = (a, b)$,*

$$(h_k(x), h_l(x))_\Lambda = \tau_k\delta_{kl}, \quad (h_k''(x), h_l(x))_\Lambda = -\delta_{kl}.$$

Proof. Due to the symmetry of G, we have

$$\begin{aligned}(h_k(x), h_l(x))_\Lambda &= \sum_{i=0}^{N-2}\sum_{j=0}^{N-2} q_{ik}q_{jl}(z_i(x), z_j(x))_\Lambda \\ &= \sum_{i=0}^{N-2}\sum_{j=0}^{N-2} q_{ik}q_{jl}g_{ij} = \sum_{i=0}^{N-2}\sum_{j=0}^{N-2} q_{jl}g_{ji}q_{ik} \\ &= (Q^TGQ)_{lk} = \Pi_{lk}.\end{aligned}$$

Similarly, the second equation can be verified. ∎

5.2. The temporal discretization. Based on the properties of the Jacobi polynomials and the ideas from [23], we choose the following basis function for time discretization:

$$\phi_j(t) = J_j^{-\alpha/2,0}\left(\frac{2t}{T}-1\right) + J_{j-1}^{-\alpha/2,0}\left(\frac{2t}{T}-1\right), j = 1, 2, \ldots, M,$$

and

$$\psi_n(t) = \frac{n}{n-\alpha/2} J_n^{0,-\alpha/2}\left(\frac{2t}{T}-1\right) + J_{n-1}^{0,-\alpha/2}\left(\frac{2t}{T}-1\right), n = 1, 2, \ldots, M.$$

It is easy to verify that

$$\begin{aligned}\mathring{P}_M(I) &= \operatorname{span}\{\phi_1(t), \phi_2(t), \ldots, \phi_M(t)\}\\ &= \operatorname{span}\{\psi_1(t), \psi_2(t), \ldots, \psi_M(t)\}.\end{aligned}$$

5.3. Implementation. Consider the approximation function space $W_L = \operatorname{span}\{\phi_i(t)h_j(x), 1 \le i \le M, 0 \le j \le N-2\}$. Set

$$u_L = \sum_{i=1}^{M}\sum_{j=0}^{N-2} u_{ij}\phi_i(t)h_j(x), \quad v_L = \psi_m(t)h_n(x),$$

where $m = 1, 2, \ldots, M$, $n = 0, 1, \ldots, N-2$.

Now, the Galerkin spectral equation (4.1) is written as

$$({}_0D_t^{\frac{\alpha}{2}}u_L, {}_tD_T^{\frac{\alpha}{2}}v_L)_\Omega + K_\alpha(\partial_x u_L, \partial_x v_L)_\Omega - (\partial_x(pu_L), v_L)_L = (f, v_L)_L, \tag{5.3}$$

where $(\cdot,\cdot)_L$ denotes numerical integration. Substituting the expansions of u_L and v_L into (5.3) we obtain the following linear system of dimension $M \times (N-1)$:

$$AU = F \tag{5.4}$$

with $U = (u_{10}, u_{20}, \ldots, u_{M0}, u_{11}, u_{21}, \ldots, u_{M2}, \ldots, u_{1,N-2}, u_{2,N-2}, \ldots, u_{M,N-2})^T$. Due to the nonsymmetric nature of A, system (5.4) can be solved by the BiCGSTAB method [37]. It can be seen that the computational magnitude of the system (5.4) is $O(M^2N^2)$.

In what follows we consider the computation of A and F. Notice that by Proposition 2.6

$$\begin{aligned}{}_0D_t^{\frac{\alpha}{2}}\phi_i(t) &= \left(\frac{\Gamma(i+1)}{\Gamma(i+1-\frac{\alpha}{2})}J_i^{0,-\frac{\alpha}{2}}\Big(\frac{2t}{T}-1\Big) + \frac{\Gamma(i)}{\Gamma(i-\frac{\alpha}{2})}J_{i-1}^{0,-\frac{\alpha}{2}}\Big(\frac{2t}{T}-1\Big)\right)t^{-\frac{\alpha}{2}}\\ &= \widetilde{\phi}_i(t)t^{-\frac{\alpha}{2}}\end{aligned}$$

and

$$\begin{aligned}{}_tD_T^{\frac{\alpha}{2}}\psi_m(t) &= \left(\frac{m\Gamma(m+1)}{(m-\frac{\alpha}{2})\Gamma(m+1-\frac{\alpha}{2})}J_m^{-\frac{\alpha}{2},0}\Big(\frac{2t}{T}-1\Big)\right.\\ &\quad \left. + \frac{\Gamma(m)}{\Gamma(m-\frac{\alpha}{2})}J_{m-1}^{-\frac{\alpha}{2},0}\Big(\frac{2t}{T}-1\Big)\right)(T-t)^{-\frac{\alpha}{2}}\\ &= \widetilde{\psi}_m(t)(T-t)^{-\frac{\alpha}{2}},\end{aligned}$$

where

$$\widetilde{\phi}_i(t) = \left(\frac{\Gamma(i+1)}{\Gamma(i+1-\frac{\alpha}{2})} J_i^{0,-\frac{\alpha}{2}}\left(\frac{2t}{T}-1\right) + \frac{\Gamma(i)}{\Gamma(i-\frac{\alpha}{2})} J_{i-1}^{0,-\frac{\alpha}{2}}\left(\frac{2t}{T}-1\right)\right)$$
$$\widetilde{\psi}_m(t) = \left(\frac{m\Gamma(m+1)}{(m-\frac{\alpha}{2})\Gamma(m+1-\frac{\alpha}{2})} J_m^{-\frac{\alpha}{2},0}\left(\frac{2t}{T}-1\right) + \frac{\Gamma(m)}{\Gamma(m-\frac{\alpha}{2})} J_{m-1}^{-\frac{\alpha}{2},0}\left(\frac{2t}{T}-1\right)\right).$$

Therefore,

$$(5.5)\quad ({}_0D_t^{\frac{\alpha}{2}} u_L, {}_tD_T^{\frac{\alpha}{2}} v_L)_\Omega = \sum_{i=1}^{M}\sum_{j=0}^{N-2} u_{ij}(h_j(x), h_n(x))_\Lambda ({}_0D_t^{\frac{\alpha}{2}}\phi_i(t), {}_tD_T^{\frac{\alpha}{2}}\psi_m(t))_I$$
$$= \sum_{i=1}^{M}\sum_{j=0}^{N-2} u_{ij}(\widetilde{\phi}_i(t), \widetilde{\psi}_m(t))_{\omega,I}\tau_j\delta_{jn},$$

where the weighted inner product is

$$(\widetilde{\phi}_j(t), \widetilde{\psi}_m(t))_{\omega,I} = \int_0^T \widetilde{\phi}_j(t)\widetilde{\psi}_m(t) t^{-\frac{\alpha}{2}}(T-t)^{-\frac{\alpha}{2}} dt.$$

Notice that $\widetilde{\phi}_j(t)\widetilde{\psi}_m(t)$ is a polynomial of degree not larger than $2M$. Thus, the weighted inner product can be obtained exactly by the Jacobi–Gauss quadrature.

Similarly,

$$(\partial_x u_L, \partial_x v_L)_\Omega = \sum_{i=1}^{M}\sum_{j=0}^{N-2} u_{ij}(h_j', h_n')_\Lambda(\phi_i(t), \psi_m(t))_I$$
$$(5.6)\qquad = \sum_{i=1}^{M}\sum_{j=0}^{N-2} u_{ij}(\phi_i(t), \psi_m(t))_I\delta_{jn}.$$

Since $\phi_i(t)$ and $\psi_m(t)$ are polynomials, $(\phi_i(t), \psi_m(t))_I$ can be obtained exactly by the Legendre–Gauss quadrature.

Finally,

$$(\partial_x(pu_L), v_L)_L = \sum_{i=1}^{M}\sum_{j=0}^{N-2} u_{ij}(\phi_i(t), \psi_m(t))_I(\partial_x(ph_j), h_n)_N.$$

Thus, using the Legendre–Gauss quadrature, $(\partial_x(ph_j), h_n)_N$ and $(f, v_L)_L$ are obtained. Thus, the system (5.4) is constructed.

Furthermore, the discretization scheme of the time fractional diffusion equation, which need not consider the third term of the right-hand side of (5.3), has a simpler form $AU = F$, where A is a block-diagonal matrix with

$$A = \begin{pmatrix} \tau_0 B_1 & & & \\ & \tau_1 B_1 & & \\ & & \ddots & \\ & & & \tau_{N-2}B_1 \end{pmatrix} + K_\alpha \begin{pmatrix} B_2 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_2 \end{pmatrix}.$$

Here, $B_1 = (b_{ij})_{M\times M}$, $B_2 = (d_{ij})_{M\times M}$ and

$$b_{ij} = (\widetilde{\phi}_i(t), \widetilde{\psi}_j(t))_{\omega,I},\ d_{ij} = (\phi_i(t), \psi_j(t))_I.$$

Therefore, the linear system (5.3) can be decomposed into $(N-1)$ uncoupled linear systems, and the computational cost is $O(M^2N)$.

Remark 5.1.

1. Our method may be extended to solve the time fractional advection-dispersion equations. The resulting scheme is unconditionally stable and spectrally accurate in time and space.
2. For long-time problems, the space-time spectral method should be implemented as a time marching scheme. We partition the time interval $[0,T]$ into K nonoverlapping subintervals and apply the space-time spectral method on each subinterval sequentially with a relatively small M to reduce the computational complexity.

6. Numerical examples. In this section, we shall examine three numerical examples for illustrating the high order accuracy which the space-time spectral method possesses for solving the time fractional diffusion and Fokker–Planck equations.

We first consider a time fractional diffusion equation to investigate the numerical convergence order and efficiency of our method. Next, we study the practical implementation of our method for a convection-diffusion equation—the Fokker–Planck equation—of different order. We consider three kinds of derivative, i.e., $\alpha = 0.9, 0.5, 0.1$, and investigate the relationship of the approximation orders between the time fractional diffusion and convection-diffusion equations (see Remark 4.1). Finally, we consider the Fokker–Planck equation that has an advective term much stronger than the diffusive one, endowed with nonhomogeneous boundary and initial conditions. We define the approximation order and convergence order by

$$\mathrm{AO} = \frac{\log(error(M))}{\log M^{-1}} \text{ or } \frac{\log(error(N))}{\log N^{-1}},$$
$$\mathrm{CO} = \frac{\log(error(M_1)/error(M_2))}{\log(M_2/M_1)} \text{ or } \frac{\log(error(N_1)/error(N_2))}{\log(N_2/N_1)},$$

where $error(N)$ denotes the error in certain norm corresponding to the polynomial degree N. When $\mathrm{AO} = r$, it means that the numerical solution has an approximation order $O(M^{-r})$ $(O(N^{-r}))$. Similarly, if $\mathrm{CO} = s$, then the numerical convergence order behaves like $O(M^{-s})$ $(O(N^{-s}))$.

The numerical solutions of the following examples are obtained by the BiCGSTAB method using zero vector as the initial solution approximation. All the computations are carried out on an ASUS laptop with the following configuration: Intel Core i3-3217U CPU, 1.80 GHz and 4.00 G RAM, with 64 bits operation system.

Example 6.1. Consider the following time fractional diffusion equation:

$$\begin{cases} \frac{\partial u}{\partial t} = {}_0D_t^{1-\alpha}\left\{\partial_{xx}u(x,t) + f(x,t)\right\}, (x,t) \in (0,1)\times(0,1], \\ u(x,0) = 0, x \in (0,1), \\ u(0,t) = u(1,t) = 0, t \in (0,1], \end{cases} \tag{6.1}$$

where

$$f(x,t) = \pi^2 t^6 \sin \pi x + \frac{\Gamma(7)t^{6-\alpha}\sin \pi x}{\Gamma(7-\alpha)}.$$

The exact solution is $u(x,t) = t^6 \sin \pi x$. Equation (6.1) has been considered in [23] by the space-time spectral method. Here, we aim to consider the convergence

TABLE 1
L^∞- and L^2-errors versus N, convergence order, and CPU time of problem (6.1), $M = 20$, $\alpha = 0.5$

N	$\|u - u_L\|_{L^\infty}$	Order	$\|u - u_L\|_0$	Order	CPU time (sec)
5	9.0193e-04		2.5366e-03		0.3191s
7	9.9782e-06	13.3863	2.7664e-05	13.4288	0.3651s
9	6.8547e-08	19.8183	1.8826e-07	19.8558	0.4001s
11	3.2073e-10	26.7337	9.0031e-10	26.6249	0.4014s
13	1.0749e-12	34.1109	3.1848e-12	33.8754	0.4385s

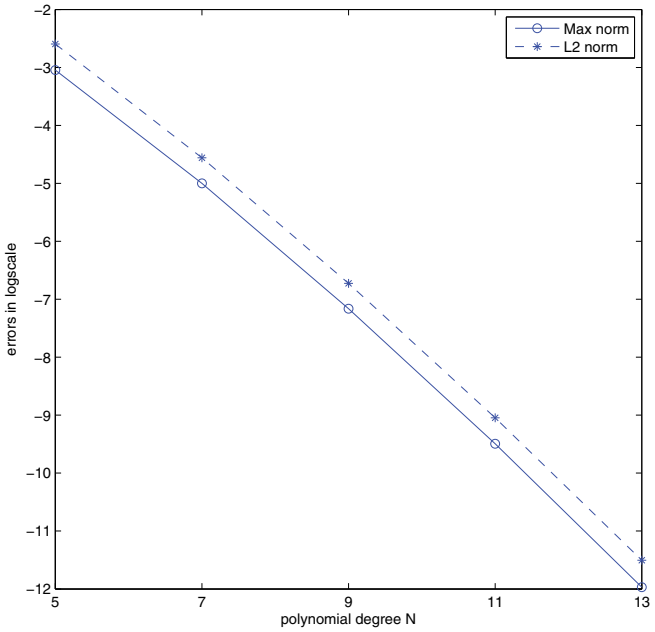

FIG. 1. L^∞- and L^2-errors versus N of problem (6.1) with $M = 20$ when $\alpha = 0.5$.

order and implementation efficiency of our method for solving the fractional diffusion equation.

We consider the case of $\alpha = 0.5$ and take $M = 20$ in order that the temporal discretization error is negligible compared with the spatial discretization error. The relationship of error with the degree N of the polynomials and CPU time is shown in Table 1. In Figure 1 we plot the log-linear error of the numerical solution as a function of the polynomial degree N. This highlights that the errors show an exponential decay as expected, since in this log-linear representation one may observe that the error variations are essentially linear versus the degree of the polynomial.

Next, we investigate the temporal discretization errors. We fix $N = 20$, a value large enough such that the spatial errors are negligible compared with the temporal errors. The relationship of the temporal errors with the polynomial degree M and CPU time is shown in Table 2. Similarly, we can see the almost exponential decay of the temporal errors from Figure 2 because the graphs are approximately linear.

TABLE 2
L^∞- and L^2-errors versus M, convergence order, and CPU time of problem (6.1), $N = 20$, $\alpha = 0.5$

M	$\|u - u_L\|_{L^\infty}$	Order	$\|u - u_L\|_0$	Order	CPU time (sec)
7	2.5281e-08		7.9946e-08		0.3159s
9	1.3347e-09	11.7038	4.2209e-09	12.5921	0.3161s
11	1.2743e-10	11.7052	4.0328e-10	11.7016	0.3475s
13	1.7746e-11	11.8010	5.6109e-11	11.8065	0.3687s
15	2.9678e-12	12.4970	9.8545e-12	12.1548	0.3725s

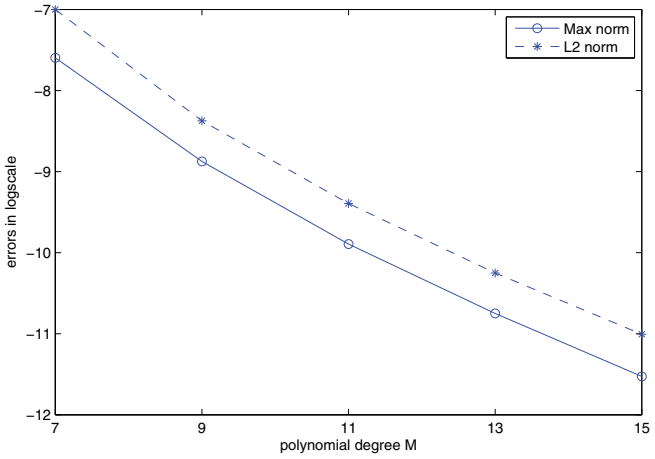

FIG. 2. *L^∞- and L^2-errors versus M of problem* (6.1) *with $N = 20$ when $\alpha = 0.5$.*

Additionally, Tables 1 and 2 tell us the convergence orders of the space-time spectral method. Due to exponential decay, the convergence orders in spatial discretization become bigger and bigger. On the other hand, because of the limited regularity of f in time, the convergence order in temporal discretization behaves like $O(M^{-12.0})$ in L^2-norm. Anyway, the implementation efficiency of our method is remarkably superior to one of the FDMs (e.g., see [32, 49]).

Now, we investigate the numerical solution of the following Fokker–Planck equation using the space-time spectral method.

Example 6.2. Consider the following equation:

$$\begin{cases} \frac{\partial u}{\partial t} = {}_0D_t^{1-\alpha}\left\{\partial_x(-u(x,t)) + \partial_{xx}u(x,t) + f(x,t)\right\}, (x,t) \in (0,1)\times(0,1], \\ u(x,0) = 0, x \in (0,1), \\ u(0,t) = u(1,t) = 0, t \in (0,1], \end{cases} \tag{6.2}$$

where

$$f(x,t) = \pi^2 t^6 \sin \pi x + \pi t^6 \cos \pi x + \frac{\Gamma(7)t^{6-\alpha}\sin \pi x}{\Gamma(7-\alpha)}.$$

Equation (6.2) has been studied in [9]. It is easy to verify that the exact solution is $u(x,t) = t^6 \sin \pi x$. Here, we investigate its numerical solutions in three cases of order $\alpha = 0.9, 0.5, 0.1$. First, fixing $M = 20$ and considering the $\|\cdot\|_{V_{\alpha/2}}$-errors versus N, the errors and numerical orders are shown in Table 3. From Figure 3, we can see that $\|\cdot\|_{V_{\alpha/2}}$-errors exponentially decay with the degree increasing.

TABLE 3
$\|\cdot\|_{V_{\alpha/2}}$-errors versus N and convergence order of problem (6.2), $M = 20$, $\alpha = 0.9, 0.5, 0.1$.

N	$\alpha = 0.9$		$\alpha = 0.5$		$\alpha = 0.1$	
	$\|u - u_L\|_{V_{\alpha/2}}$	Order	$\|u - u_L\|_{V_{\alpha/2}}$	Order	$\|u - u_L\|_{V_{\alpha/2}}$	Order
5	5.8698e-02		5.7033e-02		5.5981e-02	
7	9.6490e-04	12.1813	9.4645e-04	12.2094	9.3486e-04	12.1626
9	9.0310e-06	18.5669	8.9047e-06	18.5877	8.8256e-06	18.5534
11	5.2974e-08	25.5937	5.2375e-08	25.6072	5.1998e-08	25.5852
13	2.2285e-10	32.8262	2.1755e-10	32.7502	2.1667e-10	32.8072

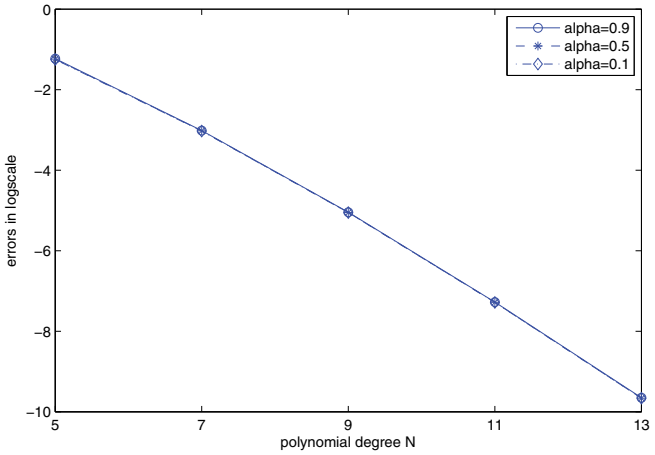

FIG. 3. $\|\cdot\|_{V_{\alpha/2}}$-errors versus N of problem (6.2) with $M = 20$ when $\alpha = 0.9, 0.5, 0.1$.

Second, fix $N = 20$ and let M vary from $7, 9, 11, 13, 15$. The errors in $\|\cdot\|_{V_{\alpha/2}}$-norm and numerical convergence order are listed on Table 4. The almost exponential decaying can been seen similarly in Figure 4.

The above results show that the space-time spectral method has a high convergence order for solving the fractional Fokker–Planck equation and is superior to the method proposed in [9, 10]. According to the theoretical analysis, the time fractional diffusion equation and advection diffusion equation have the same approximation order. In order to illustrate the analytical result, we list the approximation orders in $\|\cdot\|_{V_{\alpha/2}}$-norm in Table 5 as $\alpha = 0.5$. The results agree well with the theoretical analysis. From the approximation order we can also see that the choice of $N = 20$ or $M = 20$ is reasonable to study the convergence order versus M or N.

Our methods also can be used for solving nonhomogeneous initial-boundary problems with high convergence order. The following example is a Fokker–Planck equation in which the advective term is stronger than the diffusive one.

Example 6.3. Consider the following problem with nonhomogeneous datum:

$$\begin{cases} \frac{\partial u}{\partial t} = {}_0D_t^{0.9}\left\{\partial_x[-u(x,t)] + \frac{1}{\pi^2}\partial_{xx}u(x,t) + f(x,t)\right\}, (x,t) \in (0,1)\times(0,1], \\ u(x,0) = \sin \pi x, x \in (0,1), \\ u(0,t) = u(1,t) = 0, t \in (0,1], \end{cases}$$

where

$$f(x,t) = (1+t)(\sin \pi x + \pi \cos \pi x) + \frac{t^{0.9}}{\Gamma(1.9)} \sin \pi x.$$

TABLE 4
$\|\cdot\|_{V_{\alpha/2}}$-errors versus M and convergence order of problem (6.2), $N = 20$, $\alpha = 0.9, 0.5, 0.1$.

M	$\alpha = 0.9$		$\alpha = 0.5$		$\alpha = 0.1$	
	$\|u - u_L\|_{V_{\alpha/2}}$	Order	$\|u - u_L\|_{V_{\alpha/2}}$	Order	$\|u - u_L\|_{V_{\alpha/2}}$	Order
7	1.1380e-07		5.4872e-07		1.0823e-07	
9	5.8437e-09	11.8142	3.0965e-08	11.4388	5.1896e-09	12.0869
11	5.4579e-10	11.8148	3.1325e-09	11.4168	4.8086e-10	11.8544
13	7.5128e-11	11.8707	4.5871e-10	11.5003	6.6622e-11	11.8317
15	1.3155e-11	12.1760	8.7022e-11	11.6160	1.2001e-11	11.9779

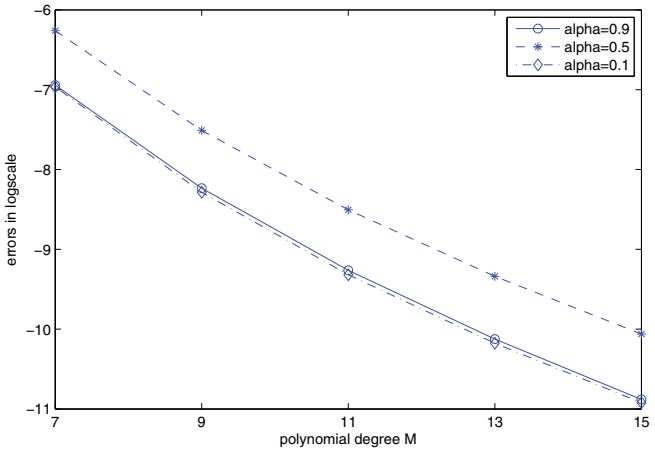

FIG. 4. *$\|\cdot\|_{V_{\alpha/2}}$-errors versus M of problem (6.2) with $N = 20$ when $\alpha = 0.9, 0.5, 0.1$.*

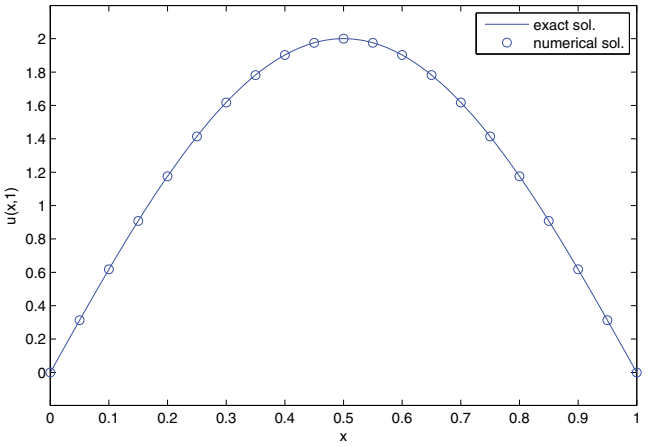

FIG. 5. *The numerical solution and the exact solution at $t = 1$ of problem (6.3), taking $M = N = 11$.*

One can obtain that the exact solution is $u(x,t) = (1+t)\sin \pi x$. We make use of the space-time spectral method for solving this nonhomogeneous initial datum

TABLE 5

Comparison of the approximation order in $\|\cdot\|_{V_{\alpha/2}}$-norm between the fractional diffusion equation (6.1) *and advection-diffusion equation* (6.2) *versus M and N as N and M fixed, respectively, $\alpha = 0.5$*

$N = 20$,M	FDE-AO	FADE-AO	$M = 20$,N	FDE-AO	FADE-AO
7	$M^{-7.4076}$	$M^{-7.4082}$	5	$N^{-1.7807}$	$N^{-1.7617}$
9	$M^{-7.8692}$	$M^{-7.8692}$	7	$N^{-3.5788}$	$N^{-3.5682}$
11	$M^{-8.1665}$	$M^{-8.1661}$	9	$N^{-5.2929}$	$N^{-5.2861}$
13	$M^{-8.3840}$	$M^{-8.3832}$	11	$N^{-6.9917}$	$N^{-6.9867}$
15	$M^{-8.5555}$	$M^{-8.5541}$	13	$N^{-8.6730}$	$N^{-8.6647}$

TABLE 6

CPU time (seconds) on Intel Core i3-3217U CPU of problem (6.3). *Here, M and N denote the degree of the spectral polynomials in time and space variable, respectively, and τ and h denote the time and space step size, respectively.*

L^2-norm	GL-BDIA		ST-SM	
	$\tau = h = n^{-1}$	Time (seconds)	M and N	Time (seconds)
$O(10^{-3})$	$n = 2^5$	0.24s	$M = N = 5$	2.12s
$O(10^{-4})$	$n = 2^8$	55.67s	$M = N = 6$	3.67s
$O(10^{-5})$	$n = 2^{11}$	515.04s	$M = N = 8$	6.72s
$O(10^{-6})$	$n = 2^{14}$	$> 1h$	$M = N = 9$	8.67s
$O(10^{-8})$	–	–	$M = N = 11$	10.83s

problem. The numerical solution (numerical sol.) and the exact solution (exact sol.) at $t = 1$ are shown in Figure 5.

Additionally, in order to highlight the efficiency of the present method, we compare the CPU time in the Grünwald–Letnikov backward difference approximation (GL-BDIA) [9] and the space-time spectral scheme (ST-SM) for solving this nonhomogeneous problem in Table 6. From Table 6, the GL-BDIA scheme cannot achieve the error level $O(10^{-8})$ because of the lack of memory. However, the efficiency of the ST-SM scheme can be seen as well as the high order accuracy.

Obviously, for the problems with advective terms dominating the diffusive ones, the implementation of space-time spectral method needs much time in comparison with the fractional diffusion equation. In practice this situation is also difficult to treat for standard advection-diffusion equations, giving rise to instability phenomena if the spatial grid size is not small enough. However, the space-time spectral method is advisable for solving such problems by choosing a suitable preconditioner.

7. Conclusion. In this paper, we investigate the application of space-time spectral methods for solving the time fractional Fokker–Planck equation. In general, the temporal discretization of the time fractional partial differential equation is obtained by a finite difference scheme. This implementation leads to a lower order accuracy, $O(\tau^{2-\alpha})$ for the case of the fractional derivative of order α, and makes the computational efficiency unsatisfactory. It is well known that the fractional derivative possesses long-range dependence. Thus, pursuing a numerical scheme of high order accuracy is a very interesting issue for numerically solving a fractional differential equation. Our method in this paper presents an exponential decay in numerical errors and enhances the computational efficiency for solving the fractional differential equation. This method is more efficient for solving a subdiffusion problem than the scheme developed in [23]. Numerical examples are in agreement with the results of our theoretical analysis, which also show high order accuracy and efficiency. However,

it is noted that the spectral discretization would cause a full stiffness matrix to be generated. In the case of the time fractional advection diffusion, a preconditioner is required for solving the full linear system.

In upcoming work, we shall investigate the numerical solution of the time-space fractional diffusion equation based on the space-time spectral method. Also, we shall extend the space-time spectral method to the high dimensional cases of time fractional diffusion equations.

Appendix. In the appendix, we shall prove the formulae for the fractional derivative of a shifted Jacobi polynomial. First, a well-known result (see Theorem 6.4 of [31]) is introduced in the following.

THEOREM A.1. *If $0 < \nu < 1$, γ is an arbitrary real number, and r and k are integer numbers such that $r > -1 + \gamma - \nu/2$, $k > -1 - \gamma - \nu/2$, then for $-1 < t < 1$ it holds that*

$$\int_{-1}^{1} \left(sign(t-\tau) + \frac{\tan \pi\gamma}{\tan \frac{\pi\nu}{2}} \right) \frac{Q_m^{-\gamma+\nu/2+r,\gamma+\nu/2+k}(\tau)}{|t-\tau|^\nu} d\tau$$
$$= \frac{\pi(-1)^{r+k+1} \sin \pi(\gamma - \nu/2) 2^{r+k+1} \Gamma(m+\nu)}{\Gamma(m+1)\Gamma(\nu) \sin \frac{\pi\nu}{2} \cos \pi\gamma \sin \pi(-\gamma + \nu/2 - k)} J_{m+r+k+1}^{\gamma+\nu/2-r-1,-\gamma+\nu/2-k-1}(t),$$

where $r + k + m + 1 \geq 0, m \in \mathbb{N}$, $Q_m^{\alpha,\beta} = J_m^{\alpha,\beta}(x)(1-x)^\alpha(1+x)^\beta$. By Theorem A.1, one obtains the next theorem.

THEOREM A.2. *If $0 < \nu < 1$, and r and k are integer numbers such that $r > -1$, $k > -1 - \nu$, then for $-1 < t < 1$ the following hold:*

$$\int_{-1}^{t} \frac{Q_m^{r,\nu+k}(\tau)}{(t-\tau)^\nu} d\tau = \frac{\pi(-1)^r 2^{r+k+1} \Gamma(m+\nu)}{m!\Gamma(\nu) \sin \pi\nu} J_{m+r+k+1}^{\nu-r-1,-k-1}(t), \tag{A.1}$$

$$\int_{t}^{1} \frac{Q_m^{r+\nu,k}(\tau)}{|t-\tau|^\nu} d\tau = \frac{\pi(-1)^{r+1} 2^{r+k+1} \Gamma(m+\nu)}{m!\Gamma(\nu) \sin \pi\nu} J_{m+r+k+1}^{-r-1,\nu-k-1}(t). \tag{A.2}$$

In particular,

$$\int_{-1}^{t} \frac{Q_m^{0,\nu-1}(\tau)}{(t-\tau)^\nu} d\tau = \frac{\pi\Gamma(m+\nu)}{m!\Gamma(\nu) \sin \pi\nu} J_m^{\nu-1,0}(t), \tag{A.3}$$

$$\int_{t}^{1} \frac{Q_m^{-1+\nu,0}(\tau)}{|t-\tau|^\nu} d\tau = \frac{\pi\Gamma(m+\nu)}{m!\Gamma(\nu) \sin \pi\nu} J_m^{0,\nu-1}(t). \tag{A.4}$$

Proof. Equation (A.1) is exactly Theorem 6.9 of [31], which is derived by taking the limit of $\gamma \to \frac{\nu}{2}$. Equation (A.2) follows by taking $\gamma = -\frac{\nu}{2}$ in Theorem A.1. ∎

Taking $\nu = 1 - \alpha$ in (A.3) and (A.4) yields Proposition 2.3. Now we present the proof of Proposition 2.5.

Proof of Proposition 2.5. In terms of Proposition 2.3,

$${}_0I_t^\alpha Q_m^{0,-\alpha}\left(\frac{2t}{T} - 1\right) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{Q_m^{0,-\alpha}\left(\frac{2t'}{T} - 1\right)}{(t-t')^{1-\alpha}} dt'.$$

A variable transform $\tau = \frac{2t'}{T} - 1$ gives

$$
\begin{aligned}
{}_0I_t^{\alpha} Q_m^{0,-\alpha}\left(\frac{2t}{T}-1\right) &= \frac{1}{\Gamma(\alpha)} \int_{-1}^{\frac{2t}{T}-1} \frac{(\frac{T}{2})^{\alpha} Q_m^{0,-\alpha}(\tau)}{(\frac{2t}{T}-1-\tau)^{1-\alpha}} d\tau \\
&= \frac{T^{\alpha}}{2^{\alpha}\Gamma(\alpha)} \frac{\pi\Gamma(m+1-\alpha)}{m!\Gamma(1-\alpha)\sin\pi(1-\alpha)} J_m^{-\alpha,0}\left(\frac{2t}{T}-1\right).
\end{aligned}
$$

Here, equality (A.3) is used. Notice that

$$
\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin\pi\alpha},
$$

so we obtain

$$
{}_0I_t^{\alpha} Q_m^{0,-\alpha}\left(\frac{2t}{T}-1\right) = \frac{T^{\alpha}\Gamma(m+1-\alpha)}{2^{\alpha}\Gamma(m+1)} J_m^{-\alpha,0}\left(\frac{2t}{T}-1\right).
$$

Similarly,

$$
\begin{aligned}
{}_tI_T^{\alpha} Q_m^{-\alpha,0}\left(\frac{2t}{T}-1\right) &= \frac{1}{\Gamma(\alpha)} \int_t^T \frac{Q_m^{-\alpha,0}\left(\frac{2t'}{T}-1\right)}{(t'-t)^{1-\alpha}} dt' \\
&= \frac{1}{\Gamma(\alpha)} \int_{\frac{2t}{T}-1}^{T} \frac{(\frac{T}{2})^{\alpha} Q_m^{-\alpha,0}(\tau)}{(\tau-\frac{2t}{T}+1)^{1-\alpha}} d\tau \\
&= \frac{T^{\alpha}\Gamma(m+1-\alpha)}{2^{\alpha}\Gamma(m+1)} J_m^{0,-\alpha}\left(\frac{2t}{T}-1\right).
\end{aligned}
$$

This finishes the proof of Proposition 2.5.

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