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Research Article

A Novel Homotopy Perturbation Method with Applications to Nonlinear Fractional Order KdV and Burger Equation with Exponential-Decay Kernel

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In this paper, we introduce the Yang transform homotopy perturbation method (YTHPM), which is a novel method. We provide formulae for the Yang transform of Caputo-Fabrizio fractional order derivatives. We derive an algorithm for the solution of Caputo-Fabrizio (CF) fractional order partial differential equation in series form and show its convergence to the exact solution. To demonstrate the novel approach, we include some examples with detailed solutions. We use tables and graphs to compare the exact and approximate solutions.

1. Introduction and Motivation

Noninteger calculus is a popular field that is aimed at explaining real-world phenomena that are modeled with operators of fractional order. It is also a field that uses noninteger order derivatives for differentiation and integration. A fractional derivative is a sort of derivative with a noninteger order that meets specific conditions: we get the primary function when the order is zero, and we get the ordinary derivative when the order is one [1]. The memory effect and conserved illustrative physical properties are two advantages of fractional derivatives. Over time, more accurate and up-to-date studies have been revealed using these types of operators. In this sense, the theory of fractional calculus and its applications are gaining popularity around the world. Fractional order models incorporate all previous knowledge from the past due to the memory effect, making it better for them to forecast and analyze dynamical models more efficiently. Fractional order calculus has a wide range of applications in a variety of areas due to its efficient properties, including biology and physics. [2–4], economics and finance [5, 6], science and engineering [7, 8], and mathematical modeling and mechanics [9–11]. Since the kernel of Caputo and Riemann derivatives is singular, they have a problem with this kernel. Since the kernel is used to explain the physical system's memory effect, it is clear that both derivatives cannot accurately interpret the memory's full effect due to this limitation. Caputo and Fabrizio (CF) [12] suggested a new fractional operator with an exponential kernel in a recent attempt around the middle of the last decade. This derivative's kernel is nonsingular, so the results are more reasonable than the classical one. We include some applications of the CF operator in [13–15].

There are two types of nonlinear equations: linear and nonlinear. Partially differential equations of fractional order are notoriously difficult to solve, and finding an exact solution is even more difficult. In applied mathematics, exact and numerical solutions to this type of equation are necessary. As a result, new techniques for obtaining analytical solutions that are relatively close to the exact solutions have been developed. Integral transformations were often used to solve the differential equations. Integral transforms are useful for solving IVPs and BVPs in differential and integral equations. Several authors implemented various kinds of integral transforms and examined their implications in different types of differential equations. The Laplace transform is the integral transform that is used the most [16]. In 1998, Watugala [17] was successfully introducing Sumudu transform to solve differential equations and control engineering problems. Recently, in 2011, T. Elzaki and S. Elzaki introduced new integral transform "Elzaki Transform" and used heavily in solving partial differential equations [18]. Also in 2013, Aboodh introduces "Aboodh Transform" and applies for solving partial differential equations [19]. There are many transform which are available in literature.

He formulated the homotopy perturbation method (HPM) [20] in 1999, which is a combination of the homotopy method and the classical perturbation technique and has been widely applied on both linear and nonlinear problems [21–23]. The importance of HPM is that it does not need a small parameter in the equation, so it reduces the drawbacks of traditional perturbation methods. The main purpose of this article is to apply newly introduced integral transform called "Yang Transform" discovered by Yang [24] with HPM to solve nonlinear fractional order PDEs. We solve two popular nonlinear PDEs through the proposed method. We obtain a power series solution in the context of a quickly convergent series, and just several iterations are needed to achieve very efficient results. There is no requirement for a method like discretizing the problem and no linearization for the nonlinear problem, and just a few iterations can lead to a solution that can be easily estimated using these techniques.

2. Preliminaries

We give the basic definitions which are needed in the rest of paper. For the sake of simplicity, we write the exponential-decay kernel as$K(\tau, \varrho) = \exp\left[-\alpha(\tau - \varrho/1-\alpha)\right]$.

Definition 1 [12]. If $\mathbb{P}(\tau) \in \mathbf{H}^1[0, T]$, $T > 0$, then the Caputo-Fabrizio (CF) derivative may be expressed as follows:

$$ {}^{CF}D_t^{\alpha}[\mathbb{P}(\tau)] = \frac{N(\alpha)}{1-\alpha}\int_0^{\tau}\mathbb{P}'(\varrho)K(\tau,\varrho)d\varrho. \quad (1)$$

$N(\alpha)$ is the normalization function with $N(1) = N(0) = 1$. However, if $\mathbb{P}(\tau) \notin \mathbf{H}^1[0, T]$, then the above derivative is defined as follows:

$$ {}^{CF}D_t^{\alpha}[\mathbb{P}(\tau)] = \frac{N(\alpha)}{1-\alpha}\int_0^{\tau}[\mathbb{P}(\tau)-\mathbb{P}(\varrho)]K(\tau,\varrho)d\varrho. \quad (2)$$

Definition 2 [12]. The CF fractional integral may be expressed as follows:

$$ {}^{CF}I_t^{\alpha}[\mathbb{P}(\tau)] = \frac{1-\alpha}{N(\alpha)}\mathbb{P}(\tau) + \frac{\alpha}{N(\alpha)}\int_0^{\tau}\mathbb{P}(\varrho)d\varrho, \tau \ge 0, \alpha \in (0,1]. \quad (3)$$

Definition 3 [7]. For $N(\alpha) = 1$, the following result represents the Laplace transform of CF derivative:

$$L\left[{}^{\mathrm{CF}}D_{\tau}^{\alpha}[\mathbb{P}(\tau)]\right] = \frac{\nu L[\mathbb{P}(\tau)] - \mathbb{P}(0)}{\nu + \alpha(1-\nu)}. \quad (4)$$

Definition 4. [24]. The Yang transform of $P(\tau)$ is defined as follows:

$$Y[\mathbb{P}(\tau)] = \chi(\nu) = \int_0^{\infty}\mathbb{P}(\tau)e^{-\tau/\nu}, \tau > 0, \quad (5)$$

where v is transform variable and for some v the integral on the right exists.

Remark 5. Yang transform of some useful functions is given below.

$$\begin{aligned} Y[1] &= \nu, \\ Y[\tau] &= \nu^2, \\ Y[\tau^q] &= \Gamma(q+1)\nu^{q+1}. \end{aligned} \quad (6)$$

3. Main Work

First, we derive formula for Yang transform of Caputo-Fabrizio fractional derivative through the Yang-Laplace duality property. At the end of this section, we give some examples with detailed solution to check the validity and efficiency of the novel method.

Lemma 6 (Laplace-Yang duality). *Let the Laplace transform of $P(\tau)$ is $F(v)$, then $\chi(v) = F(1/v)$.*

Proof. From Equation (5), we can obtain another form of the Yang transform by substituting $-\tau/u = x$ as

$$Y[\mathbb{P}(\tau)] = \chi(\nu) = \nu\int_0^{\infty}\mathbb{P}(\nu x)e^{-x}dx, x > 0. \quad (7)$$

Since $L[\mathbb{P}(\tau)] = F(v)$, this implies that

$$F(\nu) = L[\mathbb{P}(\tau)] = \int_0^{\infty}e^{-\nu\tau}\mathbb{P}(\tau)d\tau. \quad (8)$$

Put $\tau = x/\nu$ in (8), we have

$$F(\nu) = \frac{1}{\nu}\int_0^{\infty}e^{-x}\mathbb{P}\left(\frac{x}{\nu}\right)dx. \quad (9)$$

Thus, from Equation (7), we obtain

$$F(\nu) = \chi\left(\frac{1}{\nu}\right). \quad (10)$$

Also from Equations (5) and (8), we obtain

$$F\left(\frac{1}{\nu}\right) = \chi(\nu). \tag{11}$$

The relations (10) and (11) represent the duality relation between the Yang and Laplace transform. □

Lemma 7. *Let $P(\tau)$ be a continuous function; then, Yang transform of CF derivative of $P(\tau)$ is given by*

$$Y\left[{}^{CF}\mathbb{P}^{\alpha}[\tau]\right] = \frac{Y[\mathbb{P}(\tau)] - \nu\mathbb{P}(0)}{1 + \alpha(\nu - 1)}. \tag{12}$$

Proof. The Laplace transform of the CF fractional is given by

$$L\left[{}^{\mathrm{CF}}\mathbb{P}^{\alpha}(\tau)\right] = \frac{\nu L[\mathbb{P}(\tau)] - \mathbb{P}(0)}{\nu + \alpha(1 - \nu)}. \tag{13}$$

Also, we have that the relation between Laplace and Yang property, i.e., $\chi(\nu) = F(1/\nu)$. To obtain the required result, we replace υ by $1/\nu$ in Equation (13), and we get

$$\begin{aligned} Y\left[{}^{\mathrm{CF}}\mathbb{P}^{\alpha}(\tau)\right] &= \frac{1/\nu Y[\mathbb{P}(\tau)] - \mathbb{P}(0)}{1/\nu + \alpha(1 - 1/\nu)}, \\ Y\left[{}^{\mathrm{CF}}\mathbb{P}^{\alpha}(\tau)\right] &= \frac{Y[\mathbb{P}(\tau)] - \nu\mathbb{P}(0)}{1 + \alpha(\nu - 1)}. \end{aligned} \tag{14}$$

The proof is completed. □

4. Algorithm of the Proposed Method

In this part, we discuss the algorithms of fractional order differential equations involving exponential-decay kernel. We provide some examples with detailed solution and its comparison with exact solutions.

4.1. Application to Caputo-Fabrizio Fractional Differential Equations. First, we develop the solution procedure of general nonlinear Caputo-Fabrizio (CF) fractional partial differential equations through YTHPM. Let us take a general nonlinear CF PDE with nonlinear term $N(G(x,\tau))$ and linear term $L(G(x,\tau))$ as

$$\begin{cases} {}^{\mathrm{CF}}D_{\tau}^{\alpha}G(x,\tau) + L(G(x,\tau)) + N(G(x,\tau)) = g(x,\tau), \\ G(x,0) = h(x), \end{cases} \tag{15}$$

where the term $g(x,\tau)$ represents the source term. Implement Yang transform to Equation (15), and one can achieve

$$\frac{Y[G(x,\tau)] - \nu G(x,0)}{1 + \alpha(\nu - 1)} = -Y[L(G(x,\tau)) + N(G(x,\tau))] + Y[g(x,\tau)],$$

$$\begin{aligned} Y[G(x,\tau)] = \nu h(x) - (1 + \alpha(\nu - 1))[Y[L(G(x,\tau)) \\ + N(G(x,\tau))] + Y[g(x,\tau)]]. \end{aligned} \tag{16}$$

Applying inverse of Yang transform, we achieve

$$G(x,\tau) = \mathcal{G}(x,\tau) - Y^{-1}[(1 + \alpha(\nu - 1))Y[L(G(x,\tau)) + N(G(x,\tau))]], \tag{17}$$

where the term $G(x,\tau)$ represents the source term and the given I.C (initial condition). Now, we utilize HPM:

$$G(x,\tau) = \sum_{q=0}^{\infty} \rho^q G_q(x,\tau). \tag{18}$$

We decompose the nonlinear term $N(G(x,\tau))$ as

$$N(G(x,\tau)) = \sum_{q=0}^{\infty} \rho^q H_q(G), \tag{19}$$

where $H_q(G)$ represents the He's polynomial and is calculated through the formula:

$$H_q(G_1, G_2, \cdots, G_q) = \frac{1}{\Gamma(q+1)} \frac{\partial^q}{\partial \rho^q}\left[N\left(\sum_{i=0}^{\infty} \rho^i G_i\right)\right]_{\rho=0} \quad q = 0, 1, \cdots. \tag{20}$$

Putting Equations (18) and (19) in Equation (17), we achieve

$$\begin{aligned} \sum_{q=0}^{\infty} \rho^q G_q(x,\tau) = \mathcal{G}(x,\tau) - \rho\left(Y^{-1}\left[(1 + \alpha(\nu - 1))Y\left[L\sum_{q=0}^{\infty} \rho^q G_q(x,\tau)\right.\right.\right. \\ \left.\left.\left. + \sum_{q=0}^{\infty} \rho^q H_q(G) = \right]\right]\right). \end{aligned} \tag{21}$$

We achieve the following terms by comparing coefficients of ρ in (21):

$$\begin{aligned} \rho^0 &: G_0(x,\tau) = \mathcal{G}(x,\tau), \\ \rho^1 &: G_1(x,\tau) = Y^{-1}[(1 + \alpha(\nu - 1))Y[L(G_0(x,\tau)) + H_0(G)]], \\ \rho^2 &: G_2(x,\tau) = Y^{-1}[(1 + \alpha(\nu - 1))Y[L(G_1(x,\tau)) + H_1(G)]], \\ \rho^3 &: G_3(x,\tau) = Y^{-1}[(1 + \alpha(\nu - 1))Y[L(G_2(x,\tau)) + H_2(G)]], \\ &\vdots \\ \rho^q &: G_q(x,\tau) = Y^{-1}\left[(1 + \alpha(\nu - 1))Y\left[L\left(G_q(x,\tau)\right) + H_q(G)\right]\right]. \end{aligned} \tag{22}$$

Thus, we may write the acquired solution of Equation (15) as follows:

$$G(x,\tau) = G_0(x,\tau) + G_1(x,\tau) + \cdots. \tag{23}$$

4.2. Convergence and Error Analysis. The following theorems are based on the method's mechanism and address the original problem's (15) convergence and error analysis.

Theorem 8. *Let $G(x,\tau)$ be the exact solution of (15) and let $G_q(x,\tau), G_n(x,\tau) \in H$ and $\sigma \in (0,1)$, where H denotes the Hilbert space. Then, the obtained solution $\sum_{q=0}^{\infty} G_q(x,\tau)$ will converge $G(x,\tau)$ if $G_q(x,\tau) \le \sigma G_{q-1}(x,\tau) \forall q > A$, i.e., for any $\omega > 0 \exists A > 0$, such that $\|G_{q+n}(x,\tau)\| \le \beta$, $\forall m, n \in N$.*

Proof. We make a sequence of $\sum_{q=0}^{\infty} G_q(x,\tau)$.

$$\begin{aligned}
C_0(x,\tau) &= G_0(x,\tau),\\
C_1(x,\tau) &= G_0(x,\tau) + G_1(x,\tau),\\
C_2(x,\tau) &= G_0(x,\tau) + G_1(x,\tau) + G_2(x,\tau),\\
C_3(x,\tau) &= G_0(x,\tau) + G_1(x,\tau) + G_2(x,\tau) + G_3(x,\tau),\\
&\vdots\\
C_q(x,\tau) &= G_0(x,\tau) + G_1(x,\tau) + G_2(x,\tau) + \cdots + G_q(x,\tau).
\end{aligned} \tag{24}$$

To get the desired result, we have to prove that $C_q(x,\tau)$ forms a "Cauchy sequence." Further, let us take

$$\begin{aligned}
\|C_q + 1(x,\tau) - C_q(x,\tau)\| = \|G_q + 1(x,\tau)\| &\le \sigma \|G_q(x,\tau)\|\\
&\le \sigma^2 \|G_{q-1}(x,\tau)\| \le \sigma^3 \|G_{q-2}(x,\tau)\| \vdots\\
&\le \sigma_{q+1} \|G_0(x,\tau).
\end{aligned} \tag{25}$$

For $q, n \in N$, we acquire

$$\begin{aligned}
\|C_q(x,\tau) - C_n(x,\tau)\| = \|G_{q+n}(x,\tau)\| &= \|(C_q(x,\tau) - C_{q-1}(x,\tau))\\
&\quad + (C_{q-1}(x,\tau) - C_{q-2}(x,\tau))\\
&\quad + (C_{q-2}(x,\tau) - C_{q-3}(x,\tau)) + \cdots\\
&\quad + (C_{n+1}(x,\tau) - C_n(x,\tau))\|\\
&\le \|C_q(x,\tau) - C_{q-1}(x,\tau)\| + \|C_{q-1}(x,\tau)\\
&\quad - C_{q-2}(x,\tau)\| + \cdots + \|C_{n+1}(x,\tau) - C_n(x,\tau).\\
&\le \sigma^q \|G_0(x,\tau)\| + \sigma^{q-1}\|G_0(x,\tau)\| + \cdots + \sigma^{q+1}\|G_0(x,\tau)\|\\
&= \|G_0(x,\tau)\| (\sigma^q + \sigma^{q-1} + \cdots + \sigma^{q+1})\\
&= \|G_0(x,\tau)\| \frac{1-\sigma^{q-n}}{1-\sigma^{q+1}} \sigma^{n+1}.
\end{aligned} \tag{26}$$

Since $0 < \sigma < 1$, and $G_0(x,\tau)$ is bounded, let us take $\beta = 1 - \sigma/(1-\sigma^{q-n})\sigma^{n+1}\|G_0(x,\tau)\|$, and we obtain

$$\|G_{q+n}(x,\tau)\| \le \beta, \forall q, n \in N. \tag{27}$$

Thus, $\{G_q(x,\tau)\}_{q=0}^{\infty}$ forms a "Cauchy sequence" in H. It follows that the sequence $\{G_q(x,\tau)\}_{q=0}^{\infty}$ is a convergent sequence with the limit $(\lim/q \longrightarrow \infty) G_q(x,\tau) = G(x,\tau)$ for $\exists G(x,\tau) \in \mathscr{H}$. Hence, this ends the proof. □

Theorem 9. *Let $\sum_{h=0}^{k} G_h(x,\tau)$ is finite and $G(x,\tau)$ represents the obtained series solution. Let $\sigma > 0$ such that $\|G_{h+1}(x,\tau)\| \le \sigma \|G_h(x,\tau)\|$, then the following relation gives the maximum absolute error.*

$$\left\| G(x,\tau) - \sum_{h=0}^{k} G_h(x,\tau) \right\| < \frac{\sigma^{k+1}}{1-\sigma} \|G_0(x,\tau)\|. \tag{28}$$

Proof. Since $\sum_{h=0}^{k} G_h(x,\tau)$ is finite, this implies that $\sum_{h=0}^{k} G_h(x,\tau) < \infty$. Consider

$$\begin{aligned}
\left\| G(x,\tau) - \sum_{h=0}^{k} G_h(x,\tau) \right\| &= \left\| \sum_{h=k+1}^{\infty} G_h(x,\tau) \right\|\\
&\le \sum_{h=k+1}^{\infty} \|G_h(x,\tau)\|\\
&\le \sum_{h=k+1}^{\infty} \sigma^h \|G_0(x,\tau)\|\\
&\le \sigma^{k+1}(1 + \sigma + \sigma^2 + \cdots)\|G_0(x,\tau)\|\\
&\le \frac{\sigma^{k+1}}{1-\sigma} \|G_0(x,\tau)\|.
\end{aligned} \tag{29}$$

This ends the theorem's proof. □

4.3. Test Problems. The Yang transform homotopy perturbation method is applied to well-known nonlinear fractional PDEs in this section, demonstrating its ease of use and high accuracy. The space where the solution of the following examples lies is the Hilbert space H.

Example 1. We take nonlinear KdV equation as follows:

$${}^{CF}D_\tau^\alpha z(x,\tau) = -zz_x - zz_{xxx}, 0 < \alpha \in (0,1], \tag{30}$$

subjected to I.C $z(x,0) = x$.

Solution 1. Implementing the Yang transform to Equation (30), we have

$$Y[z(x,\tau)] = \upsilon z(x,0) - (1 + \alpha\upsilon - \alpha) Y[zz_x + zz_{xxx}]. \tag{31}$$

Applying Yang inverse transform, we have

$$z(x,\tau) = z(x,0) - Y^{-1}[(1 + \alpha\upsilon - \alpha) Y[zz_x + zz_{xxx}]]. \tag{32}$$

The solution via HPT is as follows:

$$z(x,\tau) = \sum_{q=0}^{\infty} \rho^q z_q(x,\tau). \tag{33}$$

Thus, Equation (32) can be written as

$$\sum_{q=0}^{\infty} \rho^q z_q(x,\tau) = x - \rho Y^{-1}\left[(1 + \alpha\nu - \alpha) Y\left[\sum_{q=0}^{\infty} \rho^q H_q(z)\right]\right], \tag{34}$$

where $H_q(z)$ is He's polynomial which represents the nonlinear term $zz_x + zz_{xxx}$. The first three terms can be written as

$$\begin{aligned} H_0(z) &= z_0 \frac{\partial}{\partial x} z_0 + z_0 \frac{\partial^3}{\partial x^3} z_0, \\ H_1(z) &= z_0 \frac{\partial}{\partial x} z_1 + z_1 \frac{\partial}{\partial x} z_0 + z_0 \frac{\partial^3}{\partial x^3} z_1 + z_1 \frac{\partial^3}{\partial x^3} z_0, \\ H_1(z) &= z_0 \frac{\partial}{\partial x} z_2 + z_1 \frac{\partial}{\partial x} z_1 + z_2 \frac{\partial}{\partial x} z_0 \\ &\quad + z_0 \frac{\partial^3}{\partial x^3} z_2 + z_1 \frac{\partial^3}{\partial x^3} z_1 + z_2 \frac{\partial^3}{\partial x^3} z_0 \\ &\vdots . \end{aligned} \tag{35}$$

Comparing the like powers of ρ, we obtain

$$\rho^0 : z_0(x, \tau) = x. \tag{36}$$

Now, using He's polynomials, we get $H_0(z) = x$. The second approximation is given by

$$\begin{aligned} \rho^1 : z_1(x, \tau) &= -Y^{-1}[(1 + \alpha v - \alpha) Y[H_0(z)]] \\ &= -Y^{-1}[(1 + \alpha v - \alpha) Y[x]] = -\left[x v + \alpha \times v^2 - \alpha \times v\right] \\ &= -Y^{-1}\left[x(1 - \alpha) v + \alpha \times v^2\right] \rho^1 : z_1(x, \tau) \\ &= -\alpha \times \tau - x(1 - \alpha). \end{aligned} \tag{37}$$

The third approximation is given by

$$\rho^1 : z_1(x, \tau) = -Y^{-1}[(1 + \alpha v - \alpha) Y[H_1(z)]]. \tag{38}$$

The second term $H_1(z)$ of He's polynomial is calculated as

$$\begin{aligned} H_1(z) &= z_0 \frac{\partial}{\partial x} z_1 + z_1 \frac{\partial}{\partial x} z_0 + z_0 \frac{\partial^3}{\partial x^3} z_1 + z_1 \frac{\partial^3}{\partial x^3} z_0 \\ &= x \frac{\partial}{\partial x}(-\alpha x \tau - x(1 - \alpha)) + (-\alpha x \tau - x(1 - \alpha)) \frac{\partial}{\partial x} x \\ &\quad + x \frac{\partial^3}{\partial x^3}(-\alpha x \tau - x(1 - \alpha)) + (-\alpha x \tau - x(1 - \alpha)) \frac{\partial^3}{\partial x^3} x, \end{aligned} \tag{39}$$

After simple calculation, we get

$$H_1(z) = -2\alpha x \tau - 2x(1 - \alpha). \tag{40}$$

Now, substituting Equation (40) into Equation (38), we get

$$\begin{aligned} \rho^2 : z_2(x, \tau) &= -Y^{-1}[(1 + \alpha v - \alpha) Y[-2\alpha x \tau - 2x(1 - \alpha)]] \\ &= -Y^{-1}[(1 + \alpha v - \alpha)(-2\alpha x \tau - 2x(1 - \alpha) v)] \\ &= 2Y^{-1}\left[\alpha^3 x v^3 + 2\alpha x(1 - \alpha) v^2 + x(1 - \alpha)^2 v\right] \\ &= \alpha^2 x \tau^2 + 4\alpha x(1 - \alpha)\tau + 2x(1 - \alpha)^2, \end{aligned} \tag{41}$$

Similarly, one can compute the other terms. Thus, the approximate solution is given.

$$\begin{aligned} z(x, \tau) &= x - \alpha x \tau - x(1 - \alpha) + \alpha^2 x \tau^2 \\ &\quad + 4\alpha x(1 - \alpha)\tau + 2x(1 - \alpha)^2 + \cdots . \end{aligned} \tag{42}$$

Remark 10. The proposed method is less computational and accurate as the obtained solution fast converges to the classical exact solution by substituting $\alpha = 1$ *in* Equation (42), i.e.,

$$\begin{aligned} z(x, \tau) &= x\left(1 - \tau + \tau^2 - \tau^3 + \cdots\right) = x \sum_{i=0}^{\infty} (-1)^i \tau^i, \\ z(x, \tau) &= \frac{x}{1 + \tau}. \end{aligned} \tag{43}$$

Equation (43) represents the exact classical solution of (30).

The absolute errors between exact solution and approximate solution for $\alpha = 1$ are given in Table 1. Also, the absolute errors of $\alpha = 1$ and $\alpha = 0.9$ are given in Table 2. The approximate solutions for $\alpha = 0.98$ and $\alpha = 1$ are represented by z_{approx} and z_{approx}, respectively.

Example 2. Consider the nonlinear time fractional order Burger equation as

$${}^{CF}D_\tau^\alpha z(x, \tau) + z \frac{\partial z}{\partial x} = \eta \frac{\partial^2 z}{\partial x^2}, \alpha \in (0, 1], \tag{44}$$

subjected to the initial condition $z(x, 0) = nx$.

Solution 2. Implementing Yang transform to Equation (44), we get

$$Y[z(x, \tau)] = v z(x, 0) - (1 + \alpha v - \alpha) Y\left[-\eta \frac{\partial^2 z}{\partial x^2} + z \frac{\partial z}{\partial x}.\right]. \tag{45}$$

Applying inverse Yang transform, we obtain

$$z(x, \tau) = xx - Y^{-1}\left[(1 + \alpha v - \alpha) Y\left[-\eta \frac{\partial^2 z}{\partial x^2} + z \frac{\partial z}{\partial x}\right]\right]. \tag{46}$$

Table 1: Absolute error between exact and approximate solutions of Example 1 at $\tau = 0.5$ and $\alpha = 1$.

x	$z_{approx}(x,\tau)$	$z_{exact}(x,\tau)$	$\lvert z_{exact} - z_{approx}\rvert$
0.1	0.0750	0.0667	0.0083
0.2	0.1500	0.1333	0.0167
0.3	0.2250	0.2000	0.0250
0.4	0.3000	0.2667	0.0333
0.5	0.3750	0.3333	0.0417
0.6	0.4500	0.4000	0.0500
0.7	0.5250	0.4667	0.0583
0.8	0.6000	0.5333	0.0667
0.9	0.6750	0.6000	0.0750
1	0.7500	0.6667	0.08333

Table 2: Absolute error between approximate solution of Example 1 at $\alpha = 0.98$ and $\alpha = 1$.

x	$z_{approx}(x,\tau)$	$z'_{approx}(x,\tau)$	$\lvert z_{exact} - z'_{approx}\rvert$
0.1	0.0770	0.07500	0.0020
0.2	0.1540	0.1500	0.0040
0.3	0.2310	0.2250	0.0060
0.4	0.3080	0.3000	0.0080
0.5	0.3851	0.3750	0.0101
0.6	0.4621	0.4500	0.0121
0.7	0.5391	0.5250	0.0141
0.8	0.6161	0.6000	0.0161
0.9	0.6931	0.6750	0.0181
1.0	0.7701	0.7500	0.0201

Using the HPT method, the approximate solution is

$$\sum_{q=0}^{\infty}\rho^q z_q(x,\tau) = nx - \rho Y^{-1}\left[(1+\alpha v-\alpha)Y\left[\sum_{q=0}^{\infty}\rho^q H_q(z) - \eta\frac{\partial^2 z}{\partial x^2}\sum_{q=0}^{\infty}\rho^q(z_q).\right]\right], \tag{47}$$

where He's polynomial $H_q(z)$ represents the nonlinear term $z(\partial z/\partial x)$. The first three terms of $H_q(z)$ are given by

$$\begin{aligned} H_0(z) &= z_0\frac{\partial z_0}{\partial x}, \\ H_1(z) &= z_0\frac{\partial z_1}{\partial x} + z_1\frac{\partial z_0}{\partial x}, \\ H_1(z) &= z_0\frac{\partial z_2}{\partial x} + z_1\frac{\partial z_1}{\partial x} + z_2\frac{\partial z_0}{\partial x}, \\ &\vdots . \end{aligned} \tag{48}$$

Table 3: Absolute error between exact and approximate solution of the Example 2 at $\tau = 0.1$ and $\alpha = 1$.

x	$z_{approx}(x,\tau)$	$z_{exact}(x,\tau)$	$\lvert z_{exact} - z_{approx}\rvert$
0.1	0.1680	0.1667	0.0013
0.2	0.3360	0.3333	0.0027
0.3	0.5040	0.5000	0.0040
0.4	0.6720	0.6667	0.0053
0.5	0.8400	0.8333	0.0067
0.6	1.0080	1.0000	0.0080
0.7	1.1760	1.1667	0.0093
0.8	1.3440	1.3333	0.0107
0.9	1.5120	1.5000	0.0120
1.0	1.6800	1.6667	0.0133

Table 4: Absolute error between approximate solution of Example 2 at $\alpha = 0.98$ and $\alpha = 1$.

τ	$z_{approx}(x,\tau)$	$z'_{approx}(x,\tau)$	$\lvert z_{approx} - z'_{approx}\rvert$
0.1	0.1988	0.1680	0.0308
0.2	0.3975	0.3360	0.0615
0.3	0.5963	0.5040	0.0923
0.4	0.7950	0.6720	0.1230
0.5	0.9938	0.8400	0.1538
0.6	1.1925	1.0080	0.1845
0.7	1.3913	1.1760	0.2153
0.8	1.5900	1.3440	0.2460
0.9	1.7888	1.5120	0.2768
1.0	1.9876	1.6800	0.3076

Similarly, other terms can be calculated. Comparing the like powers of ρ in (47), we achieve

$$\begin{aligned} \rho^0 &: z_0(x,\tau) = nx, \\ \rho^1 &: z_1(x,\tau) = -Y^{-1}\left[(1+\alpha v-\alpha)Y\left[H_0(z) - \eta\frac{\partial^2}{\partial x^2}z_0\right]\right]. \end{aligned} \tag{49}$$

Now, we compute $H_0(z)$ as follows:

$$H_0(z) = z_0\frac{\partial z_0}{\partial x} = nx\frac{\partial}{\partial x}(nx) = n^2 x. \tag{50}$$

Thus, Equation (49) can be written as

$$\begin{aligned} \rho^1 &: z_1(x,\tau) = -Y^{-1}\left[(1+\alpha v-\alpha)Y\left[n^2x-0\right]\right] = -n^2xY^{-1}\left[v+\alpha v^2-\alpha v\right], \\ \rho^1 &: z_1(x,\tau) = -n^2x(1-\alpha+\alpha\tau). \end{aligned} \tag{51}$$

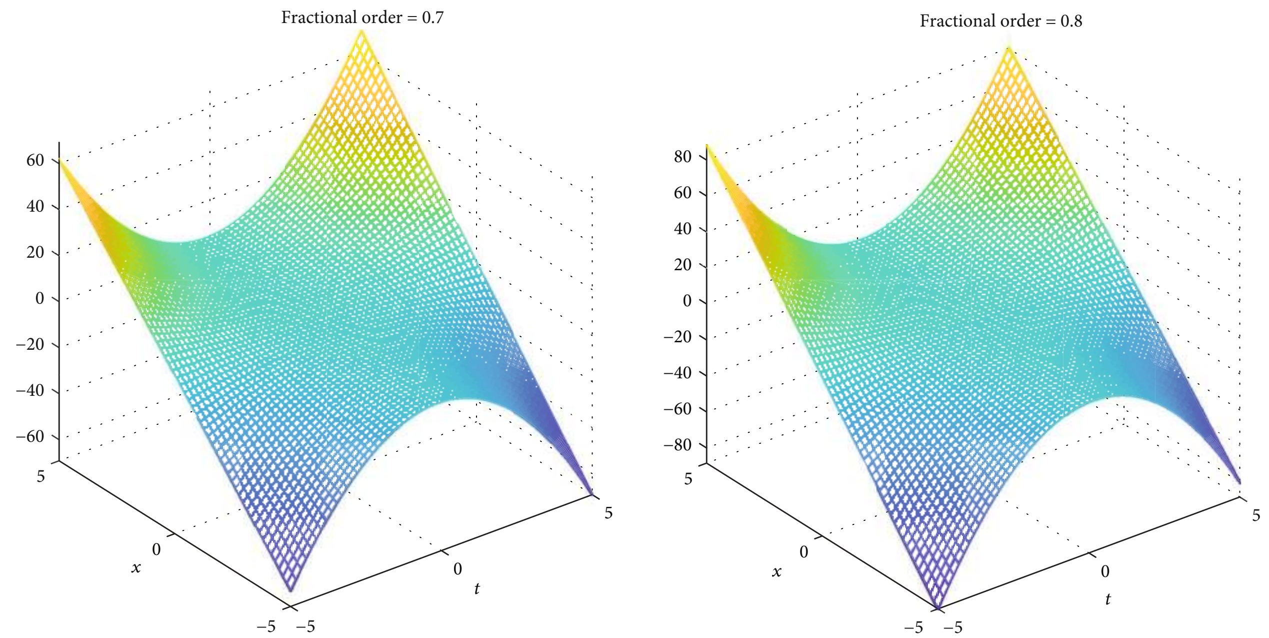

Figure 1: Approximate solution $z(x, \tau)$ for fractional orders $\alpha = 0.7$ and 0.8.

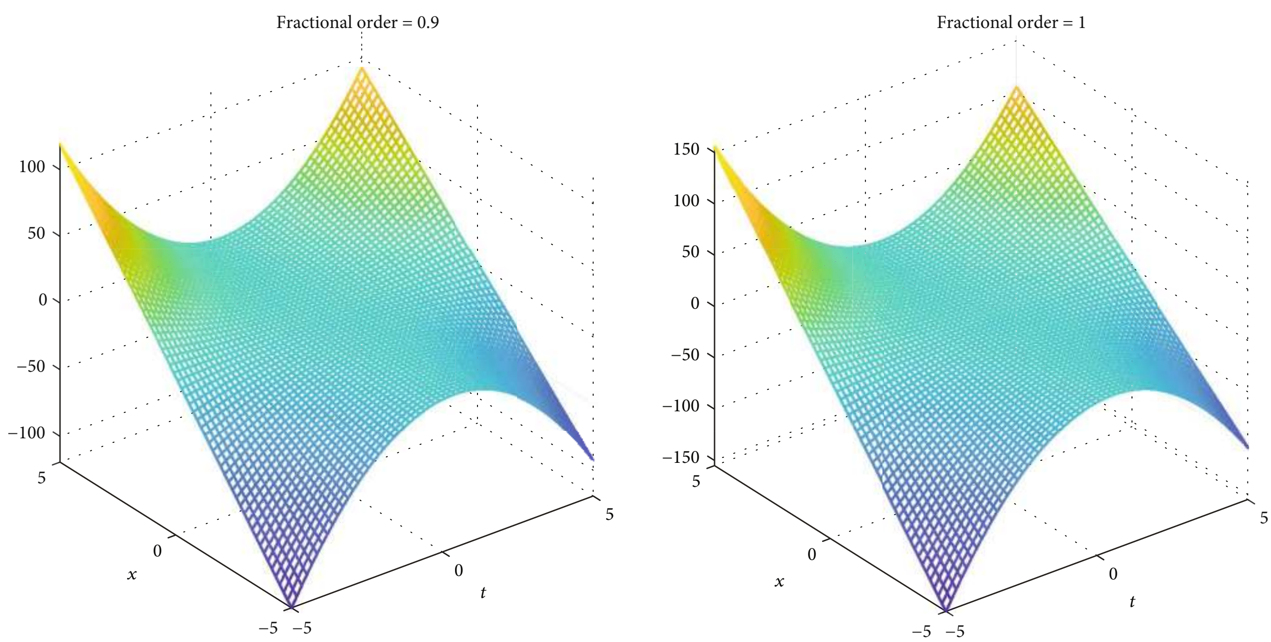

Figure 2: Approximate solution $z(x, \tau)$ for fractional orders $\alpha = 0.9$ and 1.

Similarly, the third approximation is given by

$$\rho^2 : z_2(x, \tau) = 2n^3 x\left(\alpha - 2\alpha + \alpha^2 \frac{\tau^2}{2} - 2\alpha(\alpha - 1)\tau + 1\right). \tag{52}$$

One can calculate more terms. The acquired series solution is represented as follows:

$$z(x, \tau) = nx - n^2 x(1 - \alpha + \alpha\tau) + 2n^3 x \cdot \left(\alpha - 2\alpha + \alpha^2 \frac{\tau^2}{2} - 2\alpha(\alpha - 1)\tau + 1\right) + \cdots. \tag{53}$$

Remark 11. The proposed method is less computational and accurate as the obtained solution fastly converges to the

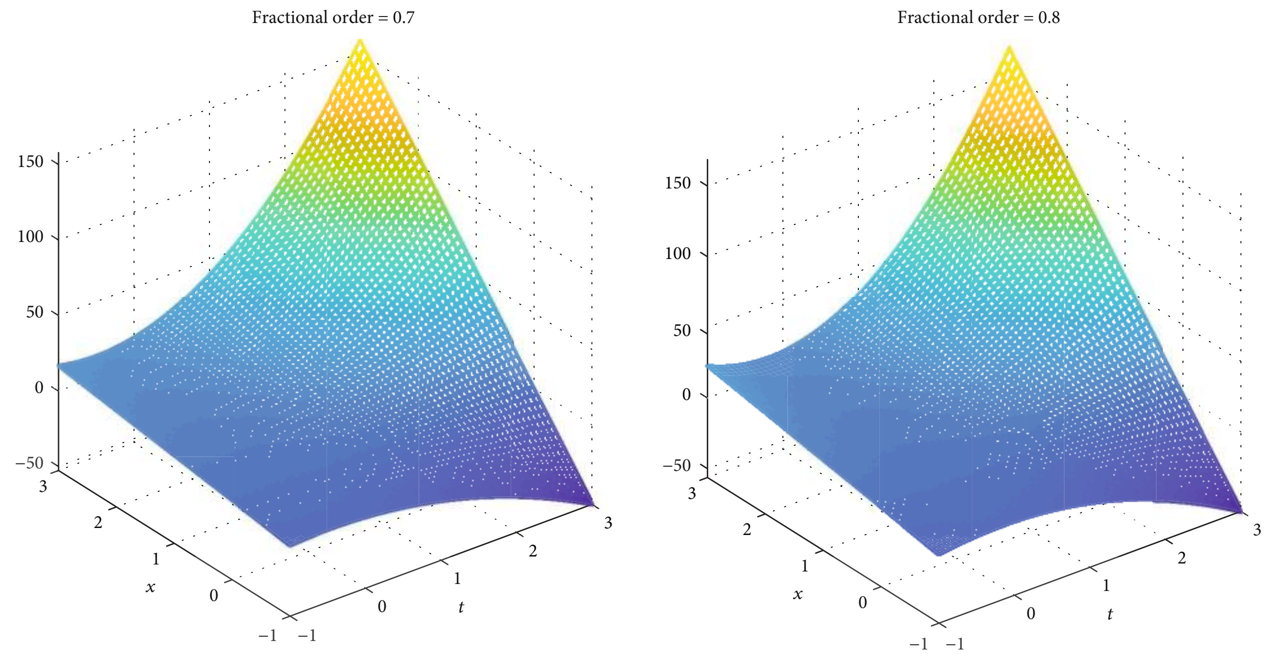

Figure 3: Approximate solution $z(x, \tau)$ for fractional orders $\alpha = 0.7$ and 0.8.

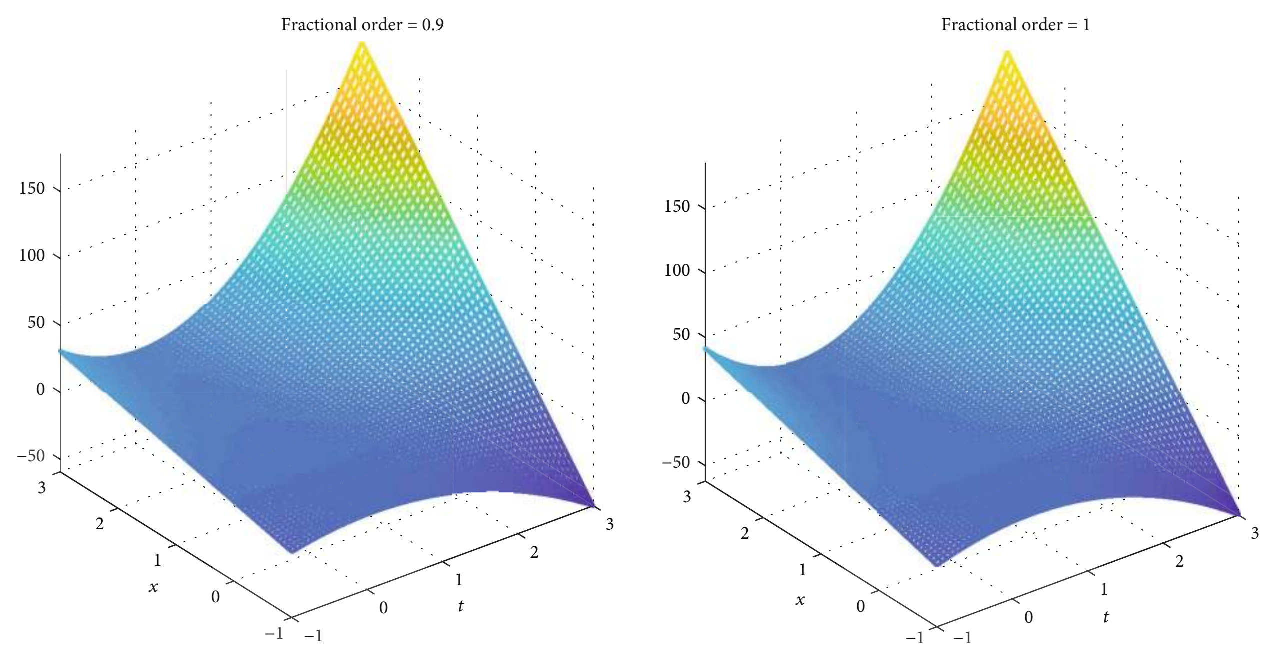

Figure 4: Approximate solution $z(x, \tau)$ for fractional orders $\alpha = 0.9$ and 1.

classical exact solution by substituting $\alpha = 1$ in Equation (53), i.e.,

$$z(x, \tau) = nx\left(1 - n\tau + n^2\tau^2 - n^3\tau^3 + \cdots\right) = x\sum_{i=0}^{\infty} (-1)^i n^{i+1}\tau^i,$$
$$z(x, \tau) = \frac{nx}{1 + n\tau}. \tag{54}$$

Equation (54) is the classical solution of the considered problem.

The absolute errors between exact solution and approximate solution for $\alpha = 1$ and $\tau = 0.5$ are given in Table 3. Also, the absolute errors of for $x = 0.5$, $\alpha = 1$, and $\alpha = 0.9$ are given in Table 4. The approximate solutions for $\alpha = 0.98$ and $\alpha = 1$ are represented by z_{approx} and z'_{approx}, respectively.

Remark 12. The proposed YTHPM method is a powerful new method which needs less computation time and is much

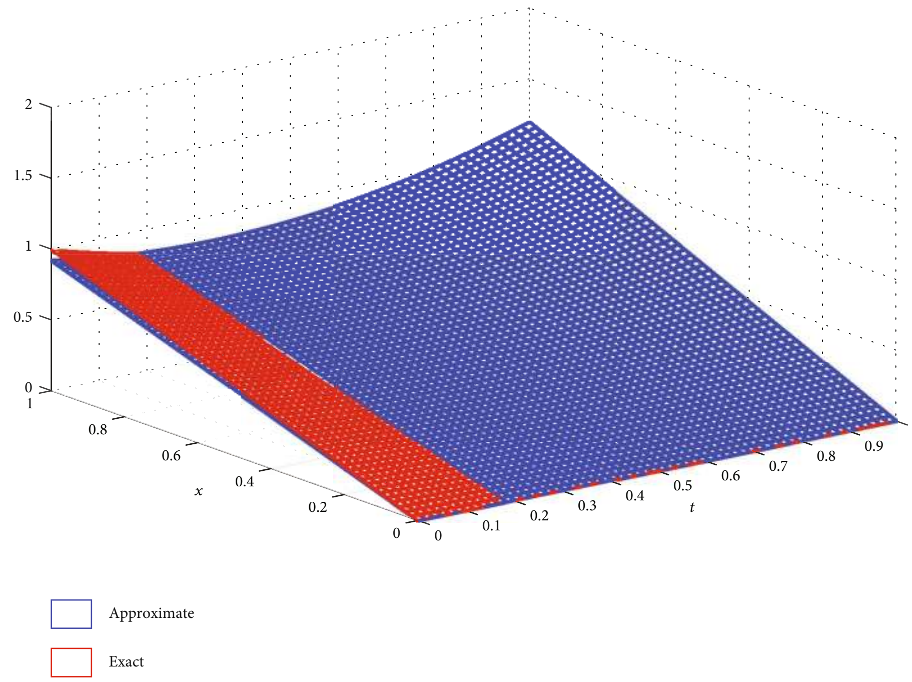

FIGURE 5: Comparison between exact and approximate solutions of Example 1.

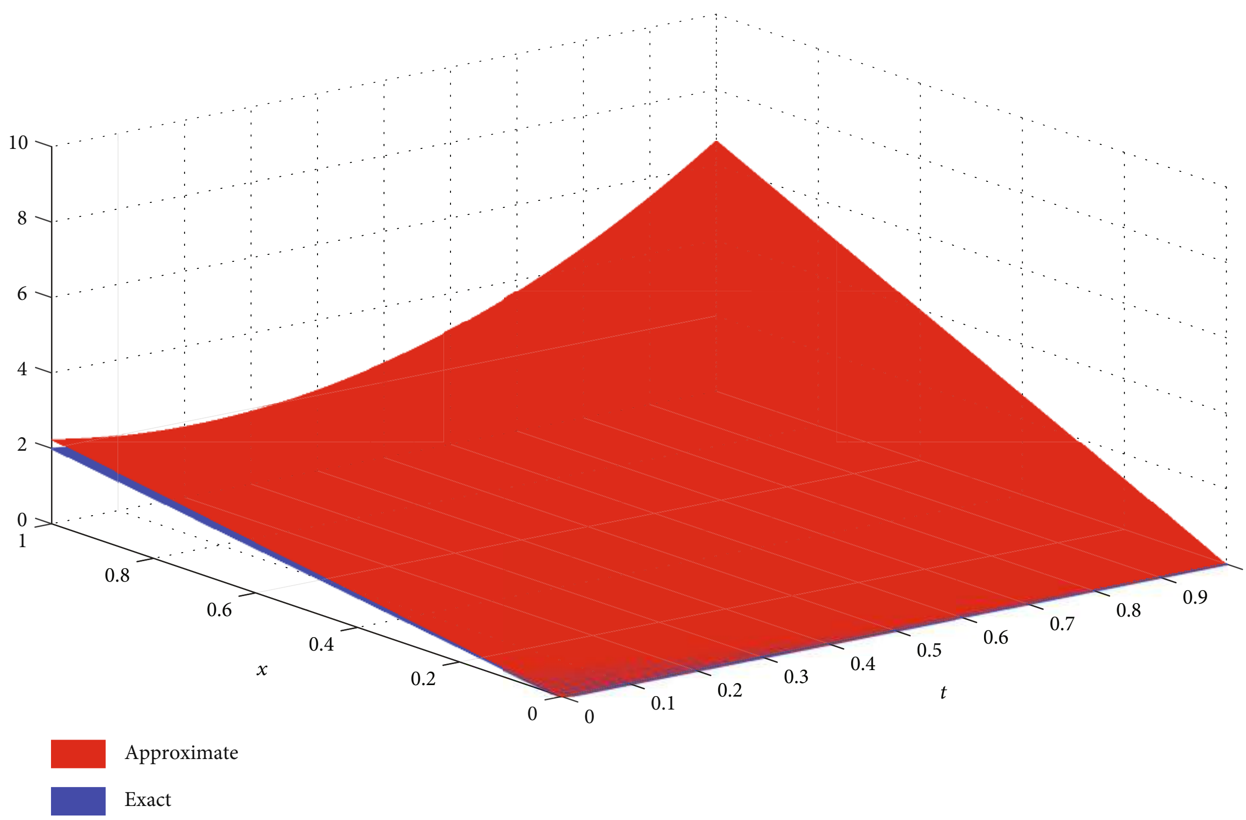

FIGURE 6: Comparison between exact and approximate solutions of the Example 2.

easier and more convenient than the other methods like HAM (homotopy analysis method). Computational time means HPM takes less time, to calculate the iterative terms for the series solution, because HPM only depends on single parameter. However, HAM depends on two parameters, i.e., h (auxiliary parameter) and p (embedding parameter). So, HAM takes a little more time for calculating the successive terms of the series solution. Also, the convergence of the HAM depends on h which is different for various approximations. When $h=-1$ in HAM, then the solution directly converges to the HPM solution; otherwise, more terms will be calculated than HPM case. So, computational labor is much in HAM instead of HPM. Actually, the Yang transform is closely related to Laplace transform which is convergent. Therefore, Yang transform combined with HPM is an efficient and valid computational method for solving DEs because the Yang transform allows one in many situations to overcome the deficiency mainly caused by unsatisfied boundary or initial conditions that appear in other semianalytical methods such as HAM. The comparison between HPM and HAM is already given in [25, 26].

5. Conclusion

We used a novel method to solve approximately nonlinear PDEs of fractional order in the Caputo-Fabrizio context in this paper. The Yang homotopy perturbation transform method (YHPTM) is a new method that combines the Yang transform and the HPM. The nonlinear term was decomposed into He's polynomial using the HPM. A general procedure for solving nonlinear PDEs described by the CF derivative has been developed. We have demonstrated the estimated solution's convergence and provided a result for the absolute error calculation. We solved well-known nonlinear PDEs such as the KdV equation and Burger's equation to test the accuracy and validity of the proposed technique. By substituting $\alpha = 1$, we were able to achieve the required solution in series form, which quickly converges to the exact solution of the problems, as seen in the remarks following each problem's solution (also see Figures 1–4). We determined the numerical values of the absolute errors between the estimated and exact solutions, indicating that the exact and approximate solutions are in agreement. We have shown 3D graphs that demonstrate the suggested technique's high precision and speed of convergence (see Figures 5 and 6). The approach also has the advantage of not requiring linearization, discretization, or additional memory. As a result, we have concluded that the proposed novel method is effective, reliable, and computationally effective. This approach will be used to solve the Atangana-Baleanu fractional-order PDEs. Further, we will use the Yang transform with HAM to solve nonlinear PDEs of fractional order in the future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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