A Novel Multiple Attribute Group Decision-Making Approach Based on Interval-Valued Pythagorean Fuzzy Linguistic Sets

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This work was supported in part by the Fundamental Research Funds for the Central Universities, Zhejiang University of Science and Technology (ZUST), under Grant F701107J08, and in part by the Funds for First-Class Discipline Construction under Grant XK1802-5.

ABSTRACT This article investigates multi-attribute group decision-making (MAGDM) problems based on interval-valued Pythagorean fuzzy linguistic sets (IVPFLSs). The IVPFLSs are regarded as an efficient tool to describe decision makers’ (DMs’) evaluation information from both quantitative and qualitative aspects. However, existing IVPFLSs based MAGDM methods are still insufficient and inadequate to deal with complicated practical situations. This article aims to propose a novel MAGDM method and the main contributions of the present work are three-fold. First, we propose new operations of interval-valued Pythagorean fuzzy linguistic numbers (IVPFLNs) based on linguistic scale function. Second, we propose new aggregation operators (AOs) of IVPFLNs based on power average operator and Muirhead mean. The proposed AOs take the interrelationship among any numbers of attributes into account and eliminate the bad influence of DMs’ unreasonable evaluation values on the final decision results. Third, based on the new operations and AOs of IVPFLNs, we introduce a novel approach to MAGDM and present its main steps. Finally, we discuss the effectiveness of the proposed approach and investigates their advantages through numerical examples.

INDEX TERMS Interval-valued Pythagorean fuzzy linguistic sets, interval-valued Pythagorean fuzzy linguistic power Muirhead mean, linguistic scale function, multiple attribute group decision-making.

I. INTRODUCTION

Multiple attribute group decision-making (MAGDM) theories and models have gained much attention and been extensively employed in practical decision-making problems, such as supplier selection [1], [2], investment selection [3], [4], smart medical device selection [5], signal processing [6] etc. How to effectively deal with the inherent fuzziness of decision-making problems and decision makers’ (DMs’) evaluation information is fundamental issue before determining the optimal alternative. In order to do this, Yager [7] proposed the concept of Pythagorean fuzzy sets (PFSs), satisfying the constraint that the square sum of membership grade (MG) and non-membership grade (NMG) is equal to or less than one. From the constraint of PFSs, we can find out that they can describe larger information space than the intuitionistic fuzzy sets (IFSs) [8]. Therefore, PFSs have been regarded as an efficient tool in portraying DMs’ complicated and fuzzy evaluation information in MAGDM process and quite a novel decision-making methods have been proposed [9]–[14]. Recently, Harish [15] proposed the interval-valued PFSs (IVPFSs), which employ interval values rather crisp numbers to represent the MG and NMG. Evidently, compared with the traditional PFS, the IVPFSs take more information into account and can better depict DMs’ judgements over alternatives in decision-making procedure. Afterwards, IVPFSs based decision-making methods have been a research focus. For example, Peng and Yang [16] investigated the basic properties of IVPFS, studied their aggregation operators (AOs), and proposed a new MAGDM
method. Harish [17] proposed a novel accuracy function for IVPFNs. Yang et al. [18] proposed novel operations of interval-valued Pythagorean fuzzy numbers (IVPFNs) under Frank t-norm and t-conorm and based on which the authors further proposed a set of interval-valued Pythagorean fuzzy (IVFPS) power average operators. Chen [19] proposed an IVPF compromise decision-making approach and applied it in bridge construction analysis. Wei and Mao [20] proposed an IVPF Maclaurin symmetric mean AO based decision-making method, which is powerful for its ability of capturing the interrelationship among multiple attributes. Sajjad Ali Khan et al. [21] proposed a hybrid IVPF decision-making method based on TOPSIS and Choquet integral. Sajjad Ali Khan and Abdullah [22] proposed an IVPF grey relational analysis method to deal with MAGDM problems with incomplete weight information. Liang et al. [23] introduced IVPF extended Bonferroni mean operators to proceed the heterogeneous interrelationship between aggregated IVFPFs. Garg [24] proposed some exponential operations of IVFPFs and introduced new IVPF operators. Tang et al. [25] proposed a series of IVPF Muirhead mean (MM) operators, which can capture the complicated interrelationship among IVPFNs flexibly. Peng and Li [26] proposed two IVPF decision making methods based on weighted distance-based approximation and multiparametric similarity measure and employed it in emergency decision making. For more recent developments of IVPFNs based decision-making methods, readers are suggested to refer [27]–[31].

More recently, Du et al. [32] proposed a new extension of IVPFS, called IVPF linguistic set (IVPLS), which is a combination of IVPF with linguistic terms set. The IVPLS is parallel to interval-valued intuitionistic linguistic sets (IVILSs) [33] and both of them can portray both DMS’ quantitative and qualitative evaluation information. But, IVPLSs are more powerful and flexible than IVILSs as they have laxer constraint, which provides DMs more freedom to express their judgments. In [32], Du et al. further defined the operations of IVPF linguistic numbers (IVPLNs), proposed their AOs and developed a new MAGDM method under IVPLSs context. However, the decision-making method proposed by Du et al. [32] still has several shortcomings. First, the operations of IVPLNs proposed by Du et al. [32] are not closed and they fail to handle the semantic translation requirements of different DMS. (We will discuss the drawbacks of these operations in Section 3 in detail.) Second, Du et al.’s [32] decision-making method cannot effectively deal with DMS’ extreme evaluation values. In other words, Du et al.’s [32] method may produce unreasonable decision results. Third, Du et al.’s [32] method is based on the simple weighted average/geometric operator, which is unable to deal with the complicated interrelationship among attributes. Hence, the reliability of decision results derived by this method is weak.

Based on above analysis, this article aims to propose a new MAGDM method with IVPL information. First, we propose novel operations of IVPLNs based on linguistic scale function (LSF). The new IVPL operational rules overcome the drawbacks of existing operations of IVPFLNs. In the addition, the operations can effectively handle the semantic translation requirements of different DMS. Besides, some other notions such as comparison method and distance measure are also presented. Second, to overcome the second and third shortcomings we propose new AOs of IVPLNs based on the power average [33] operator and Muirhead mean (MM) [34]. The PA operator was originated by Prof. Yager, and it has received much interests due to its ability of reducing the bad influence of DMS’ unreasonable evaluation information [35]–[39]. Hence, in this article we firstly propose some IVPFL operators based on PA, i.e. the IVPFL power average (IVPFLPA) operator and the IVPFL power weighted average (IVPFLPWA) operator. The MM is good at capturing the interrelationship among multiple attributes and this is the reason that it has been widely employed in information aggregation process [40]–[43]. Recently, Li et al. [44] integrated PA with MM and proposed the power Muirhead mean (PMM) operator, which takes the advantages of both PA and MM. Hence, the PMM operator has been regarded as a promising information aggregation technique [45]–[47]. So, we further propose some AOs of IVPFLNs based on PMM, i.e. the IVPFL power Muirhead mean (IVPFLPMM) operator and the IVPFL power weighted Muirhead mean (IVPFLPWMM) operator. Finally, we present a new MAGDM method based on the proposed AOs. In the new decision-making approach, the IVPFLPWA is employed to calculate the comprehensive decision matrix and the IVPFLPWMM is used to compute the overall evaluation values of alternatives. We further provide numerical experiments to show the good performance of our proposed method.

The rest of this article is organized as follows. Section 2 reviews some existing concepts and proposes new operations of IVPFLNs. Section 3 presents new AOs of IVPFLNs and discusses their properties. Section 4 puts forward a new MAGDM method. Section 5 provides numerical examples to better illustrate the performance and advantages of our proposed method. The summary of this article and the future research issues are presented in Section 6.

II. PRELIMINARIES

In this section, we review some basic concepts and proposes new operational laws of IVPFLNs based on LSF.

A. THE INTERVAL-VALUED PYTHAGOREAN FUZZY LINGUISTIC SETS

Definition 1 [32]: Let $X$ be an ordinary fixed set and $S$ be a continuous linguistic term set, then an interval-valued Pythagorean fuzzy linguistic set (IVPLS) $A$ defined on $X$ is expressed as

$$A = \left\{ \left[ x, \left[ s_{\theta}(x), (\mu_A(x), v_A(x)) \right] \right] \mid x \in X \right\},$$

where $\mu_A(x)$ and $v_A(x)$ are two interval values in $[0, 1]$, denoting the MD and NMD of the element $x \in X$ to the set $A$, such that $\sup (\mu_A(x))^2 + \sup (v_A(x))^2 \leq 1$ for $\forall x \in X$. For convenience, we call $A = (s_\theta, (\mu_A, v_A))$ an IVPFLN, which
can be denoted by $\alpha = (s_\theta, ([a, b], [c, d]))$ for simplicity, such that $0 \leq a \leq b \leq 1$, $0 \leq c \leq d \leq 1$ and $b^2 + d^2 \leq 1$.

Based on the operations of IVPLFNs, Du et al. [32] proposed some operational rules of IVPLFNs.

**Definition 4** [33]: Let $\alpha_1 = (s_\theta_1, ([a_1, b_1], [c_1, d_1]))$, $\alpha_2 = (s_\theta_2, ([a_2, b_2], [c_2, d_2]))$ and $\alpha = (s_\theta, ([a, b], [c, d]))$ be any three IVPLFNs, $\lambda$ be a positive real number, then

1. $\alpha_1 \oplus \alpha_2 = \left\langle s_{\theta_1 + \theta_2}, \left(\left[\left(\frac{a_1 + a_2 - a_1 a_2}{a_1 a_2}\right)^{1/2}, \left(\frac{b_1 + b_2 - b_1 b_2}{b_1 b_2}\right)^{1/2}\right), [c_1 c_2, b_1 b_2]\right)\rightangle$;
2. $\alpha_1 \odot \alpha_2 = \left\langle s_{\theta_1 \times \theta_2}, \left(\left[\left(\frac{c_1 + c_2 - c_1 c_2}{c_1 c_2}\right)^{1/2}, \left(\frac{d_1 + d_2 - d_1 d_2}{d_1 d_2}\right)^{1/2}\right), [c_1, d_1]\right)\right\rangle$;
3. $\lambda \alpha = \left\langle s_{\lambda \theta}, \left(\left[\left(1 - (1 - a^2)^{\lambda}\right)^{1/2}, \left(1 - (1 - b^2)^{\lambda}\right)^{1/2}\right), [c^{\lambda}, d^{\lambda}]\right)\right\rangle$;
4. $a^\lambda = \left\langle s_{\theta^\lambda}, \left(\left[\left(1 - (1 - c^2)^{\lambda}\right)^{1/2}, \left(1 - (1 - d^2)^{\lambda}\right)^{1/2}\right), [a^{\lambda}, b^{\lambda}]\right)\right\rangle$.

To compare any two IVPLFNs, Du et al. [32] proposed a comparison method.

**Definition 3** [32]: Let $\alpha = (s_\theta, ([a, b], [c, d]))$ be an IVPLFN, then the score function $S(\alpha)$ of $\alpha$ is expressed as

$$S(\alpha) = s_{\theta \times (a^2 + b^2 - c^2 - d^2 + 2)}/4,$$

and the accuracy function $H(\alpha)$ of $\alpha$ is expressed as

$$H(\alpha) = s_{\theta \times (a^2 + b^2 + c^2 + d^2)/2}.$$

Let $\alpha_1$ and $\alpha_2$ be any two IVPLFNs, then

1. If $S(\alpha_1) > S(\alpha_2)$, then $\alpha_1 > \alpha_2$;
2. If $S(\alpha_1) = S(\alpha_2)$, then
   - If $H(\alpha_1) = H(\alpha_2)$, then $\alpha_1 = \alpha_2$;
   - If $H(\alpha_1) > H(\alpha_2)$, then $\alpha_1 > \alpha_2$.

**B. THE POWER AVERAGE AND MuIRHEAD MEAN**

**Definition 4** [33]: Let $a_i (i = 1, 2, \ldots, n)$ be a collection of crisp numbers, then the power average (PA) operator is expressed as

$$PA(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^{n} (1 + T(a_i)) a_i}{\sum_{i=1}^{n} (1 + T(a_i))},$$

where $T(a_i) = \sum_{j=1,j\neq i}^{n} Sup(a_i, a_j), Sup(a_i, a_j)$ is the support of $a_i$ from $a_j$, satisfying the following properties

1. $0 \leq Sup(a, b) \leq 1$;
2. $Sup(a, b) = Sup(b, a)$;
3. $Sup(a, b) \leq Sup(c, d)$, if $[a, b] \geq [c, d]$.

**Definition 5** [34]: Let $a_i (i = 1, 2, \ldots, n)$ be a collection of crisp numbers and $P = (p_1, p_2, \ldots, p_n)^T$ be a vector of parameters. If

$$MM^P(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{\vartheta(p_j)} \frac{1}{\sum_{j=1}^{\vartheta(p_j)}}\right),$$

then $MM^P$ is the Muirhead mean (MM) operator, where $\vartheta(j) (j = 1, 2, \ldots, n)$ represents any permutation of $(1, 2, \ldots, n)$ and $L_n$ denotes all possible permutations of $(1, 2, \ldots, n)$.

**III. NEW OPERATIONS OF IVPLFNs BASED ON LINGUISTIC SCALE FUNCTIONS**

In this section, we aim to propose novel operations of IVPLFNs based on LSF. In order to do this, we first explain the necessity and motivations of proposing novel operations of IVPLFNs. Second, we briefly review the notion of LSFs. Further, we present new interval-valued Pythagorean fuzzy linguistic operations and discuss their properties.

**A. NECESSITY AND MOTIVATIONS**

In [32], Du et al. proposed some basic operations of IVPLFNs. However, Du et al.’s [32] operations have an obvious shortcoming. To better explain the shortcoming, we provide the following example.

**Example 1**: Let $\alpha_1 = (s_\theta, ([0.6, 0.7], [0.3, 0.4]))$ and $\alpha_2 = (s_\theta, ([0.4, 0.6], [0.2, 0.6]))$ be any two IVPLFNs defined on a given LTS $S = \{s_0, s_1, \ldots, s_6\}$. According to the operations proposed by Du et al. [32], we have

1. $\alpha_1 \oplus \alpha_2 = (s_8, ([0.68, 0.8207], [0.06, 0.24]))$;
2. $\alpha_1 \odot \alpha_2 = (s_8, ([0.24, 0.42], [0.3555, 0.68]))$;
3. $3 \alpha_1 = (s_9, ([0.8590, 0.9313], [0.027, 0.064]))$;
4. $a^{\lambda_1} = (s_{27}, ([0.216, 0.343], [0.4964, 0.6382]))$.

From Example 1, it is easy to find out that the operations proposed by Du et al. [32] have an evident drawback, i.e. the calculation results in Example 1 have exceed the upper limit of the given LTS $S$. Hence, it is necessary to propose some new operational rules of IVPLFNs. Actually, similar researches can be found in recent publications. For instance, Liu et al. [48] proposed novel operations of intuitionistic uncertain linguistic variables based on LSF. Liu et al. [49] proposed operations and aggregation operators of interval-valued hesitant uncertain linguistic variables based on LSF. Therefore, we can propose new operational rules of IVPLFNs based on LSF. In order to this, we first review the concept of LSFs.
B. THE CONCEPT OF LSF

Definition 6 [50]: Let \( S = \{ s_i | i = 0, 1, \ldots, 2r \} \) be a linguistic term set, \( s_i \in S \) be a linguistic term and \( \tau_i \in [0, 1] \) be a real number. A linguistic scale function (LSF) \( f \) is a mapping from \( s_i \) to \( \tau_i (i = 1, 2, \ldots, 2r) \) such that

\[
f : s_i \rightarrow \tau_i (i = 0, 1, 2, \ldots, 2r),
\]

where \( 0 \leq \tau_0 < \tau_1 < \ldots < \tau_{2r} \). Hence, \( f \) is a strictly monotonicly increasing function with regard to linguistic subscripts \( i \). Generally, there are three types of LSFs and we give a brief review in the following.

1. The most widely used LSF is expressed as

\[
f_1 (s_i) = \theta_i = \frac{i}{2t} (i = 0, 1, 2, \ldots, 2r),
\]

which is a simple average calculation of the subscripts of linguistic terms.

2. The second type of LSF is expressed \( s \) follows

\[
f_2 (s_i) = \theta_i = \begin{cases} \frac{\rho^t - \rho^{-t}}{2\rho^t - 2} (i = 0, 1, 2, \ldots, t) \\ \frac{\rho^t + \rho^{-t} - 2}{2\rho^t - 2} (i = t + 1, t + 2, \ldots, 2r) \end{cases},
\]

(8)

3. The third type of LSF is expressed as

\[
f_3 (s_i) = \theta_i = \begin{cases} \frac{t^e - (t - i)^e}{2t^e} (i = 0, 1, 2, \ldots, t) \\ \frac{t^e + (t - i)^e}{2t^e} (i = t + 1, t + 2, \ldots, 2r) \end{cases},
\]

(9)

For Eq. (8), \( y_t \in [0, 1] \) and the absolute deviation between adjacent linguistic subscripts \( i (i = 0, 1, 2, \ldots, 2r) \). We can use a subjective method to determine the value of \( \rho \). Suppose that the weight of attribute \( A \) is much greater than attribute \( B \), and the weight ratio is \( m \), then \( \rho^k = m(k \) represents the scale level) and \( \rho = \sqrt[m]{m} \). Up to now, most researchers reckon that the upper limit of the weight ratio is 9. Thus, \( \rho = \sqrt[m]{m} \approx 1.37 \) can be calculated when the scale level is 7.

Especially, if \( e = \beta = 1 \), then Eq. (9) is reduced to Eq. (7). Meanwhile, the function \( f \) can be further expanded to a continuous function such that \( f^*: \tilde{f} \rightarrow \Omega^+ (\Omega^+ = \{ d | d \geq 0 \}, d \in R \), which satisfies \( f^* (s_i) = \gamma_i \). The inverse function of \( f^* \) is depicted as \( f^*^{-1} \). Then, we can get

\[
f_1^{-1} (\gamma_i) = s_{2r+1} (i = 0, 1, 2, \ldots, 2r),
\]

(10)

\[
f_2^{-1} (\gamma_i) = \begin{cases} s_{-\log_2 (\rho^t - (2\rho^t - 2)\gamma_i)} (\gamma_i \in [0, 0.5]) \\ s_{+\log_2 (2\rho^t - 2\gamma_i - \rho^t + 2)} (\gamma_i \in [0.5, 1.0]) \end{cases},
\]

(11)

\[
f_3^{-1} (\gamma_i) = \begin{cases} s_{-(t^e - 2 \times t^{e-1} \gamma_i)^1/t^e} (\gamma_i \in [0, 0.5]) \\ s_{+(2 \times t^{e-1} \gamma_i - t^e)^1/t^e} (\gamma_i \in [0.5, 1]) \end{cases}
\]

(12)

C. NEW OPERATIONS OF IVPLFNs

Definition 7: Let \( \alpha_1 = \{ s_{\theta_1}, (\{ a_1, b_1 \}, [c_1, d_1]) \}, \alpha_2 = \{ s_{\theta_2}, (\{ a_2, b_2 \}, [c_2, d_2]) \} \) be any three IVPLFNs and \( \lambda \) be a positive real number, then

1. \( \alpha_1 \oplus \alpha_2 = f^{*^{-1}} (f^* (\theta_1) + f^* (\theta_2) - f^* (\theta_1) f^* (\theta_2)) \),

\[
\left[ \frac{a_1^2 + a_2^2 - a_1^2 a_2^2}{\left( b_1^2 + b_2^2 - b_1^2 b_2^2 \right)} \right] \cdot \left[ c_1 c_2, b_1 b_2 \right]
\]

(13)

2. \( \alpha_1 \otimes \alpha_2 = f^{*^{-1}} (f^* (\theta_1) \times f^* (\theta_2)) \),

\[
\left[ a_1 a_2, b_1 b_2 \right], \left[ c_1^2 + c_2^2 - c_1^2 c_2^2 \right], \left[ d_1^2 + d_2^2 - d_1^2 d_2^2 \right]
\]

(14)

3. \( \lambda \alpha = f^{*^{-1}} (1 - (1 - f^* (\theta))^\lambda) \),

\[
\left[ \left( 1 - 1 - d_2^2 \right)^{1/2}, \left( 1 - 1 - d_2^2 \right)^{1/2} \right], [c^\lambda, d^\lambda]
\]

(15)

4. \( \alpha^\lambda = f^{*^{-1}} ([f^*(\theta)]\lambda),

\[
\left[ 0, 1 - (1 - d_2^2) \right]^{1/2}, \left( 1 - 1 - d_2^2 \right)^{1/2}
\]

(16)

Example 2. Let \( \alpha_1 = (s_3, ([0.6, 0.7], [0.3, 0.4])) \) and \( \alpha_2 = (s_5, ([0.4, 0.6], [0.2, 0.6])) \) be any two IVPLFNs defined on a given LTS \( S = \{ s_0, s_1, \ldots, s_6 \} \). If LSF 1 is applied in Definition 7, then we have

1. \( \alpha_1 \oplus \alpha_2 = (s_5.5, ([0.68, 0.8207], [0.06, 0.24])), \)

2. \( \alpha_1 \otimes \alpha_2 = (s_5.2, ([0.24, 0.42], [0.3555, 0.68])); \)

3. \( \lambda \alpha_1 = (s_{5.25}, ([0.8590, 0.9313], [0.027, 0.0641])); \)

4. \( \alpha^\lambda = (s_{0.75}, ([0.216, 0.343], [0.4964, 0.6382])); \)

It is easy to prove that the following theorem holds.

Theorem 1: Let \( \alpha_1, \alpha_2 \) and \( \alpha \) be any three IVPLFNs, then

1. \( \alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1 \);

2. \( \alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1 \);

3. \( \lambda (\alpha_1 \oplus \alpha_2) = \lambda \alpha_2 \oplus \lambda \alpha_1 \), \( \lambda > 0 \); (17)

4. \( \lambda \alpha_1 \otimes \lambda \alpha_2 = (\lambda_1 + \lambda_2) \alpha_1, \lambda_1, \lambda_2 > 0; \)

5. \( \alpha_1^\lambda \otimes \alpha_2^\lambda = (\alpha_2 \otimes \alpha_1)^\lambda , \lambda > 0 \);

6. \( \alpha_1^\lambda \oplus \alpha_2^\lambda = \alpha_1^{\lambda + \lambda}, \lambda_1, \lambda_2 > 0; \)

D. NEW COMPARISON METHOD OF IVPLFNs

Based on LSF, we propose novel comparison method of IVPLFNs.

Definition 8: Let \( \alpha = (s_\alpha, ([a, b], [c, d])) \) be an IVPLFN, then the score function of \( \alpha \) is defined as

\[
S (\alpha) = \frac{1}{2} \times \left( 1 + \frac{a^2 + b^2}{2} - \frac{c^2 + d^2}{2} \right) \times f^* (\theta),
\]

(13)

and the accuracy function of \( \alpha \) is defined as

\[
H (\alpha) = \left( \frac{a^2 + b^2}{2} + \frac{c^2 + d^2}{2} \right) \times f^* (\theta). \]

(14)

For any two IVPLFNs \( \alpha_1 = (s_{\theta_1}, ([a_1, b_1], [c_1, d_1])) \) and \( \alpha_2 = (s_{\theta_2}, ([a_2, b_2], [c_2, d_2])) \), then
(3) If $S(\alpha_1) > S(\alpha_2)$, then $\alpha_1 > \alpha_2$;
(4) If $S(\alpha_1) = S(\alpha_2)$, then,
If $H(\alpha_1) = H(\alpha_2)$, then $\alpha_1 = \alpha_2$;
If $H(\alpha_1) > H(\alpha_2)$, then $\alpha_1 > \alpha_2$.

Example 3: Let $\alpha_1 = (s_3, ([0.6, 0.7], [0.4, 0.5]))$ and $\alpha_2 = (s_4, ([0.6, 0.75], [0.35, 0.5]))$ be any two IVPLFNs defined on a given LTS $S = \{s_0, s_1, \ldots, s_6\}$. If we employ LSF 1, we can obtain $S(\alpha_1) = 0.305$, $H(\alpha_1) = 0.315$, $S(\alpha_2) = 0.425$ and $H(\alpha_2) = 0.4317$. Hence, $\alpha_1 < \alpha_2$. If we employ LSF 2 and $\rho = 1.37$, then we can obtain $S(\alpha_1) = 0.305$, $H(\alpha_1) = 0.315$, $S(\alpha_2) = 0.3938$ and $H(\alpha_2) = 0.4$. Hence, $\alpha_1 < \alpha_2$. If we use LSF 3 and $\varepsilon = \beta = 1.25$, then we get $S(\alpha_1) = 0.305$, $H(\alpha_1) = 0.315$, $S(\alpha_2) = 0.3995$ and $H(\alpha_2) = 0.4057$. Thus, $\alpha_1 < \alpha_2$.

E. DISTANCE BETWEEN TWO IVPLFNLS BASED ON LSF
Based on LSF, we propose a new concept of distance between two IVPLFNs.

Definition 9: Let $\alpha_1 = \{s_{i_1}, ([a_1, b_1], [c_1, d_1])\}$, $\alpha_2 = \{s_{i_2}, ([a_2, b_2], [c_2, d_2])\}$ be any two IVPLFNs, then the distance between $\alpha_1$ and $\alpha_2$ is expressed as

$$ d(\alpha_1, \alpha_2) = \frac{1}{4} \times |f(\theta_1) - f(\theta_2)|$$
$$ \times \left( |a_1^2 - a_2^2| + |b_1^2 - b_2^2| + |c_1^2 - c_2^2| + |d_1^2 - d_2^2| \right).$$

Example 4: Let $\alpha_1 = (s_3, ([0.6, 0.7], [0.3, 0.4]))$ and $\alpha_2 = (s_5, ([0.4, 0.6], [0.2, 0.6]))$ be two IVPLFNs defined on a given LTS $S = \{s_0, s_1, \ldots, s_6\}$. If we use LSF 1, then the distance between $\alpha_1$ and $\alpha_2$ is

$$ d(\alpha_1, \alpha_2) = \frac{1}{4} \times \begin{vmatrix} 3 \cr 6 \end{vmatrix}$$
$$ \times \left( 0.6^2 - 0.4^2 + 0.7^2 - 0.6^2 
+ 0.3^2 - 0.2^2 + 0.4^2 - 0.6^2 \right) = 0.0483.$$
where \( T(\alpha_i) = \sum_{j=1, i \neq j}^{n} \text{Sup}(\alpha_i, \alpha_j), \) \( \text{Sup}(\alpha_i, \alpha_j) \) denotes the support for \( \alpha_i \) from \( \alpha_j \), satisfying the properties presented in Definition 10. If we assume
\[
\varsigma_i = \frac{w_i (1 + T(\alpha_i))}{\sum_{i=1}^{n} w_i (1 + T(\alpha_i))},
\]
then (22) can be transformed into
\[
\text{IVPFLPWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_{i=1}^{n} \varsigma_i \alpha_i,
\]
such that \( 0 \leq \varsigma_i \leq 1 \) and \( \sum_{i=1}^{n} \varsigma_i = 1 \).

The following theorem can be obtained based on the operations of IVPFLNs.

**Theorem 5:** Let \( \alpha_i = (s_{i_1}, \{[a_i, b_i], [c_i, d_i]\}) \) \((i = 1, 2, \ldots, n)\) be a collection of IVPFLNs, then the aggregated value by the IVPFLPWA operator is an IVPFLN and
\[
\text{IVPFLPWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left( f^{-1} \left( 1- \left( \prod_{i=1}^{n} (1-f^*(\theta_i))^{\varsigma_i} \right) \right) \right) \left( \left[ \left( \prod_{i=1}^{n} T(\alpha_j) \right) \right] \right),
\]
where
\[
T(\alpha_j) = \sum_{i=1, i \neq j}^{n} \text{Sup}(\alpha_i, \alpha_j),
\]
and \( d(\alpha_i, \alpha_j) \) is the distance between \( \alpha_i \) and \( \alpha_j \), \( \vartheta(j)(j = 1, 2, \ldots, n) \) represents any permutation of \( (1, 2, \ldots, n) \), \( L_n \) denotes all possible permutations of \( (1, 2, \ldots, n) \), \( n \) is the balancing coefficient, and \( \text{Sup}(\alpha_i, \alpha_j) \) denotes the support for \( \alpha_i \) from \( \alpha_j \), satisfying the properties in Definition 10. To simplify Eq. (26), let
\[
\eta_j = \frac{1 + T(\alpha_j)}{\sum_{j=1}^{n} (1 + T(\alpha_j))},
\]
then Eq. (26) can be written as
\[
\text{IVPFLPMM}^p(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left( \frac{1}{n^!} \otimes \sum_{j=1}^{n} \left( \eta_j \vartheta(j) \alpha_j \right)^p \right)^{\frac{1}{n^!}}.
\]

where \( 0 \leq \eta_j \leq 1 \) and \( \sum_{j=1}^{n} \eta_j = 1 \).

**Theorem 6:** Let \( \alpha_i = (s_{i_1}, \{[a_i, b_i], [c_i, d_i]\}) \) \((i = 1, 2, \ldots, n)\) be a collection of IVPFLNs, then the aggregated value by the IVPFLPMM operator is still an IVPFLN and (30), as shown at the bottom of the next page.

**Proof:** According to Definition 7, we can get
\[
\left( \eta_j \vartheta(j) \alpha_j \right)^p = \left( f^{-1} \left( \prod_{i=1}^{n} (1-f(\theta_i))^{\eta_j \vartheta(j) \alpha_j} \right) \right) \left( \left[ \left( \prod_{i=1}^{n} T(\alpha_j) \right) \right] \right),
\]
and
\[
\sum_{j=1}^{n} \left( \eta_j \vartheta(j) \alpha_j \right)^p = \left( f^{-1} \left( \prod_{i=1}^{n} (1-f(\theta_i))^{\eta_j \vartheta(j) \alpha_j} \right) \right) \left( \left[ \left( \prod_{i=1}^{n} T(\alpha_j) \right) \right] \right).
\]
and

\[
\bigg[ \prod_{\theta \in \varepsilon} \bigg( 1 - \prod_{j=1}^{n} \left( 1 - \left( 1 - f * (\theta_{\theta(\ell)}) \right)^{n_{\theta(\ell)}} \right) \right) \bigg]^{1/2},
\]

Finally, \( \left( \frac{1}{n} \bigoplus \bigotimes_{\theta \in \varepsilon} (n_{\theta(\ell)} \alpha_{\theta(\ell)}) \right)^{1/2} \), as shown at the bottom of the next page.

**IVPFLPMM** \( \alpha_1, \alpha_2, \ldots, \alpha_n \) = \( \bigg[ \prod_{\theta \in \varepsilon} \bigg( 1 - \prod_{j=1}^{n} \left( 1 - \left( 1 - f * (\theta_{\theta(\ell)}) \right)^{n_{\theta(\ell)}} \right) \right) \bigg]^{1/2} \).
Theorem 7 (Idempotency): Let \( \alpha_j (j = 1, 2, \ldots, n) \) be a set of IVPFLNs, if \( \alpha_j = \alpha = (s_0, (a, b), ([c, d])) \) of all \( j \), then

\[
IVPFLPMM^P (\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha.
\] (31)

Proof: As \( \alpha_j = \alpha = (s_0, (a, b), ([c, d])) \) \((j = 1, 2, \ldots, n)\), then \( \eta_j = 1/n \) \((j = 1, 2, \ldots, n) \). Thus, \( \eta_j = 1/n \) \((j = 1, 2, \ldots, n) \) holds for all \( j \). According to Theorem 5, we have

\[
\alpha^- \leq IVPFLPMM^P (\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+.
\] (32)

In addition

Hence, we have \( \alpha^- \leq IVPFLPMM^P (\alpha_1, \alpha_2, \ldots, \alpha_n) \). Similarly, we can get \( IVPFLPMM^P (\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+ \). Therefore \( \alpha^- \leq IV_d - ROULPMM^H (\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+ \).

In the following, we investigate some special cases of the proposed IVPFLPMM operator with respect to the parameter vector \( P \).

Case 1. If \( P = (1, 0, \ldots, 0) \), then the IVPFLPMM operator reduces to the IVPFLPA operator, i.e.

\[
IVPFLPMM^{(1,0,0,\ldots,0)} (\alpha_1, \alpha_2, \ldots, \alpha_n)
= \left\{ f^{-1} \left( \prod_{j=1}^{n} \left( 1 - \prod_{j=1}^{n} (1 - f \ast (\theta_j))^{\eta_j} \right)^{1/2} \right) \right\}.
\]

In this case, if \( \text{Sup} (\alpha_i, \alpha_j) = t (t > 0) \) for \( i, j = 1, 2, \ldots, n \) \((i \neq j)\), then the IVPFLPMM operator reduces to the interval-valued Pythagorean fuzzy linguistic average
\(IVPFLPM\) operator, i.e.

\[
IVPFLPM^{(1,0,0,0,0)}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left\{ f^{-1} \left( 1 - \prod_{j=1}^{n} \left( 1 - f \left( \theta_j \right) \right)^{1/n} \right), \right. \\
\left. \left( \prod_{j=1}^{n} \left( 1 - a_j^{1/n} \right) \right)^{1/2}, \right. \\
\left. \left( 1 - \prod_{j=1}^{n} \left( 1 - b_j^{1/n} \right) \right)^{1/2} \right\}
\]

Case 2. If \( P = (1, 1, 0, 0, 0, 0) \), then the \(IVPFLPMM\) operator reduces to the interval-valued Pythagorean fuzzy linguistic power Bonferroni mean (IVPFLPBM) operator, i.e.

\[
\left[ \prod_{j=1}^{n} c_j^{1/n}, \prod_{j=1}^{n} d_j^{1/n} \right] = \frac{1}{n} \sum_{j=1}^{n} \alpha_j = IVPFLA(\alpha_1, \alpha_2, \ldots, \alpha_n).
\]
Case 3: If \( P = \left( 1, 1, \ldots, 1, 0, 0, \ldots, 0 \right) \), then the IVPFLPMM operator reduces to the interval-valued Pythagorean fuzzy linguistic power Maclaurin symmetric mean (IVPFLPMSM) operator, i.e.

In this case, In this case, if \( \text{Sup}(\alpha_i, \alpha_j) = t (t > 0) \) for \( i, j = 1, 2, \ldots, n (i \neq j) \), then the IVPFLPMM operator reduces to the interval-valued Pythagorean fuzzy linguistic Maclaurin symmetric mean (IVPFLMSM) operator, i.e.

Case 4: If \( P = (1, 1, \ldots, 1) \) or \( P = (1/n, 1/n, \ldots, 1/n) \), then the IVPFLPMM operator reduces to the interval-valued Pythagorean fuzzy linguistic power geometric (IVPFLPG) operator, i.e.

\[
IVPFLPMM^{(1,1,\ldots,1,1/n,1/n,\ldots,1/n)}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \nbigotimes_{j=1}^{n} \left( \frac{\alpha_j^n}{\sum_{j=1}^{n} \alpha_j^n} \right) \]

D. THE INTERVAL-VALUED PYTHAGOREAN FUZZY LINGUISTIC POWER WEIGHTED MUHREAD MEAN OPERATOR

Definition 13: Let \( \alpha_j (j = 1, 2, \ldots, n) \) be a collection of IVPFLNs and \( P = (p_1, p_2, \ldots, p_n) \in R^n \) be vector of parameter. Let \( w = (w_1, w_2, \ldots, w_n)^T \) be the weight vector, such that \( \sum_{j=1}^{n} w_j = 1 \) and \( 0 \leq w_j \leq 1 \). The interval-valued Pythagorean fuzzy linguistic power weighted Muirhead mean (IVPFLPWMW) operator is defined as

\[
IVPFLPWMW^P(\alpha_1, \alpha_2, \ldots, \alpha_n) = \nbigotimes_{\theta \in L_n} \left( \frac{1}{n!} \sum_{j=1}^{n} \left( \frac{w_{\theta(j)} (1 + T (\alpha_{\theta(j)}))}{\sum_{j=1}^{n} w_j (1 + T (\alpha_j))} \right)^{p_j} \right) \]

In this case, if \( \text{Sup}(\alpha_i, \alpha_j) = t (t > 0) \) for \( i, j = 1, 2, \ldots, n (i \neq j) \), then the IVPFLPMM operator reduces to the interval-valued Pythagorean fuzzy linguistic geometric (IVPFLG) operator, i.e.

\[
IVPFLPG(\alpha_1, \alpha_2, \ldots, \alpha_n) = \nbigotimes_{j=1}^{n} \left( \frac{1}{n!} \sum_{j=1}^{n} \left( \frac{w_{\theta(j)} (1 + T (\alpha_{\theta(j)}))}{\sum_{j=1}^{n} w_j (1 + T (\alpha_j))} \right)^{p_j} \right) \]

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where \( T(\alpha_j) = \sum_{i=1, i \neq j}^{n} \text{Sup}(\alpha_i, \alpha_j), \text{Sup}(\alpha_i, \alpha_j) = 1 - d(\alpha_i, \alpha_j) \), and \( d(\alpha_i, \alpha_j) \) is the distance between \( \alpha_i \) and \( \alpha_j \). \( \vartheta(j) (j = 1, 2, \ldots, n) \) represents any permutation of \((1, 2, \ldots, n)\), \( L_n \) denotes all possible permutations of \((1, 2, \ldots, n)\), \( n \) is the balancing coefficient, and \( \text{Sup}(\alpha_i, \alpha_j) \) denotes the support for \( \alpha_i \) from \( \alpha_j \), satisfying the properties of Definition 10.

To simplify Eq. (41), let

\[ \vartheta_1 = \sum_{j=1}^{n} w_j (1 + T(\alpha_j)) \],

then Eq. (41) can be written as

\[
IVPFLPWMM^p(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left( \frac{1}{n!} \oplus_{\theta \in L_n} (n\vartheta_1(\alpha_\theta)\vartheta_1(\alpha_\theta))^p \right)^{\frac{1}{n!}}. \tag{43}
\]

**V. A NEW MAGDM METHOD UNDER IVPFLS**

Based on the proposed AOs of IVPFLNs, we propose a new MAGDM method to deal with decision-making problems under IVPFLSs. Let’s consider a MAGDM problem with IVPFL information. There are \( m \) feasible alternatives \((A_1, A_2, \ldots, A_m)\) that to be evaluated under \( n \) attributes, i.e. \((C_1, C_2, \ldots, C_n)\). The weight vector of attributes is \( w = (w_1, w_2, \ldots, w_n)^T \), such that \( \sum_{i=1}^{n} w_i = 1 \) and \( 0 \leq w_i \leq 1 \). A group of DMs are required to evaluate the performance of all the possible alternatives. Let \((D_1, D_2, \ldots, D_t)\) be the DM set, with the weight vector being \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)^T \), such that \( 0 \leq \lambda_e \leq 1 \) and \( \sum_{e=1}^{t} \lambda_e = 1 \). For attribute \( C_j (j = 1, 2, \ldots, n) \) for alter-

\[
IVPFLPBM^{1,1}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left( \frac{1}{n(n-1)} \oplus_{i,j=1}^{n} (m\lambda_i \alpha_i \otimes m\lambda_j \alpha_j) \right)^{\frac{1}{2}}. \tag{35}
\]
native $A_i (i = 1, 2, \ldots, m)$, DM $D_e (e = 1, 2, \ldots, t)$ utilizes $\alpha_{ij}^e = \left( s_{ij}^e, \left[ a_{ij}^e, b_{ij}^e \right], \left[ c_{ij}^e, d_{ij}^e \right] \right)$ to express his/her evaluation value, where $\alpha_{ij}^e$ is an IVPFLN defined on the linguistic set $S$. Hence, $t$ interval-valued Pythagorean fuzzy decision matrices are obtained. In the following, based on the proposed AOs, we give the main steps of determining the rank of feasible alternatives.

**Step 1.** Generally, there are two types of attributes, i.e., benefit type and cost type. Hence, the original decision matrices should be normalized. If $C_j$ is benefit type, then the original decision matrices do not need to be normalized. If $C_j$ is cost type, then the original decision matrices should be changed according to the following formula

$$\alpha_{ij}^e = \left( s_{ij}^e, \left[ c_{ij}^e, d_{ij}^e \right], \left[ a_{ij}^e, b_{ij}^e \right] \right).$$

**Step 2.** Compute $\text{Sup} \left( \alpha_{ij}^k, \alpha_{ij}^d \right) (k, d = 1, 2, \ldots, t; k \neq d)$ according to the following equation

$$\text{Sup} \left( \alpha_{ij}^k, \alpha_{ij}^d \right) = 1 - d \left( \alpha_{ij}^k, \alpha_{ij}^d \right),$$

where $d \left( \alpha_{ij}^k, \alpha_{ij}^d \right)$ is the distance between the two IVPFLNs $\alpha_{ij}^k$ and $\alpha_{ij}^d$. Definition 8 illustrates how to calculate the distance between any two IVPFLNs.

**Step 3.** Calculate the overall supports $T \left( \alpha_{ij}^k \right)$ by

$$T \left( \alpha_{ij}^k \right) = \sum_{g=1, g \neq k}^t \text{Sup} \left( \alpha_{ij}^k, \alpha_{ij}^g \right).$$

**Step 4.** For DM $D_k$, compute the power weight associated with the IVPFLN $\alpha_{ij}^k$ by

$$\delta_{ij}^k = \frac{\lambda_k \left( 1 + T \left( \alpha_{ij}^k \right) \right)}{\sum_{k=1}^t \lambda_k \left( 1 + T \left( \alpha_{ij}^k \right) \right)}.$$ 

**Step 5.** Use the IVPFLPM operator to determine the comprehensive decision matrix

$$\alpha_{ij} = \text{IVPFLPM} \left( \alpha_{ij}^1, \alpha_{ij}^2, \ldots, \alpha_{ij}^t \right).$$

**Step 6.** Calculate the support between the two IVPFLNs $\alpha_{ij}$ and $\alpha_{ij}'$ by

$$\text{Sup} \left( \alpha_{ij}, \alpha_{ij}' \right) = 1 - d \left( \alpha_{ij}, \alpha_{ij}' \right).$$

---

IVPFLPM$^{(1,1,0,\ldots,0)} (\alpha_1, \alpha_2, \ldots, \alpha_n)$

$$= \left\{ f \ast^{-1} \left( \left( 1 - \prod_{i,j=1}^n \left( 1 - f \ast \left( \theta_i \right) f \ast \left( \theta_j \right) \right)^{\frac{1}{n(n-1)}} \right) \right) \right\}^{\frac{1}{2}}$$

$$= \left\{ \left[ 1 - \prod_{i,j=1}^n \left( 1 - a_{ij}^2 b_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right]^{\frac{1}{2}}, \left[ 1 - \prod_{i,j=1}^n \left( 1 - b_{ij}^2 b_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right]^{\frac{1}{2}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( c_{ij}^2 - c_{ij}^2 c_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( d_{ij}^2 - d_{ij}^2 d_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( e_{ij}^2 - e_{ij}^2 e_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( f_{ij}^2 - f_{ij}^2 f_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( g_{ij}^2 - g_{ij}^2 g_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( h_{ij}^2 - h_{ij}^2 h_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( i_{ij}^2 - i_{ij}^2 i_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( j_{ij}^2 - j_{ij}^2 j_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( k_{ij}^2 - k_{ij}^2 k_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( l_{ij}^2 - l_{ij}^2 l_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( m_{ij}^2 - m_{ij}^2 m_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( n_{ij}^2 - n_{ij}^2 n_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( o_{ij}^2 - o_{ij}^2 o_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( p_{ij}^2 - p_{ij}^2 p_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( q_{ij}^2 - q_{ij}^2 q_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( r_{ij}^2 - r_{ij}^2 r_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( s_{ij}^2 - s_{ij}^2 s_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( t_{ij}^2 - t_{ij}^2 t_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( u_{ij}^2 - u_{ij}^2 u_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( v_{ij}^2 - v_{ij}^2 v_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( w_{ij}^2 - w_{ij}^2 w_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( x_{ij}^2 - x_{ij}^2 x_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( y_{ij}^2 - y_{ij}^2 y_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( z_{ij}^2 - z_{ij}^2 z_{ij}^2 \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \left\{ 1 - \prod_{i,j=1}^n \left( \alpha_i \ast \alpha_j \right)^{\frac{1}{n(n-1)}} \right\}^{\frac{1}{2}}$$

$$= \text{IVPFLBM}_{1,1}^{(1,1,0,\ldots,0)} (\alpha_1, \alpha_2, \ldots, \alpha_n).$$
wherein \( l, f = 1, 2, \ldots, n; l \neq j \)

**Step 7.** Compute the overall \( T(α_{ij}) \) by

\[
T(α_{ij}) = \sum_{j=1; j \neq i}^{n} \text{Sup}(α_{ij}, α_{ik}) .
\]

**Step 8.** Compute the power weight associated with the IVPFLN \( α_{ij} \) by

\[
γ_{ij} = \frac{w_j (1 + T(α_{ij}))}{\sum_{j=1}^{n} w_j (1 + T(α_{ij}))} .
\]

**Step 9.** For each alternative, use the proposed IVPFLP-WMM to compute the overall evaluation values, i.e.

\[
α_i = IVPFLPWMM^Y(α_{i1}, α_{i2}, \ldots, α_{in}) .
\]

and a series of overall evaluation values \( α_i (i = 1, 2, \ldots, m) \) are obtained.

**Step 10.** Compute the scores of \( α_i (i = 1, 2, \ldots, m) \) according to Definition 8.

**Step 11.** Rank the corresponding alternatives and select the optimal one.

**VI. NUMERICAL EXAMPLES**

Example 5 (Revised from [33]): Suppose that there are four command and control systems \( \{A_1, A_2, A_3, A_4\} \) that to be evaluated under three attributes, i.e. system availability \( (G_1) \), information accuracy \( (G_2) \) and picture completeness \( (G_3) \). The weight vector of the attributes is \( λ = (0.3727, 0.3500, 0.2773)^T \). Let \( S = \{s_0, s_1, \ldots, s_6\} \) be a linguistic term set and three decision experts \( D_1, D_2, \text{and } D_3 \) utilize IVPFLNs defined on \( S \) to express their decision ideas. Hence, three IVPFL decision matrices \( R_1, R_2 \text{ and } R_3 \) are obtained, which are shown in Tables 1-3. The weight vector of DMs is \( λ = (0.3, 0.4, 0.3)^T \).

**A. THE DECISION-MAKING PROCESS**

We use the MAGDM method introduced in Section V to resolve Example 5, and the particular methods procedure is presented as follows.

**Step 1:** It is clearly that all attributes are benefit type, there is no need to normalize the original decision matrix.

**Step 2:** Calculate the \( \text{Sup}(α_{ij}^k, α_{ij}^d) \) according to Eq. (46) (Suppose LFS1 is utilized as the specified LSF in the calculation process). For convenience, we utilize the symbol \( S_{ij}^k \) to represent the support between \( α_{ij}^k \) and \( α_{ij}^d (i = 1, 2, 3, 4, 5; j = 1, 2, 3; k, d = 1, 2, 3; k \neq d) \).
Hence, we obtain the following results

\[
T \left( \alpha_{ij}^k \right) (i = 1, 2, 3, 4, 5; j = 1, 2, 3; k = 1, 2, 3)
\]

\[
S_2^1 = S_1^2 = \begin{bmatrix}
0.9842 & 0.9858 & 0.9863 \\
0.9588 & 1.0000 & 0.9875 \\
0.9255 & 0.9963 & 1.0000 \\
0.9525 & 0.9408 & 0.9842 \\
1.0000 & 0.9483 & 0.9800
\end{bmatrix}
\]

\[
S_3^1 = S_1^3 = \begin{bmatrix}
0.9550 & 1.0000 & 0.9717 \\
0.9592 & 0.9929 & 0.9750 \\
0.9683 & 0.9783 & 1.0000 \\
0.9729 & 1.0000 & 1.0000 \\
0.9733 & 0.9833 & 0.9750 \\
0.9850 & 0.9933 & 0.9888 \\
0.9758 & 0.9733 & 0.9854 \\
0.9929 & 0.9904 & 1.0000 \\
0.9942 & 0.9900 & 0.9908 \\
0.9896 & 0.9908 & 0.9900
\end{bmatrix}
\]

\[
S_2^3 = S_3^2 = \begin{bmatrix}
1.9392 & 1.9858 & 1.9579 \\
1.9179 & 1.9929 & 1.9625 \\
1.8908 & 1.9746 & 2.0000 \\
1.9254 & 1.9408 & 1.9842 \\
1.9733 & 1.9317 & 1.9550 \\
1.9692 & 1.9792 & 1.9750 \\
1.9346 & 1.9733 & 1.9729 \\
1.9154 & 1.9867 & 2.0000 \\
1.9467 & 1.9308 & 1.9750 \\
1.9896 & 1.9392 & 1.9700 \\
1.9400 & 1.9933 & 1.9604 \\
1.9350 & 1.9663 & 1.9604 \\
1.9613 & 1.9688 & 2.0000 \\
1.9671 & 1.9900 & 1.9908 \\
1.9629 & 1.9742 & 1.9650
\end{bmatrix}
\]

**Step 3:** Calculate \( T \left( \alpha_{ij}^k \right) \) according to Eq. (47). For convenience, we use the symbol \( T^k \) to represent the values \( \sigma^k \) to represent the values \( \delta_{ij}^k \) according to Eq. (49). For convenience, we use the symbol \( \delta^k \) to represent the values \( \delta_{ij}^k \) (i, j = 1, 2, 3; k = 1, 2, 3).

**Step 4:** For DM \( D_k \), calculate his/her power weight associated with the IVPFLV \( \alpha_{ij}^k \) on the basis of his/her weight \( \gamma_k \) according to Eq. (49). For convenience, we use the symbol \( \delta^k \) to represent the values \( \delta_{ij}^k \) (i, j = 1, 2, 3).
Therefore, we can obtain the following results.

Step 5. Utilize the IVPFLPWA operator to aggregate the individual decision matrices into a collective one, which is listed in Table 4.

Step 6. For Table 4, calculate the support between $\alpha_{il}$ and $\alpha_{lf}$, that is, $Sup(\alpha_{il}, \alpha_{lf})$ according to Eq. (50). For convenience, we utilize the symbol $S^{il}$ to represent the value $Sup(\alpha_{il}, \alpha_{lf})$ ($i = 1, 2, 3, 4, 5; l, f = 1, 2, 3; l \neq f$). Hence, we can obtain the following results

\[
S^{12} = S^{21} = (0.9547, 0.9747, 0.9853, 0.9946),
\]
\[
S^{13} = S^{31} = (0.9782, 0.9550, 0.9988, 0.9928),
\]
\[
S^{23} = S^{32} = (0.9975, 0.9983, 0.9863, 0.9899).
\]

Step 7: Calculate the support $T(\alpha_{ij})$ according to Eq. (51). Similarly, we use the symbol $T_{ij}$ to denote the value $T(\alpha_{ij})$

$\text{IVPFLPWWM}^{\mu}(\alpha_1, \alpha_2, \ldots, \alpha_n)$

\[
= f^{-1} \left( \prod_{\theta \in L_n} \left( \prod_{j=1}^{n} \left( 1 - \left( 1 - f \ast \left( \theta_{\theta(i,j)}^{\mu} \right)^{\gamma} \right) \right) \right)^{1/\mu} \right),
\]

\[
= \left( \prod_{\theta \in L_n} \left( \prod_{j=1}^{n} \left( 1 - \left( 1 - \alpha_{\theta(i,j)}^{\mu} \right)^{\gamma} \right) \right) \right)^{1/\mu},
\]

\[
= \left( \prod_{\theta \in L_n} \left( \prod_{j=1}^{n} \left( 1 - \left( 1 - \beta_{\theta(i,j)}^{\mu} \right)^{\gamma} \right) \right) \right)^{1/\mu},
\]

\[
= \left( \prod_{\theta \in L_n} \left( \prod_{j=1}^{n} \left( 1 - \left( 1 - \gamma_{\theta(i,j)}^{\mu} \right)^{\gamma} \right) \right) \right)^{1/\mu},
\]

\[
= \left( \prod_{\theta \in L_n} \left( \prod_{j=1}^{n} \left( 1 - \left( 1 - \delta_{\theta(i,j)}^{\mu} \right)^{\gamma} \right) \right) \right)^{1/\mu}.
\]
for simplicity, and we can obtain the following matrix

$$T = \begin{bmatrix} 3.8276 & 3.9303 & 3.9501 \\ 3.8738 & 3.9576 & 3.9757 \\ 3.9233 & 3.9300 & 3.9890 \end{bmatrix}$$

**Step 8:** Calculate the power weight $\gamma$ associated with the IVPFLV $\alpha_{ij}$ according to Eq. (52), and we have

$$\gamma = \begin{bmatrix} 0.3694 & 0.3720 & 0.3712 \\ 0.3503 & 0.3513 & 0.3504 \\ 0.2803 & 0.2768 & 0.2784 \end{bmatrix}$$

**Step 9:** For alternative $A_i$ ($i = 1, 2, 3, 4, 5$), utilize the IVPFLPWMM operator to calculate the overall evaluation $\alpha_i$ ($i = 1, 2, 3, 4, 5$). Without the loss of generality, let $H = (1, 1, 1)$ and the overall evaluation values are shown as follows

$$\alpha_1 = \langle 3.9562, (0.5967, 0.7387), (0.1010, 0.2119) \rangle$$
$$\alpha_2 = \langle 3.2478, (0.6468, 0.8531), (0.0852, 0.1634) \rangle$$
$$\alpha_3 = \langle 3.8006, (0.6022, 0.7373), (0.0000, 0.2272) \rangle$$
$$\alpha_4 = \langle s_{3.3774}, (0.5241, 0.6682), (0.1282, 0.3110) \rangle$$

**Step 10:** Calculate the score values $S(\alpha_i)$ ($i = 1, 2, 3, 4$) and we can get

$$S(\alpha_1) = 0.4693$$
$$S(\alpha_2) = 0.4211$$
$$S(\alpha_3) = 0.4520$$
$$S(\alpha_4) = 0.3670$$
$$S(\alpha_5) = 0.4431$$

**Step 11:** According to the score values $S(\alpha_i)$ ($i = 1, 2, 3, 4, 5$), the ranking order of the alternatives can be determined, that is, $A_1 > A_3 > A_5 > A_2 > A_4$. Therefore, $A_3$ is the best alternative.

**B. THE INFLUENCE OF THE PARAMETERS ON THE RESULTS**

In this section, we try to investigate the influence of the parameter vector $P$ and LSF $f$ on the decision results.
1) THE INFLUENCE OF THE PARAMETER VECTOR \( P \)
We assign different values in \( P \) in the IVPFLPWWMM operator and present the decision results in Table 5 (LSF 1 is used in the calculation process).

As we can see from Table 5, when \( P = (1, 0, 0) \) the ranking order of alternatives is \( A_3 > A_1 > A_2 > A_5 > A_4 \), and the optimal alternative is \( A_3 \). When \( P = (1, 1, 0) \) and \( P = (1, 1, 1) \), the ranking order of alternatives is \( A_1 > A_3 > A_5 > A_2 > A_4 \), and the best alternative is \( A_1 \). This is because when \( P = (1, 0, 0) \), our proposed decision-making method does not consider the interrelationship among attributes. In other word, when \( P = (1, 0, 0) \), it is assumed that all the attributes are independent. When \( P = (1, 1, 0) \) and \( P = (1, 1, 1) \), our proposed method takes the interrelationship among attributes. In this Example 5, there exists evident interrelationship between attributes. Let \( N_P \) denote the number of related attributes (\( N_P = 1, 2, 3 \)). We notice that with the increase of \( N_P \), the score values of alternatives will decrease, which illustrates the flexibility of our proposed decision-making method. In actual decision situations, DMs can select the proper parameter \( P \) according to practical needs.

2) THE IMPACT OF THE LSF ON THE RESULTS
Then, we investigate how the LSF \( f \) affects the decision results. We take different LSFs in the operations of IVPFLNs and presented the decision results in Table 6. As seen from Table 6, different score values of the overall evaluation values are obtained by using different LSF in the calculation process, which further leads to different ranking orders of all the feasible alternatives. In real decision-making situations, DMs and select a proper LSF according to actual needs.

C. VALIDITY OF OUR PROPOSED METHOD
In this subsection, we attempt to prove the effectiveness of our proposed method through comparison method. First, we compare our method with some other methods based on IVPFLSs. Then, as IVILS is a special case of IVPFLSs we then compare our method with those based on IVILSs to further explain the effectiveness of our method.

1) COMPARED WITH DECISION-MAKING METHOD BASED ON IVFLS
We compare our proposed method based on the IVPFLPWWMM operator with Du et al.’s method based on interval-valued Pythagorean fuzzy linguistic weighted average (IVFPLWA) operator. The two methods are used to solve the following decision-making problem and we compare their final results.

Example 6 (Adopted from Du et al. [32]): The government wants to evaluate the performance of five departments during the rescue work after the earthquake occurred. Let \( A_i (i = 1, 2, \ldots, 5) \) be the five departments and DMs evaluate their emergency response capabilities under five attributes, i.e. the emergency forecasting capability \((G_1)\), the emergency process capability \((G_2)\), the after-disaster loss evaluation capability \((G_3)\), the emergency support capability \((G_4)\), and the after-disaster reconstruction capability \((G_5)\). Let \( w = (0.15, 0.28, 0.18, 0.25, 0.14) \) be the weight vector of the attributes. Assume \( S = (s_0, s_1, \ldots, s_6) \) to be a predefined linguistic term set and the DMs employ IVPFLNs defined on \( S \) to express their evaluation values. The decision matrix is presented in Table 7. We use our proposed method and Du et al.’s [32] method to solve Example 6 and present in the decision results in Table 8.
As we can see from Table 8, although the score values of alternatives produced by the two different methods are different, the ranking orders of alternatives are the same and the best alternative is $A_3$, which illustrates the validity of our proposed method.

2) COMPARED WITH DECISION-MAKING METHODS UNDER IVILS

We continue to compare our proposed method with that proposed by Dong and Wan’s [33] based on the interval-valued intuitionistic linguistic weighted arithmetic average (IVILWAA) operator. We employ our propose method and Dong and Wang’s method to solve Example 5 and present the decision results in Table 9. As we can see from Table 9, the ranking order produced by our proposed method is the same as that obtained by Dong and Wan’s [33] decision-making method, and that illustrates the validity of our proposed method.

D. ADVANTAGES OF OUR METHOD

In this section, we detailly investigate the advantages and superiorities of our proposed method though numerical examples.

1) IT EFFECTIVELY DEALS WITH DMS’ UNREASONABLE EVALUATION VALUES

In some practical decision-making problems, some DMs may provide unreasonable decision-making values. The reasons are usually two-fold. First, due to the intricacy of MAGDM problems, DMs can hardly get all the information of all the feasible alternatives in a limited time and they may provide unduly high or low evaluation values. Second, as DMs usually have different background and priori knowledge some of them maybe prejudiced against some alternatives and they probably provide tendentious evaluation values. Evidently, the biased evaluation information offered by prejudiced DMs have negative influence on the final decision results. If such bad impact is not eliminated or reduces, unreasonable decision outcomes may be gained. It is noted that our method is based on the PA operator. The PA operator was originated to deal with possible unreasonable evaluation values and hence, our proposed method can also effectively handle the unduly high or low evaluation values.

2) IT CONSIDERS THE COMPLICATED INTERRELATIONSHIP BETWEEN ATTRIBUTES

As we can see from Table 8, when $P = (1, 0, 0, 0, 0)$, then our proposed method produces the same ranking order as Du et al.’s [32] method. This is because Du et al.’s [32] method is based on the simply weighted average operator, which does not consider the interrelationship between attributes. When $P = (1, 0, 0, 0, 0)$, our proposed method does not take the interrelationship between the attributes, which is the same as the weighted average operator. In addition, when $P = (1, 1, 0, 0, 0)$, our method takes the interrelationship between any two attributes into consideration.

### TABLE 7. The interval-valued Pythagorean fuzzy linguistic decision matrix of Example 6.

<table>
<thead>
<tr>
<th>$G_i$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_{G_i}([0.6,0.7],[0.4,0.5])$</td>
<td>$s_{A_1}([0.6,0.75],[0.35,0.5])$</td>
<td>$s_{A_2}([0.7,0.85],[0.45,0.5])$</td>
<td>$s_{A_3}([0.65,0.78],[0.4,0.55])$</td>
<td></td>
</tr>
<tr>
<td>$s_{G_i}([0.55,0.7],[0.5,0.65])$</td>
<td>$s_{A_1}([0.55,0.75],[0.5,0.65])$</td>
<td>$s_{A_2}([0.7,0.75],[0.35,0.45])$</td>
<td>$s_{A_3}([0.6,0.75],[0.4,0.55])$</td>
<td></td>
</tr>
<tr>
<td>$s_{G_i}([0.5,0.7],[0.55,0.65])$</td>
<td>$s_{A_1}([0.65,0.7],[0.45,0.6])$</td>
<td>$s_{A_2}([0.7,0.75],[0.4,0.5])$</td>
<td>$s_{A_3}([0.65,0.72],[0.42,0.55])$</td>
<td></td>
</tr>
<tr>
<td>$s_{G_i}([0.7,0.75],[0.4,0.5])$</td>
<td>$s_{A_1}([0.6,0.75],[0.4,0.5])$</td>
<td>$s_{A_2}([0.65,0.75],[0.45,0.5])$</td>
<td>$s_{A_3}([0.7,0.8],[0.4,0.5])$</td>
<td></td>
</tr>
<tr>
<td>$s_{G_i}([0.6,0.75],[0.4,0.5])$</td>
<td>$s_{A_1}([0.6,0.75],[0.4,0.5])$</td>
<td>$s_{A_2}([0.6,0.7],[0.4,0.5])$</td>
<td>$s_{A_3}([0.75,0.8],[0.3,0.5])$</td>
<td></td>
</tr>
</tbody>
</table>

### TABLE 8. The decision results of Example 6 by different methods.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Score functions $S(\alpha_i)(i = 1,2,3,4)$</th>
<th>Ranking orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>Du et al.’s [32] method</td>
<td>$S(\alpha_1) = 1.9769 , S(\alpha_2) = 2.2098 , S(\alpha_3) = 2.7942 , S(\alpha_4) = 2.4756$</td>
<td>$A_3 &gt; A_4 &gt; A_2 &gt; A_1$</td>
</tr>
<tr>
<td>The proposed method when $P = (1,1,1,1)$</td>
<td>$S(\alpha_1) = 0.3323 , S(\alpha_2) = 0.3424 , S(\alpha_3) = 0.4436 , S(\alpha_4) = 0.3846$</td>
<td>$A_3 &gt; A_4 &gt; A_2 &gt; A_1$</td>
</tr>
</tbody>
</table>

### TABLE 9. The ranking results with different methods of Example 5.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Score functions $S(\alpha_i)(i = 1,2,3,4,5)$</th>
<th>Ranking orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dong and Wan’s [51] method</td>
<td>$S(\alpha_1) = 2.7478 , S(\alpha_2) = 3.0466 , S(\alpha_3) = 2.9457 , S(\alpha_4) = 2.4734 , S(\alpha_5) = 2.8129$</td>
<td>$A_1 &gt; A_4 &gt; A_3 &gt; A_2 &gt; A_5$</td>
</tr>
<tr>
<td>The proposed method when $P = (1,1,1,1,1)$</td>
<td>$S(\alpha_1) = 0.4211 , S(\alpha_2) = 0.4693 , S(\alpha_3) = 0.4520 , S(\alpha_4) = 0.3670 , S(\alpha_5) = 0.4431$</td>
<td>$A_1 &gt; A_4 &gt; A_3 &gt; A_2 &gt; A_5$</td>
</tr>
</tbody>
</table>
When \( P = (1, 1, 1, 1, 1) \), the interrelationship among multiple attribute values is taken into consideration. Similarly, as seen from Table 8 our proposed method produces the same ranking order as Du et al.’s [32] method when \( P = (1, 0, 0) \). In this case, both our proposed method and Dong and Wang’s [33] method assume that the attributes are independent and there is no interrelationship between attributes. When \( P = (1, 1, 0) \), our method reflects the interrelationship among any two attributes. When \( P = (1, 1, 1) \), the interrelationship among all the attributes is taken into account. In most real decision-making problems, the attributes are usually correlated and there exists complicated interrelationship among attributes. To determine the optimal alternative, the interrelationship among attributes should be taken into consideration.

If there indeed no interrelationship between attributes, we can assign \( P = (s, 0, 0, \ldots, 0) (s > 0) \) in our method. Hence, our method is more powerful and flexible than those proposed by Du et al. [32] and Dong and Wan [33].

3) THE PROPOSED OPERATIONS OF IVPFLNs ARE MORE REASONABLE AND FLEXIBLE

In addition, the operations in this article are more flexible. First, they consider satisfy the constraint of LTS. Second, they consider DMs’ attitude toward optimism and pessimism. Hence, they can more suitable and powerful to deal with practical MAGDM problems. However, it should be noted that the operations proposed by Du et al. [32] and Dong and Wan [33] are based on simple operators and they have shortcomings in dealing with practical MAGDM problems.

VII. CONCLUSION

This article investigates MAGDM method wherein DMs’ evaluation information is expressed as IVPFLNs. The main contributions of this article include three aspects. First, we proposed new operational laws of IVPFLNs based on LTS. The new operations are more reasonable and can depict DMs’ attitude towards optimism and pessimism. Second, we proposed novel AOs of IVPFLNs based on the PA and PMM operators. These operators can more effectively deal with DMs’ biased evaluation values and take the interrelationship among attributes into consideration. Third, we presented a new MAGDM method. In the new decision-making method, the IVPFLPWA operator is used to compute the collective decision matrix and the IVPFLPWWMM operator is employed to calculate the overall evaluation value of each alternative. Finally, we showed the validity of the proposed method. We also conducted comparison analysis to illustrate the advantages and superiorities of our method.

In the future, we will continue our research from two aspects. First, we shall study more applications of our decision-making methods in more practical MAGDM problems. Second, we will study more MAGDM methods based on IVPFL information to provide DMs more options of manners to select the best alternative.

REFERENCES


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