A NOVEL REPRESENTATION AND ALGORITHMS FOR (QUASI) STABLE MARRIAGES

B. Y. Zavidovique
Institut d'Electronique Fondamentale, Université Paris 11
Bâtiment 220 - 91405 Orsay Cedex
Email: bertrand.zavidovique@ief.u-psud.fr

N. Suvonvorn
Institut d'Electronique Fondamentale, Université Paris 11
Bâtiment 220 - 91405 Orsay Cedex
Email: nikom.suvonvorn@ief.u-psud.fr

Guna S. Seetharaman
Air Force Institute of Technology
AFIT/ENG 2950 P Street Wright Patterson AFB OH 45433-7765
Email: guna@cacs.louisiana.edu

Keywords: stable marriages, matching algorithms.

Abstract: In this paper, we propose "stable marriages" algorithms based on a novel representation called *marriage table*. After explaining how properties as global satisfaction, sex equality and stability show in the representation, we define 3 algorithms corresponding to 3 different scans of the *marriage table* to meet progressively all constraints. The performance is evaluated in front of the population size for 200 instances in each case. That supports qualitative statistic analysis. Two matching examples in image processing are displayed for illustration.

1 INTRODUCTION

The problem of stable marriage was first studied by Gale and Shapley (Gale and Shapley, 1962). In this problem, two finite sub-sets M and W of two respective populations, say men and women, have to match. Assume n is the number of elements, $M = \{m_1, m_2, ..., m_n\}$ and $W = \{w_1, w_2, ..., w_n\}$. Each element x creates its preference list $l(x)$ i.e. it sorts all members of the opposite sex from most to less preferred. A matching $\mathcal{M}$ is a one to one correspondence between men and women. If (m, w) is a matched pair in $\mathcal{M}$, we note $\mathcal{M}(m) = w$ and $\mathcal{M}(w) = m$ and ρ_m is the rank of m in the list of w (resp. ρ_w the rank of w in the list of m) . Man m and woman w form a blocking pair if (m, w) is not in $\mathcal{M}$ but m prefers w to $\mathcal{M}(m)$ and w prefers m to $\mathcal{M}(w)$. If there is no blocking pair, then the matching $\mathcal{M}$ is stable (Abeledo and Rothblum, 1995), (Diamantoudi et al., 2004). Gale and Shapley proved that there is always at least one stable matching $\mathcal{M}$ whichever the instance $\{M, W, l(m), l(w)\}$. They proposed the algorithm of Gale-Shapley (GS) to find $\mathcal{M}$ with complexity $O(n^2)$.

Since then, this optimization problem was constantly one among the most popular in combinatorics from both theoretical (McVitie and Wilson, 1971), (K. Iwama and Morita, 1999),(D.F. Manlove et al., 2002), (Gent and Prosser, 2002), (McVitie and Wilson, 1971) and practical points of view (D. Bianco and Larimer, 2001), (C.P. Teo and Tan, 1999), (T. Kavitha and Paluch, 2004). According to (K. Iwama and Morita, 1999), the stable marriage problem was studied and generalized in 4 directions: (i) stable marriage with complete list and total order, the case of Gale and Shapley; (ii) stable marriage with incomplete list and total order (Manlove, 1999), (iii) stable marriage with complete list and indifference (Irving, 1994), and (iv) stable marriage with incomplete list and indifference (K. Iwama and Morita, 1999).

GS has usually two different solutions, *men-optimal* and *women-optimal* depending whom is asked first to choose. *Men-optimal* brings a stable matching in which men have the best possible partner and women may have the worst and conversely.

In many applications of such optimization on bipartite graphs, as resource scheduling, there might be reasons why to favour one sub-population: for instance demand constraints are economically more important than supply ones, or teachers constraints might be more strict than class-rooms ones. In many other such problems like segment-pairing in robot vision for stereo reconstruction, motion understanding or object recognition there is an a priori equal importance of both sets of segments respectively extracted from a couple of images or from the model (Bouchafa and Zavidovique,), (J.L.Lisani et al., 2001), (Monasse and Guichard, 1998), (Ballester et al., 1998), (Caselles et al., 1999) : then sex equality is likely worth accounting for, leading to a *fair* algorithm. Moreover, some global satisfaction from the matching may translate a better balanced solution among the many possible ones. Neither one is guaranteed by GS : the obtained stable matching can be such that everybody is unsatisfied. A last difficulty comes from the order in which men or women are considered inside their own sub-set, it can influence the result.

In this article, we propose a stable marriages algorithm based on a novel representation, called *marriage table*. The new algorithm fits stable marriages with complete/incomplete list and total order. It aims at stability, sex equality and global satisfaction.

2 NOVEL REPRESENTATION OF THE STABLE MARRIAGES PROBLEM

In order to build an algorithm that had a chance to meet the three criteria of stability, sex equality and global satisfaction, we first change representation. The so-called *marriage table* translates and supplements the preference lists. Stable matchings are looked for by scanning this latter array and suitable properties of the solution are associated to the type of scan. The *marriage table* is a table with $(n+1)$ lines and $(n+1)$ columns. Lines (resp. columns) frame the preference orders of men, $\{1 \cdots p \cdots N\ \infty\}$ (resp. women, $\{1 \cdots q \cdots N\ \infty\}$). The cell (p, q) contains pairs (m, w) such that w is the p^{th} choice of m, and m is the q^{th} choice of w. Cells can thus contain more than one pair or none. The cell (p, ∞) (resp. (∞, q)) contains the pairs where the woman is the p^{th} choice of the man (resp the q^{th} choice of the woman) but the man does not exist in her preference list (resp. the woman is not in his preference list). A key feature of this table in the "complete list" case is that each line contains all men once and each column contains all women once. The figure 1 shows a typical marriage table.

The table 1 is the example of an instance of three men and women. Every man or woman made their

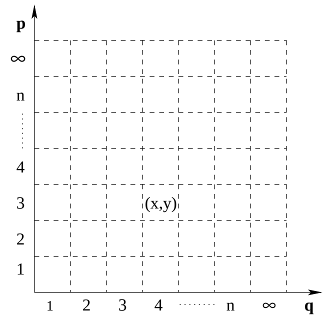
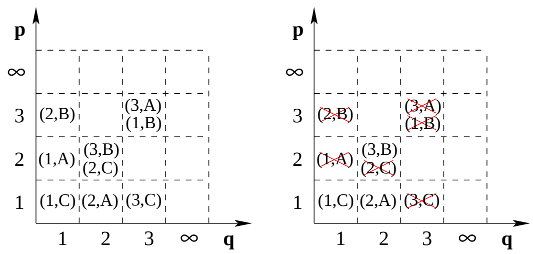


Figure 1: Marriage table : the pair (x,y), y is the 3^{rd} choice of x and x is the 4^{th} choice of y

preference list. The figure 2 is the marriage table and stable matching established from the population 1.

Men	Women
1 : C, A, B	A : 1, 2, 3
2 : A, C, B	B : 2, 3, 1
3 : C, B, A	C : 1, 2, 3

Table 1: An instance of 3 men and women and their preference lists



(a) (b)

Figure 2: (a) Marriage table established from the table 1 (b) Matching result : (1,C), (2,A) and (3,B).

One advantage of the marriage table is that satisfaction and equality of sex show concurrently in the same representation.

For instance let us define a global satisfaction by:

$$S = \sum_{(m,w) \in M} (\rho_m + \rho_w) \qquad (1)$$

Intuitively, the closer S to zero the greater global satisfaction: in average more people are satisfied. A solution with maximum global satisfaction would display matched pairs as close around the origin (table bottom-left) as mutual exclusion allows. More generally the table representation is indicative of a result global satisfaction through the lay out of the selected couples. Satisfaction is constant along antidiagonals (straight lines of equation $p+q =$ constant) and decreasing with the distance to the origin . And that provides some criteria to design scans of the marriage table that could favour better global solutions.

Conversely, sex equality tends to fit the diagonal of the marriage table. Let us define it as

$$S = \sum_{(m,w)\in M} |\rho_m - \rho_w| \quad (2)$$

Intuitively the closer to the diagonal the more balanced treatment. Elements of a pair in a cell close to the diagonal are equally satisfied or unsatisfied, depending on the distance to the origin. The smaller the greater equity. And again that provides some criteria to design scans of the marriage table that could favour more equitable solutions.

Stability gets a graphic translation too in the marriage table. In the case of complete preference lists, a blocking situation is represented figure 3. Assuming (x,t) and (z,y) were respectively paired, then (x,y) cannot be in the grey rectangle and (z,t) cannot be in the dashed one. These constraints will help building a new algorithm. In the case of uncomplete preference lists an additionnal blocking situation is when m and w are not matched in $\mathcal{M}$ but m finds w acceptable and w finds m acceptable.

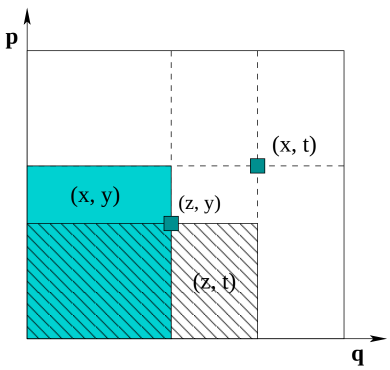


Figure 3: Blocking situation in a marriage table.

3 MARRIAGES AND TABLE SCANS

Algorithms can now be designed to find marriages that would globally guarantee one or the other property. Any selection process is a scan of the marriage table along which couples are stored or not and then released or not depending on circumstances. After the analysis above, suitable scans to meet all constraints should be zig-zag ones that trade off between the first and second diagonal directions. Considering a priori symmetries of the marriage table, scanning from left to right or conversely does not matter statistically and same for scanning from origin to top or conversely. Indeed, given an instance, changing man's lists into woman's lists makes it for the right/left invariance and complementing preferences to the population-size makes it for the top-bottom invariance. We study experimentally three strategies here (zigzag ZZ with man-optimal or woman-optimal, optimal (symmetric) zigzag OZ , blocked zigzag BZ) and then compare the results with GS in a systematic way: 200 instances are built at random for populations of 5, 10, 50, 100, 150 and 200 respectively. Each algorithm is run on the populations and for each one the following plots are displayed and analyzed: global satisfaction vs instances index, fairness vs instances index and, in case there are, number of blocking pairs vs populations.

3.1 Zigzag with man-optimal or woman-optimal (ZZ)

It appears from the global satisfaction graph (figure 4) that its patterns present some similarity (min/max, σ) by sample of 50. We pick up 40 consecutive instances at random for display to show results more in details. From the figure 5, one can see that the global satisfaction obtained by ZZ is better in average than the one by GS. Its variation is also smaller, meaning that results from ZZ are more consistent (hence more reliable to global matching) than by GS.

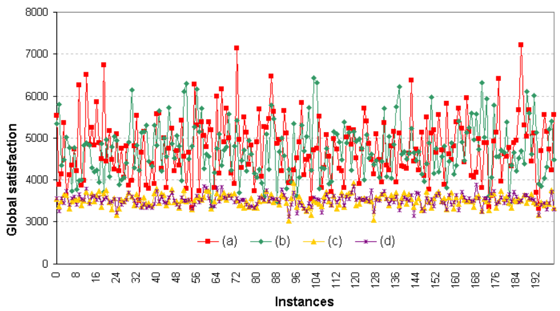


Figure 4: Comparing global satisfaction between methods: (a) GS man-optimal (b) GS woman-optimal (c) Zigzag man-optimal and (d) Zigzag woman-optimal, in case of 150 large populations.

Figure 6 shows the same sample of 40 instances for sex equality. Trends of that type of plot are very similar to global satisfaction trends. The jittery pattern of GSs are similar for global satisfaction and sex equality, with average standard deviation in the range of 40 to 50%. In both cases, strong variations from one instance to the next one show qualitative same

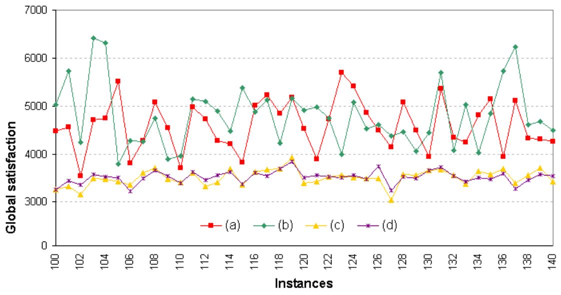


Figure 5: Close up on global satisfaction between methods: (a) GS man-optimal (b) GS woman-optimal (c) Zigzag man-optimal and (d) Zigzag woman-optimal

tendancies, contrasting with comparatively bounded variations by ZZ (15 to 20%). In average ZZ performs twice as well as GS (3500 vs. 4800 and 1800 vs. 3800). However the improvement is much more significant regarding sex equality that is genuinely denied by GS . Moreover, if one considers the number b of instances where ZZ is better than GS, defined as

$$b = \sum_{all\ instances} Y_{[min(GS_m, GS_w) - max(ZZ_m, ZZ_w)]} \tag{3}$$

$$with \quad \begin{matrix} Y_x = 1\ if\ x > 0 \\ Y_x = 0\ \ else \end{matrix} \quad and$$

$$\beta = \frac{100 \times b}{number\_of\_instances}$$

$\beta = 96\%$ for global satisfaction and $\beta = 99\%$ for sex equality. This percentage depends directly on the population size. The larger population, the greater improvement. Experimentally, beyond 200 large populations ZZ becomes 100 percent better than GS for both global satisfaction and sex equality.

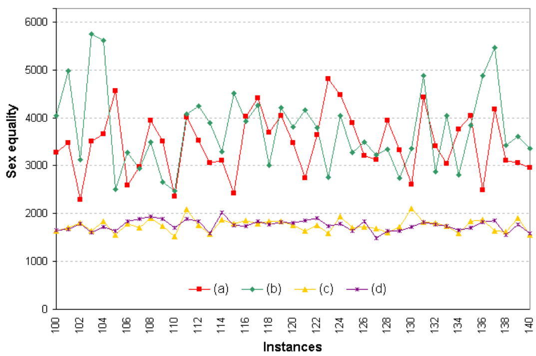


Figure 6: Comparison of sex equality between the methods: (a) GS man-optimal (b) GS woman-optimal (c) Zigzag man-optimal and (d) Zigzag woman-optimal

However, matching stability is not guaranteed by ZZ. As a gauge of unstability, figure 7 (a) and (b) display the average number of blocking pairs with their standard deviations vs. the population size. Note that cases (a) and (b), respectively ZZ man first and ZZ women first, are unseparable at that representation scale. It appears that the larger, the less stable.

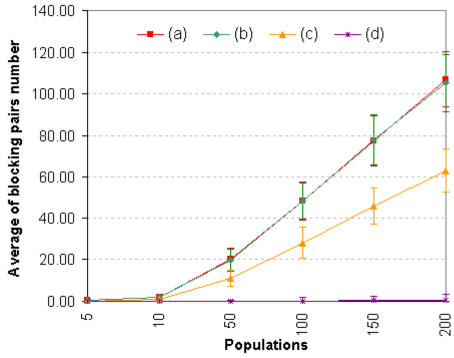


Figure 7: The average number of blocking pairs using: (a) Zigzag man-optimal and (b) Zigzag woman-optimal (c) Optimal zigzag (d) Blocked zigzag

3.2 Optimal (Symmetric) zigzag (OZ)

The primary implementation of a diagonal scan of the marriage table proves significantly better than GS as for global satisfaction and fairness. But the scan start direction - men or women first - still matters in extreme cases, although its impact is less in general. We now propose an algorithm (algo.1) that targets optimal zigzag from bottom-left to top-right (forward). Here again, anti-diagonals of the table are scanned forward from maximum to minimum global satisfaction but each one is read in swinging from center to sides meaning maximum to minimum sex equality. With this algorithm global satisfaction is considered first and then sex equality. Figures 8(c) and 9(c) show the global satisfaction and sex equality compared with GS 8(a)(b) and 9(a)(b) respectively. Both global satisfaction and sex equality slightly worsen compared to ZZ ,but the main result is blocking pair numbers decrease significantly by about 40% (figure 7 (c)). Still, the method does not guarantee any matching stability. The complexity for all these algorithms so far remains in $O(n^2)$, table building included.

3.3 Blocked zigzag (BZ)

Both previous algorithms provide matching solutions that are globally satisfactory and fair but unstable: they might even be such that everybody would like to move!. In this section an algorithm (algo.2) is designed to meet all three criteria concurrently, to the price of reasonnable increase in complexity due to

Algorithm 1: Optimal zigzag algorithm

```
begin
    foreach anti-diagonal, maximum to miminum global satisfaction do
        foreach diagonal, maximum to miminum sex equality in alternate directions do
            foreach pair (m, w) do
                if m et w are free then
                    Marry m with w
end
```

Algorithm 2: Blocked zigzag algorithm

```
begin
    while there is a bloking pair and rescan number < population size do
        foreach anti-diagonal, maximum to miminum global satisfaction do
            foreach diagonal, maximum to miminum sex equality back and forth do
                foreach pair (m, w) do
                    if m and w are free then
                        Marry m with w
        foreach anti-diagonal, maximum to miminum global satisfaction do
            foreach diagonal, maximum to miminum sex equality back and forth do
                foreach pair (m, w) do
                    if (m, w) is blocking pair then
                        Free m and w and their spouse
                        Marry m with w
end
```

systematic test added. We scan anti-diagonals same as before. In each cell, all pairs are accepted for marriage if their components are free. After all cells have been considered, the table is then revisited up to complete removal of blocking situations as follows: potential blocking pairs are matched upon detection (test according to figure 3) while both blocked couples are broken and complementary elements are freed. To overcome cycles in the assignment the number of rescanning is limited to the population size. Scan directions together with questionning all previous marriages on demand guarantees the better at end. Global satisfaction and sex equality are comparable to former ones (figure 8(d) and 9(d)) if not even better. The main improvement is matching stability now obtained in a great majority of cases, with the number of unstabilities significantly lowered otherwise (see figure 7(d)). More precisely, the number of blocking pairs is null until 50 man-or-woman large populations. It is still 0 in an average 96% of the 200 instances for populations larger than 50, and its maximum ranges in the 15 blocking pairs for 100. However, the algorithm complexity is in the order $O(n^3)$.

In figure 10, the algorithm performances are compared wrt. population size. The improvement β from GS to BZ increases with the population size. Again, beyond 100 large populations ZZ becomes 100 percent better than GS for both global satisfaction and sex equality.

4 APPLICATIONS

For sake of illustration we outline here three image processing applications in stereovision, registration and motion analysis. Matching relies on level-lines. Features as simple as junctions or sequences of line segments are extracted from each image and then selected into primitives. Each primitive is given its preference list containing primitives of the other image (see for instance figure 11). The preference list is incomplete and sorted by features similarity (e.g. contrast, length, relative orientation, relative position etc.). Blocked zigzag is then run. Figure 12(a)(b)(c)(d) show the original stereo images and the features in them respectively. The result of feature matching by BZ shows as an optical flow in figure 12(e). And the figure 12(f) supports comparison with GS. Results are quite comparable to the naked eye due to the "incomplete list" nature of the implementation.

The same process can do for motion. Figures 13 show the image sequence and the matching result respectively.

5 CONCLUSION

In this paper, we proposed and evaluated "stable marriages" algorithms based on a novel representation, called *marriage table*. Its definition and properties are presented. Algorithms follow different scan styles of the latter table, defining result properties accordingly. We introduce three different ones to progressively meet three criteria: global satisfaction, sex equality and stability. The three criteria together

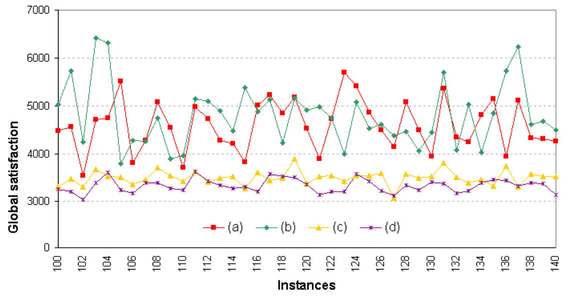


Figure 8: Comparison of global satisfaction between the methods: (a) GS man-optimal (b) GS woman-optimal (c) Optimal zigzag and (d) Blocked zigzag, with 200 instances of 150 large populations.

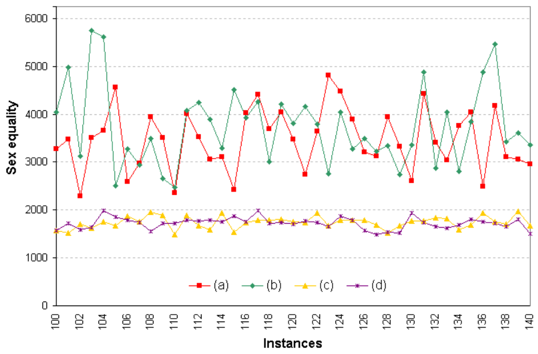


Figure 9: Comparison of sex equality between the methods: (a) GS man-optimal (b) GS woman-optimal (c) Optimal zigzag and (d) Blocked zigzag

(quasi) satisfied change complexity from $O(n^2)$ to $O(n^3)$. Matching results obtained are systematically and experimentally compared to $GS's$. Some application results in motion analysis, registration and stereovision are also given just for illustration.

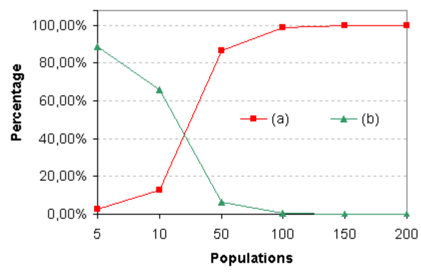


Figure 10: Performance comparison between GS and BZ algorithms

REFERENCES

Abeledo, H. and Rothblum, U. G. (1995). Paths to marriage stability. *Elsevier : Discrete Applied Mathematics*, 63(1-12).

Ballester, C., Castan, E., Gonzalez, M., and Morel, J. (1998). Contrast invariant image intersection. *C.M.L.A*, (9817).

Bouchafa, S. and Zavidovique, B. Stratgie de vote pour la mise en correspondance de lignes de niveaux. *Universit Paris-Sud*, AXIS.

Caselles, V., Coll, B., and Morel, J. (1999). Topographic maps and local contrast changes in natural images. *International Journal of Computer Vision*, 33(1):5–27.

C.P. Teo, J. S. and Tan, W. (1999). Gale-shapley stable marriage problem revisited: Strategic issues and applications. *IPCO'99*, pages 429–438.

D. Bianco, S. H. and Larimer, A. (2001). Stable matchings in the couples problem. *Morehead Electronic Journal of Applicable Mathematics*, 2.

D.F. Manlove, R.W. Irving, K. I., Miyazaki, S., and Morita, Y. (2002). Hard variants of stable marriage. *Theoretical Computer Science*, 276(1-2):261–279.

Diamantoudi, E., Miyagama, E., and Xue, L. (2004). Random paths to stability in the roommate problem. *Elsevier : Games and Economic Behavior*, 48(18-28).

Gale, D. and Shapley, L. (1962). College admissions and the stability of marriage. *American Mathematical Monthly*, 69:9–15.

Gent, I. and Prosser, P. (2002). An empirical study of the stable marriage problem with ties and incomplete lists. Technical Report APES-40-2002, APES Research Group. Available from http://www.dcs.st-and.ac.uk/ apes/apesreports.html.

Irving, R. (1994). Stable marriage and indifference. *Discrete Applied Mathematics*, 48:261–272.

J.L.Lisani, Monasse, P., and Rudin, L. (2001). Fast shape extraction and applications. *C.M.L.A*, (2001-16).

K. Iwama, D. Manlove, S. M. and Morita, Y. (1999). Stable marriage with incomplete lists and ties. *Proceedings of ICALP '99: the 26th International Colloquium on Automata, Languages and Programming*, 1644:443–452.

Manlove, D. (1999). Stable marriage with ties and unacceptable partners. Technical Report TR-1999-29, Computing Science Department of Glasgow University.

McVitie, D. G. and Wilson, L. B. (1971). Three procedures for the stable marriage problem. *Communications of the ACM*, 14,7:491–492.

Monasse, P. and Guichard, F. (1998). Fast computation of a contrast-invariant image representation. *IEEE Trans. on Image Proc.*, 9(5):860–872.

T. Kavitha, K. Mehlhorn, D. M. and Paluch, K. (2004). Strongly stable matching in time o(nm) and extension to the hospitals-residents problem. *STACS'04*, pages 222–233.

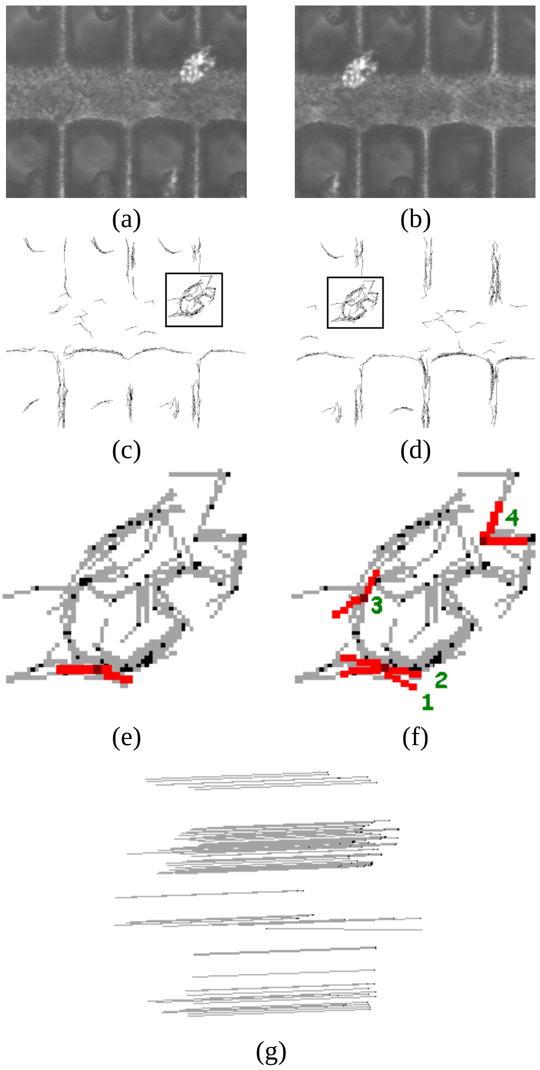


Figure 11: Stable marriages for MEMS images registration in electron beam microscopy, (a) 1^{st} field part scaned, (b) 2^{nd} field part scaned to be super imposed into a larger field. Let us underline the VLSI implementation artefact : this defect will eventually support the perfect match between (a) and (b), despiste the cumb ambiquity from periodicity, (c)(d) the primitives extracted from (a) and (b) with primitives from the defect underlined in the frame, (e) primitives in the defect that supports the perfect piecing, (f) potential mates of the primitive underlined in bold in (e) with their rank, (g) final matching results by BZ : the are in perfect agreement with the known reality

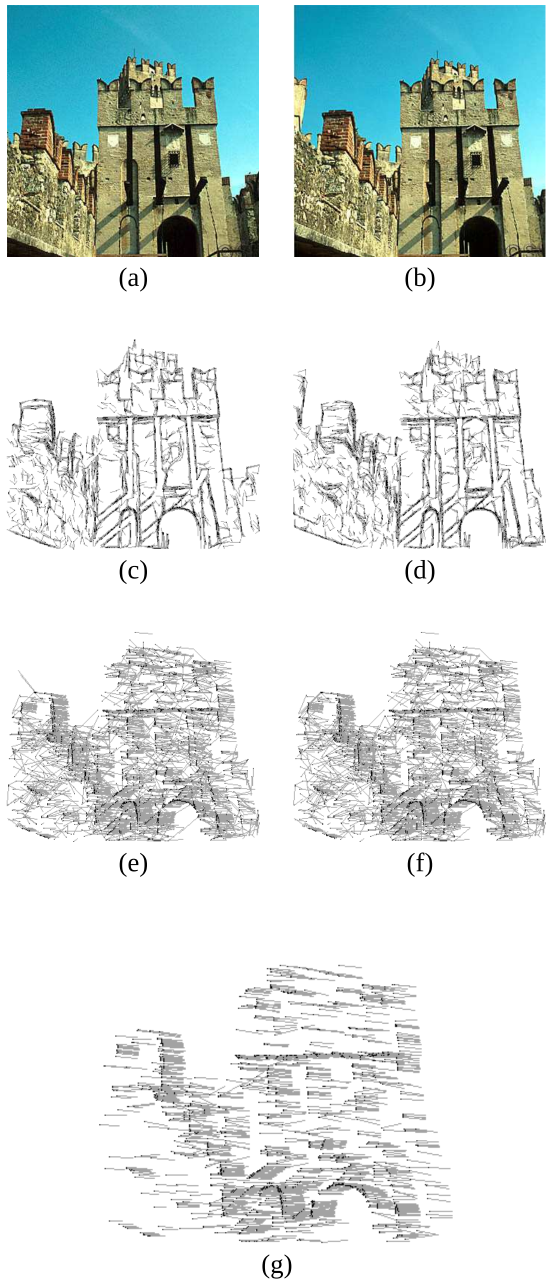
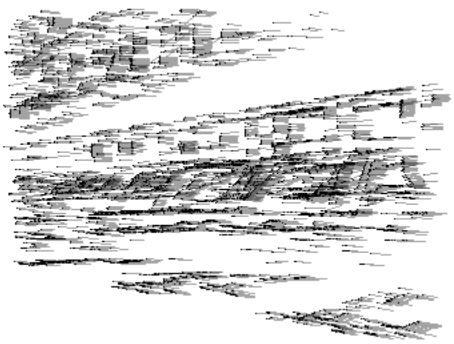


Figure 12: Stable marriages matching for stereovision, (a)(b) stereo images, (c)(d) features extracted from (a)(b) respectively, (e) matching results by BZ, (f) matching results by GS, (g) final matching results by BZ

(a)

(b)

(c)



(d)

Figure 13: (a)(b)(c) Image sequence for motion detection, (d) Matching result by using the stable marriages algorithm.