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A NUMERICAL APPROACH TO ELASTIC-PLASTIC PLANE-STRAIN PROBLEMS OF THICK-WALLED CYLINDERS

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A new finite-difference approach has been developed for solving the elastic-plastic plane-strain problems of thick-walled cylinders subjected to internal or external pressure. The material is assumed to obey the von Mises' yield criterion, the Prandtl-Reuss flow theory and the isotropic hardening rule. The numerical scheme is stable for ideally-plastic as well as straining-hardening materials. The approach is simpler than others, and very accurate solutions can be obtained by reducing the grid sizes and load increments. The results

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for the stresses and strains in all principal directions have been computed as functions of loading history.

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INTRODUCTION

Based on a detailed study reported in Reference 1, the best material model for gun tubes under high pressure operation is an elastic-plastic material which obeys the Mises' yield criterion and the Prandtl-Reuss incremental stress-strain relation. A literature survey indicates that no closed form solution exists even for the axisymmetric plane-strain problems. And, in such situations, one has to rely on numerical methods. Both the finite-difference method[2,3] and the finite-element method[4] have been used to solve the elasto-plastic problems considered here. The finite-element method is more powerful and can be used to solve more general elastoplastic problems. Since the displacement function is assumed and the programming is complicated, the accuracy of the finite-element approach has to be verified. This is usually done by comparing more rigorous solutions to simpler problems. For the problem considered here, rigorous solutions based on the finite-difference method were obtained by Hodge and White[2] for ideally-plastic materials and by Chu[3] for strain-hardening materials.

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[1]Davidson, T. E., and Kendall, D. P., "The Design of Pressure Vessels For Very High Pressure Operation," Watervliet Arsenal Report WVT-6917. Also in Mechanical Behavior of Materials Under Pressure (edited by Pugh, H. L. D.), Elsevier Co., 1970, Chapter 2.

[2]White, P. G. and White, G. N., "A Quantitative Comparison of Flow and Deformation Theories of Plasticity," J. Appl. Mech., Vol. 17, 1950, pp. 180-184.

[3]Chu, S. C., "A More Rational Approach to the Problem of an Elasto-Plastic Thick-Walled Cylinder," J. of the Franklin Institute, Vol. 294, 1972, pp. 57-65.

[4]Chen, P. C. T., "The Finite Element Analysis of Elastic-Plastic Thick-Walled Tubes," Proceedings of Army Symposium on Solid Mechanics, 1972, The Role of Mechanics in Design-Ballistic Problems, pp. 243-253.

In the present paper, a new finite-difference approach is developed for solving the axisymmetric plane strain problems subjected to internal or external pressure beyond the elastic limit. An incremental approach is used, and the numerical scheme is stable for ideally-plastic as well as strain-hardening materials. The approach is simpler than others, yet very accurate solutions can be obtained by reducing the grid sizes and load increments.

BASIC EQUATIONS

Assuming small strain and no body forces in the axisymmetric state of plane strain, the radial and tangential stresses, σ_r and σ_θ, must satisfy the equilibrium equation,

$$r(\partial\sigma_r/\partial r) = \sigma_\theta - \sigma_r \; ; \qquad (1)$$

and the corresponding strains, ε_r and ε_θ, are given in terms of the radial displacement, u, by

$$\varepsilon_r = \partial u/\partial r \quad , \quad \varepsilon_\theta = u/r \; . \qquad (2)$$

It follows that the strains must satisfy the equation of compatibility

$$r(\partial\varepsilon_\theta/\partial r) = \varepsilon_r - \varepsilon_\theta \; . \qquad (3)$$

Whereas the differential equations (1), (2) and (3) hold throughout the tube regardless of the material properties, the constitution equations assume various forms according to the adopted form of yield function, hardening rule, total or incremental theory of plasticity. In the present paper, the material is assumed to be elastic-plastic, obeying the Mises' yield criterion, the Prandtl-Reuss flow theory and the isotropic hardening law. The complete stress-strain relations are:[5]

[5]Hill, R., Mathematical Theory of Plasticity, Oxford University Press, 1950.

$$d\varepsilon_i' = d\sigma_i'/2G + (3/2)\sigma_i' d\sigma/(\sigma H') \tag{4}$$

$$d\sigma \geq 0 \quad \text{for} \quad i = r,\theta,z$$

$$d\varepsilon_m = E^{-1}(1-2\nu)d\sigma_m \tag{5}$$

where E, ν Young's modulus, Poisson's respectively,

$$2G = E/(1+\nu)$$

$$\varepsilon_m = (\varepsilon_r+\varepsilon_\theta+\varepsilon_z)/3 \quad , \quad \varepsilon_i' = \varepsilon_i-\varepsilon_m \ ,$$

$$\sigma_m = (\sigma_r+\sigma_\theta+\sigma_z)/3 \quad , \quad \sigma_i' = \sigma_i-\sigma_m \ , \tag{6}$$

$$\sigma = (1/\sqrt{2})[(\sigma_r-\sigma_\theta)^2 + (\sigma_\theta-\sigma_z)^2 + (\sigma_z-\sigma_r)^2]^{1/2} \geq \sigma_o \ ,$$

and σ_o is the yield stress in simple tension or compression. For a strain hardening material, H' is the slope of the effective stress/plastic strain curve

$$\sigma = H(\int d\varepsilon^{\rho}) \ . \tag{7}$$

For an ideally-plastic material (H'=0), the quantity $(3/2)d\sigma/(\sigma H')$ is be replaced by $d\lambda$, a positive factor of proportionality. When $\sigma < \sigma_o$ or $d\sigma < 0$, the state of stress is elastic and the second term in equation (4) disappears. Following Yamada et al,[6] equations (4) and (5) can be rewritten in an incremental form

$$d\sigma_i = d_{ij}d\varepsilon_j \quad \text{for} \quad i,j = r,\theta,z$$

and

$$d_{ij}/2G = \nu/(1-2\nu) + \delta_{ij} - \sigma_i'\sigma_j'/S \ , \tag{8}$$

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[6] Yamada, Y., Yoshimura, N., and Sakurni, T., "Plastic Stress-Strain Matrix and Its Application for the Solution of Elastic-Plastic Problems by the Finite Element Method," Int. J. Mech. Sci., Vol. 10, 1968, pp. 343-354.

where

$$S = \frac{2}{3}(1 + \frac{1}{3} H'/G)\sigma^2 , \tag{9}$$

and δ_{ij} is the Kronecker delta.

This form was used in the finite-element formulation for solving elastic-plastic thick-walled tube problems.[4] In the following section, the incremental stress-strain matrix will be used in the finite difference formulation.

FINITE-DIFFERENCE FORMULATION

Consider a thick-walled cylinder of inner radius a and external radius b. The tube is subjected to inner pressure p and/or external pressure q. The elastic solution for this problem is well-known and the pressure p* or q* required to cause initial yielding can be determined by using the Mises' yield criterion. For pressure beyond the elastic limit, an incremental approach of the finite-difference formulation is used. The analysis starts with the applied pressure p or q and the loading path is divided into m increments with

$$\Delta p = (p-p^*)/m \quad , \quad \Delta q = (q-q^*)/m . \tag{10}$$

The cross section of the tube is divided into n rings with

$$r_1=a, r_2, \ldots, r_k=\rho, \ldots, r_{n+1}=b, \tag{11}$$

[4]Chen, P. C. T., "The Finite Element Analysis of Elastic-Plastic Thick-Walled Tubes," Proceedings of Army Symposium on Solid Mechanics, 1972, The Role of Mechanics in Design-Ballistic Problems, pp. 243-253.

where ρ is the radius of the elastic-plastic interface. At the beginning of each increment of loading, the distribution of displacements, strains and stresses are assumed to be known and we want to determine Δu, $\Delta\varepsilon_r$, $\Delta\varepsilon_\theta$, $\Delta\sigma_r$, $\Delta\sigma_\theta$, $\Delta\sigma_z$ at all grid points. Since the incremental stresses are related to the incremental strains by the incremental form (Eq. (8)) and $\Delta u = r\Delta\varepsilon_\theta$, there exists only two unknowns at each station that have to be determined for each increment of loading. The unknown variables in the present formulation are $(\Delta\varepsilon_\theta)_i$, $(\Delta\varepsilon_r)_i$, for $i = 1,2,\ldots n,n+1$.

The equation of equilibrium (1) and the equation of compatibility (3) are valid for both the elastic and the plastic regions of a thick-walled tube. The finite-difference forms of these two equations at $i = 1,\ldots,n$ are given in Reference 3 by

$$(r_{i+1}-2r_i)(\Delta\sigma_r)_i - (r_{i+1}-r_i)(\Delta\sigma_\theta)_i + r_i(\Delta\sigma_r)_{i+1} = (r_{i+1}-r_i)(\sigma_\theta-\sigma_r)_i - r_i[(\sigma_r)_{i+1} - (\sigma_r)_i] \qquad (12)$$

for the equation of equilibrium, and

$$(r_{i+1}-2r_i)(\Delta\varepsilon_\theta)_i - (r_{i+1}-r_i)(\Delta\varepsilon_r)_i + r_i(\Delta\varepsilon_\theta)_{i+1} = (r_{i+1}-r_i)(\varepsilon_r-\varepsilon_\theta)_i - r_i[(\varepsilon_\theta)_{i+1} - (\varepsilon_\theta)_i] \qquad (13)$$

for the equation of compatibility.

[3]Chu, S. C., "A More Rational Approach to the Problem of an Elasto-Plastic Thick-Walled Cylinder," J. of the Franklin Institute, Vol. 294, 1972, pp. 57-65.

With the aid of the incremental stress-strain relations (Eq. (8)), equation (12) can be rewritten as

$$[(r_{i+1}-2r_i)(d_{12})_i + (-r_{i+1}+r_i)(d_{22})_i](\Delta\varepsilon_\theta)_i$$
$$+ [(r_{i+1}-2r_i)(d_{11})_i + (-r_{i+1}+r_i)(d_{21})_i](\Delta\varepsilon_r)_i$$
$$+ r_i(d_{12})_{i+1}(\Delta\varepsilon_\theta)_{i+1} + r_i(d_{11})_{i+1}(\Delta\varepsilon_r)_{i+1}$$
$$= (r_{i+1}-r_i)(\sigma_\theta-\sigma_r)_i - r_i[(\sigma_r)_{i+1} - (\sigma_r)_i] \ . \qquad (14)$$

The boundary conditions for the problem are

$$\Delta\sigma_r(a,t) = -\Delta p \quad , \quad \Delta\sigma_r(b,t) = -\Delta q \ . \qquad (15)$$

Using the incremental relations (Eq. (8)), we rewrite equation (15) as

$$(d_{12})_1(\Delta\varepsilon_\theta)_1 + (d_{11})_1(\Delta\varepsilon_r)_1 = -\Delta p \ , \qquad (16)$$

and

$$(d_{12})_{n+1}(\Delta\varepsilon_\theta)_{n+1} + (d_{11})_{n+1}(\Delta\varepsilon_r)_{n+1} = -\Delta q \ . \qquad (17)$$

Now we can form a system of 2(n+1) equations for solving 2(n+1) unknowns, $(\Delta\varepsilon_\theta)_i$, $(\Delta\varepsilon_r)_i$, for i = 1,2,...,n,n+1. Equations (16) and (17) are taken as the first and last equations, respectively, and the other 2n equations are set up at i = 1,2,...,n using (13) and (14). The final system is an unsymmetric band matrix with the nonzero terms clustered about the main diagonal, two below and one above. The storage required is 8(n+1) words for this system and at least 35(n+1) words for the approach.[3] In the computer program which was developed, the Gaussian elimination method was used to solve these equations. All calculations were carried out on IBM 360/Model 44 with double precision to reduce round-off errors.

[3]Chu, S. C., "A More Rational Approach to the Problem of an Elasto-Plastic Thick-Walled Cylinder," J. of the Franklin Institute, Vol. 294, 1972, pp. 57-65.

NUMERICAL RESULTS

The plane-strain problems of thick-walled cylinders subjected to internal pressure p or external pressure q beyond the elastic limit were solved. The elastic-perfectly-plastic as well as strain-hardening materials were considered. The numerical results were based on the following parameters: $b/a = 2$, $\nu = 0.3$, $H' = 0$ or $E/19$. Various values of m and n were used to test the convergence of the numerical results. The incremental loadings were applied until the fully plastic state was reached. The values for p or q corresponding to this final state were denoted by p^{**} or q^{**}. It was found that the results for these values converge by increasing m and/or n. For an ideally-plastic tube under internal pressure only, we have $(n, m, p^{**}/\sigma_o)$ = (20, 1277, 0.81341), (50, 1000, 0.80455), (100, 998, 0.80155), (200, 920, 0.80029). Since the storage needed for this approach is only 22% of that in Reference 3, much larger n can be used to yield better results. Additional results shown in Figures 1 to 5 were based on $n = 100$ and $m = 900\text{-}1000$. Figure 1 shows the relation between internal pressure p, external pressure q and elastic-plastic boundary ρ in an elastic-perfectly-plastic tube. Figure 2 shows the bore radial and tangential strains as functions of internal pressure p in an

[3]Chu, S. C., "A More Rational Approach to the Problem of an Elasto-Plastic Thick-Walled Cylinder," J. of the Franklin Institute, Vol. 294, 1972, pp. 57-65.

ideally-plastic as well as a strain-hardening tube. Figure 3 shows the distributions of radial, tangential and axial stresses in an ideally-plastic tube subjected to internal pressure. The dotted curves correspond to initial yielding. The solid and broken curves correspond to the cases when half and all of the tube is plastic, respectively. For an ideally-plastic tube subjected to external pressure q, the results were presented graphically in Figures 1, 4, and 5.

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5. Hill, R., Mathematical Theory of Plasticity, Oxford University Press, 1950.

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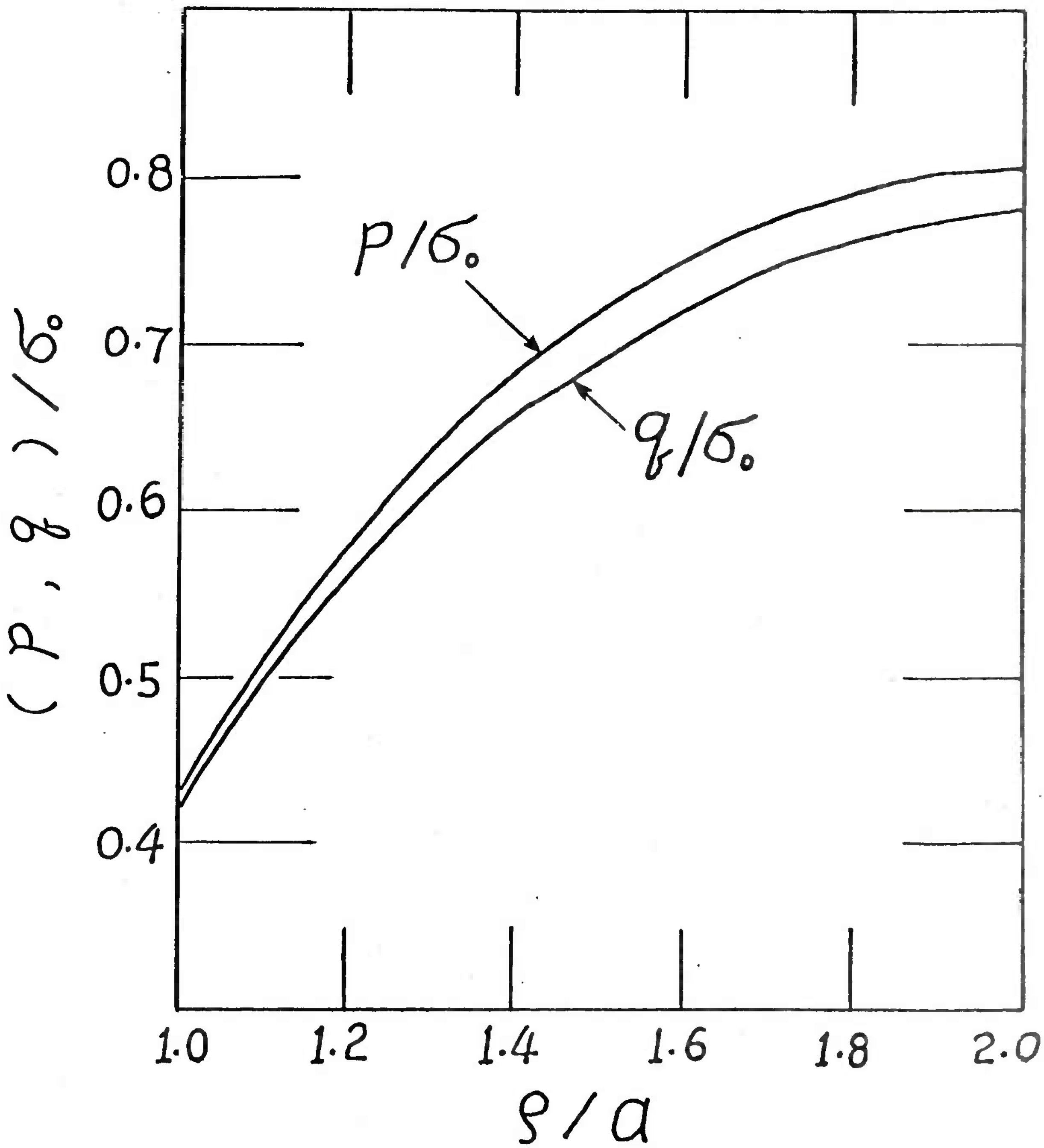

Figure 1. Internal pressure p or external pressure q as functions of elastic-plastic boundary ρ in an ideally-plastic tube

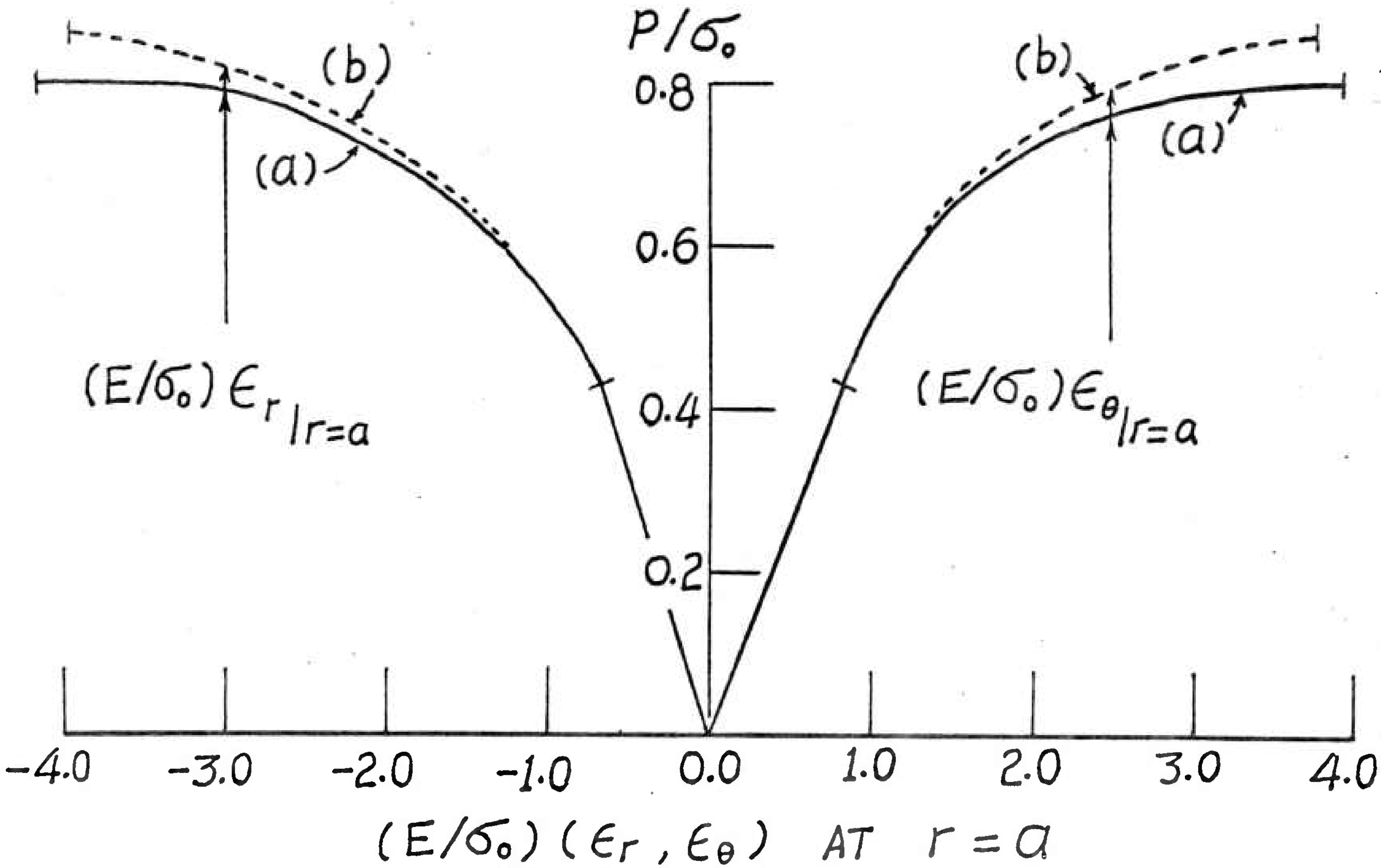

Figure 2. Bore radial and tangential strains as functions of internal pressure p (a) ideally-plastic tube, (b) strain-hardening tube, $H' = E/19$

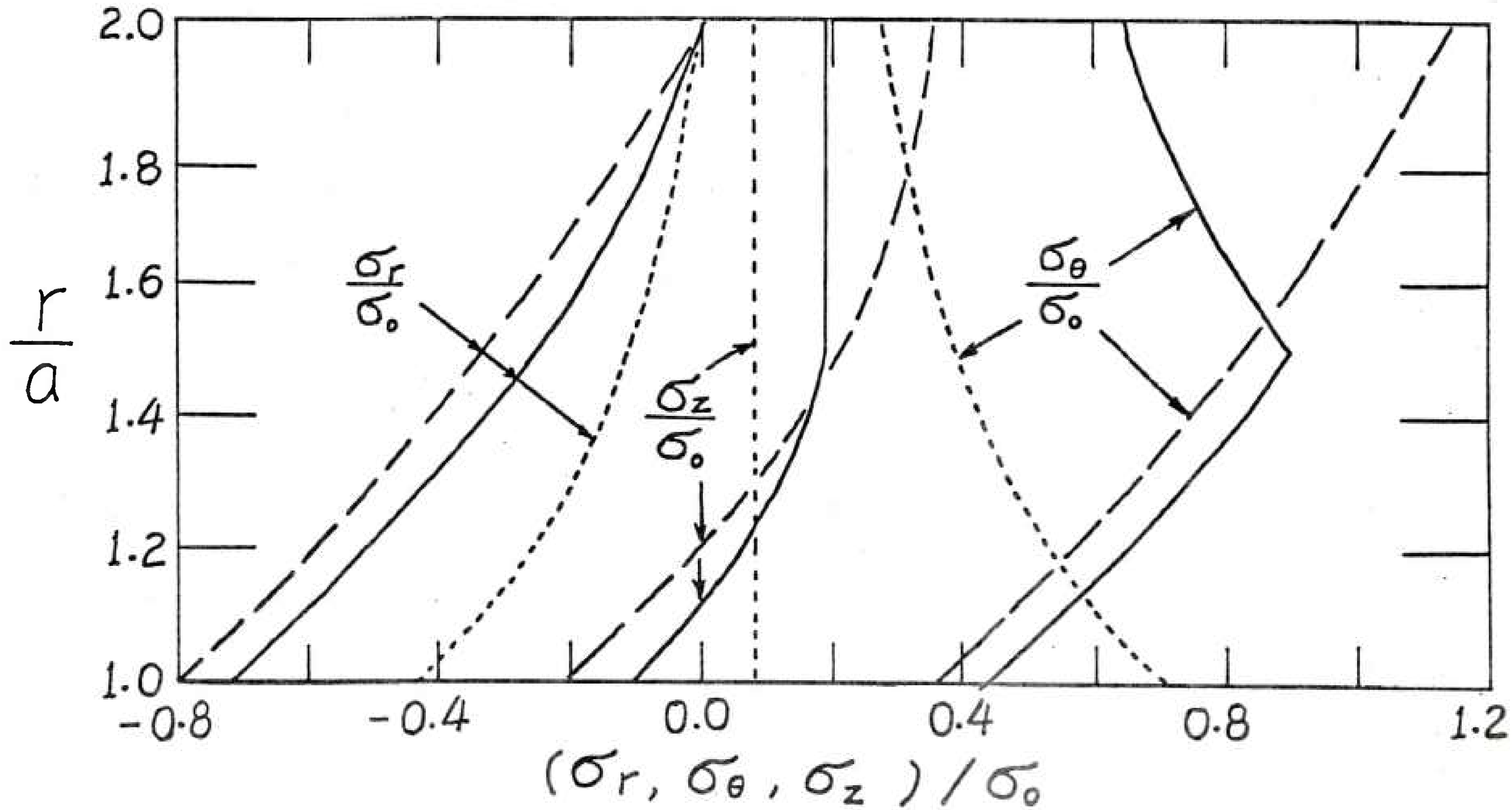

Figure 3. Distributions of radial, tangential and axial stress components in an ideally-plastic tube subjected to internal pressure (ρ/a = 1.0 ....., 1.5 ______, 2.0 — —)

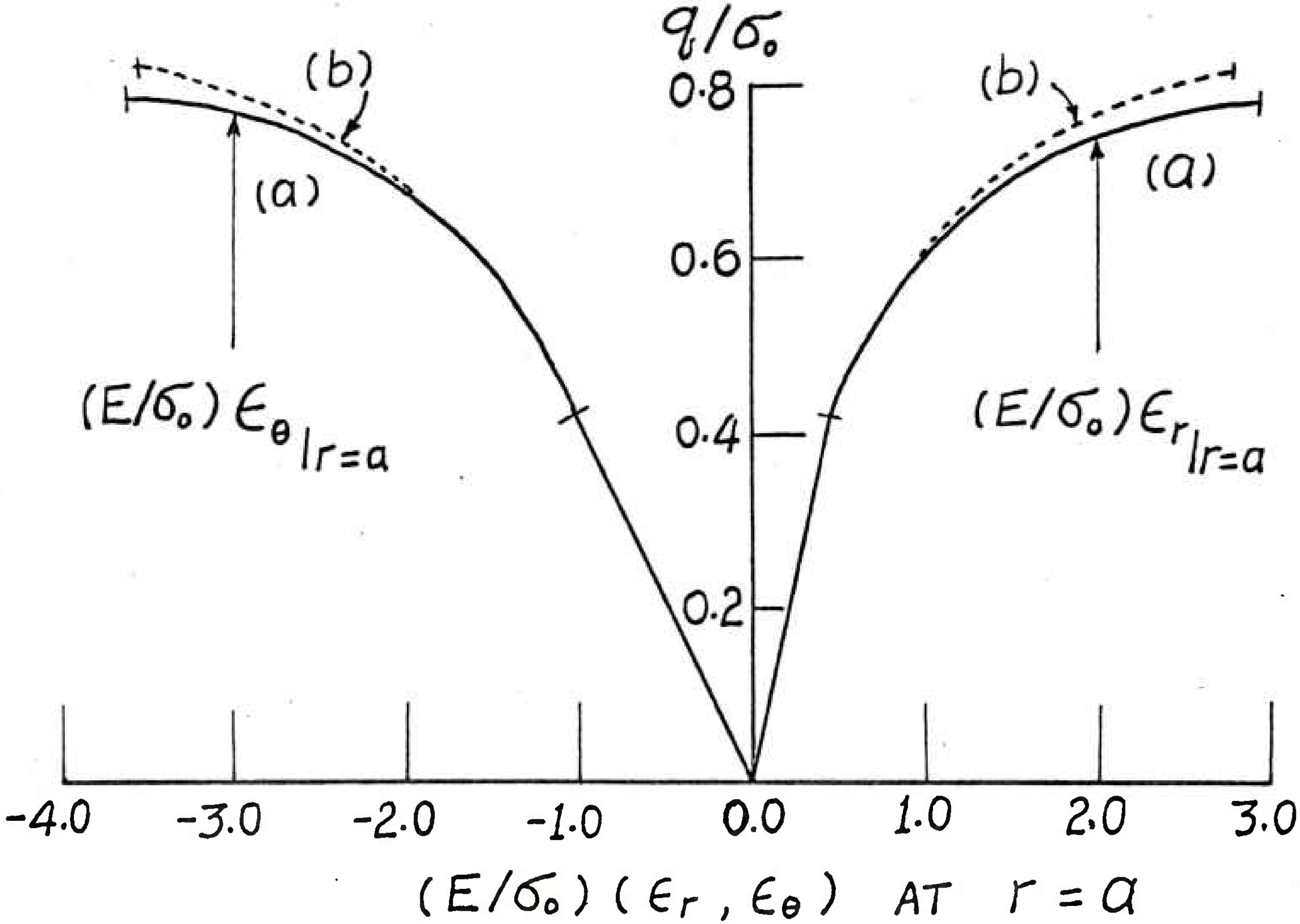

Figure 4. Bore radial and tangential strains as functions of external pressure q
(a) ideally-plastic tube, (b) strain-hardening tube, H' = E/19

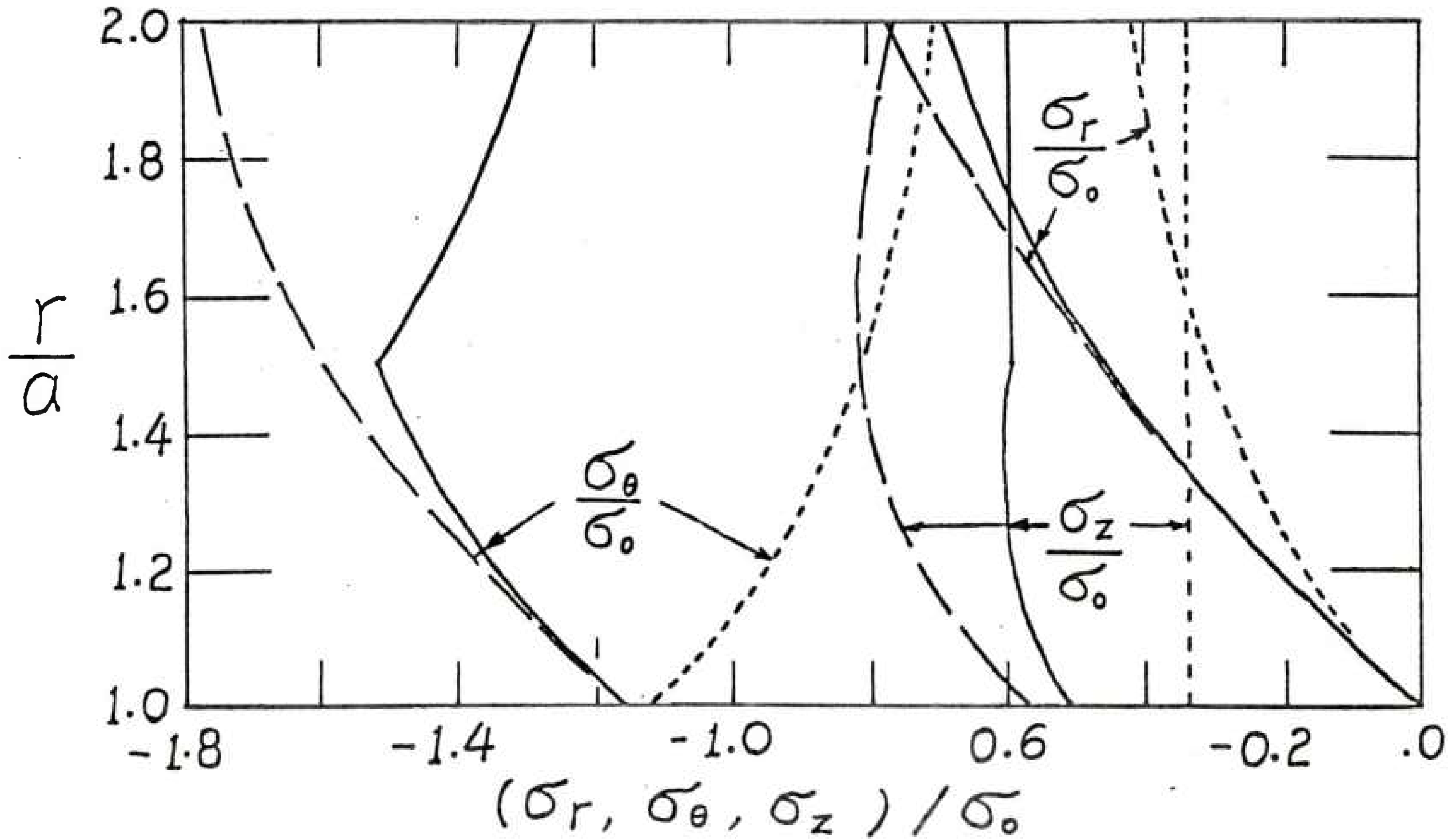

Figure 5. Distributions of radial, tangential and axial stress components in an ideally-plastic tube subjected to external pressure (ρ/a = 1.0 ....., 1.5 _____, 2.0 — —)

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