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A Numerical Approach to Ergodic Problem of Dissipative Dynamical Systems

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Based on the Lyapunov characteristic exponents, the ergodic property of dissipative dynamical systems with a few degrees of freedom is studied numerically by employing, as an example, the Lorenz system. The Lorenz system shows the spectra of $(+, 0, -)$ type concerning the 1-dimensional Lyapunov exponents, and the exponents take the same values for orbits starting from almost of all initial points on the attractor.

This result suggests that the ergodic property for general dynamical systems not necessarily belonging to the category of the axiom-A may also be characterized in the framework of the spectra of the Lyapunov characteristic exponents.

§ 1. Introduction

Recently, chaotic motions that arise due to non-linearities of dissipative dynamical systems have received a great concern in physical and non-physical fields.[1] However, in general dynamical systems which do not satisfy the axiom-A, little progress has been made to analyse those chaotic motions by theoretically well-established methods.[2)~4)]

One of the purposes of this paper is to present numerical methods, by which wide-spread chaotic motions in dissipative dynamical systems would be characterized in a systematic manner. Our basic idea for this aim is to utilize the complete set of 1-dimensional Lyapunov exponents, which characterize the asymptotic orbital instability of dynamical systems.[5)~9)] The second purpose is to show that the concept of Lyapunov exponents presents a practical tool to discuss problems of bifurcation of those chaotic solutions.

For these purposes, it becomes an important problem to estimate the Lyapunov exponents by some numerical methods, because it may not be expected, in general, that the equations for orbits exhibiting chaotic motions have globally single-valued analytic solutions. In case of measure preservig diffeomorphisms, Benettin et al.[11),*] have recently pointed out almost the same method as developed in this paper.

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*) The reader should not confuse two reference 7) and 11) by Benettin et al..

In § 2, brief discussion on the existence and properties of the k-dimensional Lyapunov exponent is presented. The relations of the characteristic exponents to the invariant measure and also to the measure-theoretic entropy (Kolmogorov entropy[10]) are commented. In the Appendix, a general scheme for estimating numerically the k-dimensional Lyapunov exponent is briefly developed, and discussion on the relation between our method and another approximate method[7] for estimating the Lyapunov exponent is also given.

In § 3, the famous turbulent model, i.e., the Lorenz model,[12] is studied in the light of our method, and the result of the complete set of the 1-dimensional Lyapunov exponents for this system is explicitly given. As an application of our approach based on the Lyapunov exponents, § 4 is devoted to problems of bifurcation of chaotic solutions in the Lorenz system.

§ 2. A partial summary of Lyapunov characteristic exponents for irregular motions

It is considered that dynamical systems exhibiting chaotic motions without any contact with external disturbance may possess some unstable properties of orbits. From this point of view, it is worth while noting that there is a fundamental method for investigating the time-dependent behavior of small deviations from an orbit. The method is called Lyapunov's method which uses the first variational equation of orbits.*)

Now, let us consider the system of which time evolution is described by a set of differential equations in N-dimensional Euclidian space,

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}) . \tag{1}$$

The solution of Eq. (1) under the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ is written as

$$\boldsymbol{x}(t) = T^t \boldsymbol{x}_0 , \tag{2}$$

where T^t is the map which describes time-t evolution of all phase points.

On the other hand, the time evolution equation for the first variation of the orbit obeys the following set of non-autonomous linear differential equations:

$$\delta\dot{\boldsymbol{x}} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}(T^t \boldsymbol{x}_0)\, \delta \boldsymbol{x} . \tag{3}$$

The solution of Eq. (3) can be written as

$$\delta \boldsymbol{x}(t) = U^t_{\boldsymbol{x}_0} \delta \boldsymbol{x}_0 , \tag{4}$$

where $U^t_{\boldsymbol{x}_0}$ is the fundamental matrix[5] of Eq. (3), and $\delta \boldsymbol{x}_0$ is an initial deviation at $t=0$. The fundamental matrix in Eq. (4) satisfies the following chain rule:

*) As a general introduction to this chapter, one may refer to the book cited in Ref. 5).

$$U_{x_0}^{t+s} = U_{T^s x_0}^t \circ U_{x_0}^s . \tag{5}$$

It is apparent that the asymptotic behavior of a small deviation is described by the asymptotic behavior of the fundamental matrix for $t \to \infty$. Now, the asymptotic behavior of this matrix for $t \to \infty$ can be characterized by the following exponents:[5),6),11)]

$$\lambda(e^k, x_0) = \lim_{t\to\infty} \frac{1}{t} \log \frac{\| U_{x_0}^t e_1 \wedge U_{x_0}^t e_2 \wedge \cdots \wedge U_{x_0}^t e_k \|}{\| e_1 \wedge e_2 \wedge \cdots \wedge e_k \|} \tag{6}$$

for $k=1, 2, \cdots, N$. The symbols in (6) have the following meanings: e^k is a k-dimensional subspace in the tangent space E_{x_0} at x_0, $\{e_i\}$ $(i=1, 2, \cdots, k)$ are a set of bases of e^k, $\wedge$ is an exterior product and $\|\circ\|$ is a norm with respect to some Riemannian metric. The exponent defined by (6) represents an expanding rate of volume of the k-dimensional parallelepiped in the tangent space along the orbit which starts at x_0, and is called the k-dimensional Lyapunov exponent. It is clear from this definition that the exponent does not depend on a choise of a set of bases nor norms, but depends only on the k-dimensional subspace e^k.[13)]

It may be useful to summarize the properties of the Lyapunov exponents, which will be utilized in the subsequent discussion.

1) 1-dimensional exponent $\lambda(e^1, x)$ may take, at most, N distinct values, and we will use the notations $\{\lambda_i\}_{1\le i\le N}$ and suppose $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N$.

2) k-dimensional exponent $\lambda(e^k, x)$ may take, at most, ${}_N C_k$ distinct values, and each value is connected with a sum of k distinct 1-dimensional exponents. For instance, in the case $N=3$, the k-dimensional exponents $\lambda(e^k, x)$ $(k=1, 2, 3)$ may take the following values respectively:

$$\lambda(e^1, x) = \text{one of the values in } \{\lambda_1, \lambda_2, \lambda_3\},$$

$$\lambda(e^2, x) = \text{one of the values in } \{(\lambda_1+\lambda_2), (\lambda_1+\lambda_3), (\lambda_2+\lambda_3)\},$$

$$\lambda(e^3, x) = (\lambda_1+\lambda_2+\lambda_3).$$

3) If a set bases $\{e_i\}$ $(i=1, 2, \cdots, N)$ is chosen at random in tangent space, then the k-dimensional exponents $\lambda(e^k, x)$ for $k=1, 2, \cdots, N$ converge respectively, with probability 1, to the maximal values among sets of values which are allowed to possess ${}_N C_k$ distinct values. (This proposition was proved by Benettin et al. in case of diffeomorphisms.[11)])

It must be mentioned here that the above discussion on the Lyapunov exponents becomes meaningful only if the existence of the limit of the quantity defined on the r.h.s. of (6) would be guaranteed.

The proof of the existence of such limits has been made by Oseledec. Here we describe his theorem in the form suited to our discussion.

The multiplicative ergodic theorem of Oseledec:[6)] If there is a T^t-invariant measure μ and $\|\partial F/\partial x\| \in L^1(\mu)$, then the k-dimensional Lyapunov exponents

$\lambda(e^k, \boldsymbol{x}_0)$ $(k=1, 2, \cdots, N)$ exist for μ-almost all $\boldsymbol{x}_0$. In this theorem, notations have the same meaning as the one used in the preceding discussion.

Hereafter, we would like to remark some relations between the Lyapunov exponents and the measure-theoretic entropy (K-entropy) of dynamical systems. It has been known that the existence of the Lyapunov exponents is directly related to the K-entropy. Of the relations obtained so far, the weakest relation may be the following:

$$H(\mu) - \int \sum_{\lambda_i>0} \lambda_i(x)\, d\mu \leq 0\,, \tag{7}$$

where $H(\mu)$ is the K-entropy of the dynamical system with invariant measure μ.

This inequality has been proved by Ruelle.[14),15)] In cases of Hamiltonian systems and axiom-A dynamical systems, the equality in (7) does hold.[17)] If the equality in (7) would be assured, then the phase average of the sum of positive Lyapunov exponents becomes the K-entropy itself. Furthermore, it is expected that the category of dynamical systems which do satisfy the equality in (7) would be much more extensive than the dynamical systems mentioned.

§ 3. 1-, 2- and 3-dimensional Lyapunov exponent of the Lorenz system

In this section, based on the numerical method developed in the Appendix, we present the explicit result of a complete set of the 1-dimensional Lyapunov exponents for the famous turbulent model due to Lorenz.

The Lorenz model is described by the following set of differential equations:

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} -\sigma X & +\sigma Y & \\ (\gamma - Z)X & -Y & \\ XY & & -bZ \end{pmatrix} = \boldsymbol{F}(\boldsymbol{x})\,, \tag{8}$$

where (σ, b, γ) are the parameters. The Lorenz model is a dissipative system, and therefore it does not have a priori invariant measure in contrast to Hamiltonian systems. It is believed in this system that the high-dimensional attractor with very complicated geometrical structure like Cantor set comes forth beyond a certain value of the parameter γ (σ and b are suitably chosen), and orbits on the attractor are non-periodic.

It should be mentioned that the high-dimensional attractor of the Lorenz system does not belong to a well-established mathematical category like the axiom-A strange attractor[17)] because on the edge of the attractor, there is a fixed point $(0, 0, 0)$, and therefore the uniform hyperbolic structure of the attractor is not materialized.

In our calculations, parameters are set as $\sigma=16.0$, $b=4.0$ and $\gamma=40.0$. These values of parameters are not equal to the original Lorenz's values, i.e., $\sigma=10.0$, $b=8/3$ and $\gamma=28.0$, but the geometrical structure of orbits on the attractor is

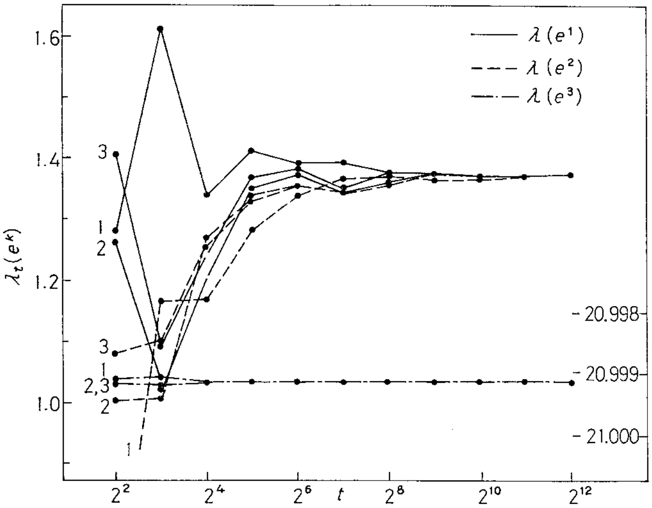

Fig. 1. Temporal convergence of the 1-, 2- and 3-dimensional Lyapunov exponent for the Lorenz system ($\sigma=16.0$, $b=4.0$ and $\gamma=40.0$). Exponents are calculated, based on the formula (A·2) given in the Appendix, under the following initial conditions $(\boldsymbol{x}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$:
1. ((10.0, 0.0, 30.0), (1.0, 0.0, 0.0), (0.0, 1.0, 0.0), (0.0, 0.0, 1.0))
2. ((10.0, 10.0, 30.0), (1.0, 0.0, 0.0), (0.0, 1.0, 0.0), (0.0, 0.0, 1.0))
3. ((10.0, 10.0, 30.0), (0.0, 1.0, 1.0), (1.0, 0.0, 1.0), (1.0, 1.0, 0.0)).

qualitatively the same as the original one within the region $50\gtrsim\gamma\gtrsim\gamma_T=\sigma(\sigma+b+3)/(\sigma-b-1)$. Our numerical integration scheme is the usual Runge-Kutta-Gill method in double precision, and typical time difference is taken as 0.01. By integrating the orbit equations and the first variational equations, according to the method investigated in the Appendix (the renormalization time was chosen as $\tau=1.0$), the 1-, 2- and 3-dimensional Lyapunov exponent have been calculated. They converged respectively to certain definite values. It should be noted here that three exponents $\lambda(e^1, \boldsymbol{x}_0)$, $\lambda(e^2, \boldsymbol{x}_0)$ and $\lambda(e^3, \boldsymbol{x}_0)$ for an orbit starting at $\boldsymbol{x}_0$ did not depend on the initial choice of the subspace $e^k (k=1, 2, 3)$. The numerical results are shown in Fig. 1. Following the statement for the Lyapunov exponents described in § 2·3), the estimated values of the exponents $\lambda(e^k, \boldsymbol{x}_0)$ $(k=1, 2, 3)$ select, respectively, the maximal values undoubtedly. Therefore, we can expect that the following relations hold:

$$\lambda(e^1, \boldsymbol{x}_0)=\lambda_1,\ \lambda(e^2, \boldsymbol{x}_0)=\lambda_1+\lambda_2 \text{ and } \lambda(e^3, \boldsymbol{x}_0)=\lambda_1+\lambda_2+\lambda_3, \tag{9}$$

where $\lambda_i (i=1, 2, 3)$ are the 1-dimensional Lyapunov exponents and are assumed to be $\lambda_i\geq\lambda_j$ for $j>i$. Therefore we can estimate the 1-dimensional Lyapunov exponents as

$$\lambda_1=\lambda(e^1, \boldsymbol{x}_0),\ \lambda_2=\lambda(e^2, \boldsymbol{x}_0)-\lambda(e^1, \boldsymbol{x}_0) \quad\text{and}\quad \lambda_3=\lambda(e^3, \boldsymbol{x}_0)-\lambda(e^2, \boldsymbol{x}_0). \tag{10}$$

Figure 1 indicates that the Lyapunov exponents $\lambda(e^k, \boldsymbol{x}_0)$ $(k=1,2,3)$ do not depend not only on the initial choice of a set of bases in the subspace e^k, but also on the initial position $\boldsymbol{x}_0$ in the state space.

Contrary to the statement of the last paragraph, there are other trivial Lyapunov exponents for the Lorenz system, i.e., the exponents for orbits which tend to the fixed point at (0, 0, 0). The 1-dimensional Lyapunov exponents for these orbits become the spectra of the linearized vector field at (0, 0, 0) and take different values from $\lambda_i (i=1,2,3)$ obtained by our numerical experiment. Therefore, it should be considered that orbits tending to the fixed point at (0, 0, 0) would be negligible in some sense.

Relating to the result presented above, we would like to make the following two remarks. The first remark is concerned with the ability of our numerical scheme for estimating the Lyapunov exponents, which has been developed in the Appendix. It has been known that the divergence of the vector field of the Lorenz system takes a constant value, i.e., div $\boldsymbol{F}(\boldsymbol{x}) = -(\sigma+b+1) = -21.0$. This property of the Lorenz system can be utilized to test the ability of our numerical scheme.

That is, as the 3-dimensional exponent $\lambda(e^3, \boldsymbol{x}_0)$ represents the rate of expansion of a volume element, it is directly comparable to the divergence of the vector field. The explicit result representing the temporal convergence of the 3-dimensional exponent, together with that of the 1- and 2-dimensional characteristic exponents, is shown in Table I. From the result of the 3-dimensional exponents $\lambda(e^3, \boldsymbol{x}_0)$, it might be concluded that we can expect, in principle, the same order

Table I. An explicit data representing temporal convergence of the 1-, 2- and 3-dimensional Lyapunov exponent for the Lorenz system ($\sigma=16.0$, $b=4.0$ and $\gamma=40.0$). In this table, it should be mentioned that the 3-dimensional exponent $\lambda(e^3)$ has converged to -21.0 within the accuracy of 0.01%.

t	$\lambda(e^1)$	$\lambda(e^2)$	$\lambda(e^3)$
2.	1.475172	−0.219607	−20.99911
4.	1.281321	0.543842	−20.99913
8.	1.610772	1.166704	−20.99910
16.	1.339753	1.171347	−20.99914
32.	1.411494	1.283690	−20.99913
64.	1.391993	1.340821	−20.99914
128.	1.392513	1.364981	−20.99914
256.	1.378710	1.371627	−20.99914
512.	1.371741	1.365655	−20.99914
1024.	1.370685	1.367358	−20.99914
2048.	1.373692	1.371871	−20.99914
4096.	1.374207	1.373337	−20.99914

Table II. The complete set of the 1-dimensional Lyapunou exponent for the Lorenz system ($\sigma=16.0$, $b=4.0$ and $\gamma=40.0$). The value in ( ) represents the maximal Lyapunov exponent obtained by Benettin's procedure.[7] In the latter procedure, we have set $\|\alpha\|=0.005$ and $\tau=0.11$.

λ_1	1.37	(1.36)
λ_2	0.00	
λ_3	−22.37	

accuracy in the calculations of $\lambda(e^2, \boldsymbol{x}_0)$ and $\lambda(e^1, \boldsymbol{x}_0)$ as that for $\lambda(e^3, \boldsymbol{x}_0)$, if we take a sufficiently long time to estimate these exponents.

According to the relation (10), we can estimate the complete set of the 1-dimensional Lyapunov exponents $\lambda_i(i=1, 2, 3)$ for the Lorenz system. The result is presented in Table II, together with the result for the maximal 1-dimensional exponents estimated by the method which uses the orbit equations (A·4) given in the Appendix.[7] As is understood from the discussion in the Appendix, two values for the maximal 1-dimensional exponent estimated by the present method and that uses the orbit equations showed a good coincidence, when conditions mentioned in the Appendix were satisfied. This makes our second remark.

§ 4. The attractor of another type and the Lyapunov exponents in the Lorenz model

It can easily be imagined that the spectral type of the Lyapunov exponents and the type of attractor are closely related with each other. Non-periodic motions of the Lorenz system possess a positive exponent, and the spectra of the 1-dimensional exponents $\lambda_i(i=1, 2, 3)$ show the hyperbolicity of $(+, 0, -)$ type. It is clear that stable periodic orbit is characterized by the Lyapunov exponents of $(0, -, -)$ type. In this sense, the spectral type of the 1-dimensional Lyapunov exponents is a very useful tool for investigating the appearance of attractors of a new type. From this point of view, we reported in a previous paper that the Lorenz system (σ and b fixed) ends up with a stable periodic attracted for large γ.[18]

Hereafter, we would like to add some new features that were not reported in the previous paper. In the intermediate region of γ, i.e., $50 \lesssim \gamma \lesssim 330$, the solutions in the Lorenz system show a very complicated behavior of bifurcation, when the parameter γ is changed ($\sigma=16.0$ and $b=4.0$).[19] The situations are illustrated in Fig. 2.

In order to discuss or predict the global bifurcation scheme of attractors, one needs the precise knowledge of the Poincaré mapping for this system. In the range of the parameter γ discussed here, there might happen to occur a violation of the tansversality of orbits in the sense that if we make the Poincaré mapping on a certain surface of the section, i.e., $Z=\gamma-1$, the orbit does not cross the surface transeversely. Therefore we cannot state, at the present stage, the global structure of bifurcation scheme for the Lorenz system in a precise manner.

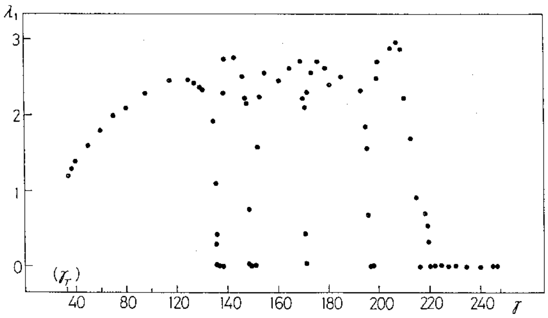

Fig. 2. γ-dependence of the maximal exponent λ_1 for the Lorenz system ($\sigma=16.0$ and $b=4.0$). In this figure, it means that if λ_1 would converge to zero at a certain value of γ, the corresponding dyamical system has a stable limit cycle.

However, for periodic attractors, there are the following clearly distinct bifurcations in some restricted ranges of the parameter γ. The first type might be called the symmetry breaking type and is illustrated in Fig. 3(a). After the original stable limit cycle becomes unstable, a pair of stable limit cycles come forth in this bifurcation. Under the map $(X, Y, Z) \overset{S}{\mapsto} (-X, -Y, Z)$, one of the limit cycles is mapped into the other limit cycle. The orbit is attracted to one of these limit cycles depending on initial conditions. Therefore the symmetry of the Lorenz equation for the transformation S breaks after the bifurcation of this type. The second is the usual Brunovsky bifurcation[20] and is also illustrated in Fig. 3(b). After the Brunovsky bifurcation of this type, the period of the limit cycles becomes twice the original period ω. The symmetry does not change in this bifurcation.

If we decrease the parameter γ from a certain value, where the system possesses an attracting periodic orbit with some periodicity and the symmetry under the transformation S, there occurs at first a symmetry breaking bifurcation. After the bifurcation of this type, there occurs a series of the Brunovsky bifurcations, through which periods of the limit cycles become longer and longer in such manner as $\omega \cdot 2^n\,(n=0, 1, \cdots)$.

Chaotic solutions are considered to appear beyond the limiting value of the parameter γ, at which the interval of the parameter consisting of a stable limit cycle with period n seems to vanish. Analogous bifurcation phenomena mentioned above have been also observed in such dynamical systems as Rössler's model[21] and a certain chemical reaction model.[22] This series of bifurcations leading to chaos may be considered to be an example of the generalized catastrophe introduced by Thom.[23]

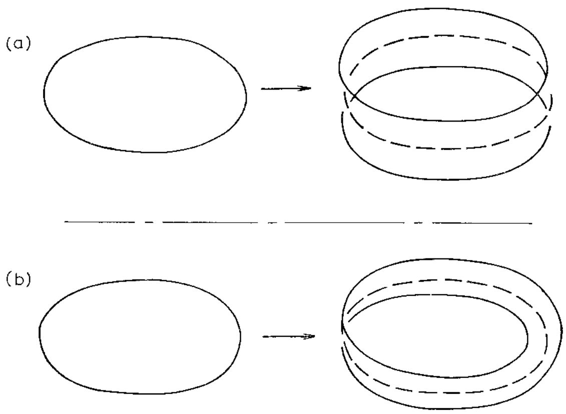

Fig. 3. Bifurcations of periodic orbits in the Lorenz system;
a) a symmetry breaking bifurcation,
b) a Brunovsky bifurcation.

It should be noted here that it includes a very delicate problem in analyzing the series of these bifurcations quantitatively. Namely, as the spectral type of the Lyapunov exponents $\lambda_i (i=1, 2, 3)$ for stable limit cycles and that of chaotic motions must be different from each other, it is clear that there is a critical point at which the spectral type changes between $(+, 0, -)$ and $(0, -, -)$. At this critical point, one of the exponents changes its sign and degenerates to zero. When this degeneracy of the exponents occurs, the dynamical system becomes structurally unstable. So, near this critical point, we must take scrupulous care of analyzing problems by any numerical method.

§ 5. Conclusions and discussion

It is shown numerically that the complete set of the 1-dimensional Lyapunov exponents $(\lambda_1, \lambda_2, \lambda_3)$ for the Lorenz attractor should exist, and the exponents take the same values for almost all orbits stating at the neighbourhood of the Lorenz attractor. The above result supports strongly that there should exist an invariant measure on the attractor, and the Lorenz system should be ergodic on the attractor with respect to this invariant measure. Furthermore, from the fact that the spectra of the 1-dimensional exponents $(\lambda_1, \lambda_2, \lambda_3)$ is of $(+, 0, -)$ type, the motion on the attractor should take the positive-definite K-entropy and therefore possesses the property of mixing.

It should be mentioned here that there is another trivial set of the 1-dimensional exponents, which implies a violation of the uniform hyperbolicity of the non-wandering set for the Lorenz system. This contradictory fact might be considered as follows: Although the Lorenz system does not satisfy the axiom-A in the strict sense,[2] but the trouble is not so severe in a measure theoretic meaning, i.e., orbits, which belong to the trivial Lyapunov exponents, would occupy a space of measure zero in the non-wandering set.

The result for the Lorenz system described here has already pointed out in the previous paper.[8] However, in the previous paper, we have employed the method which uses the orbit equation (A·4) and is more appropriate to visualizing the existence of the exponential orbital separation in state space.[24)~26)] By solving the tangent equation, it becomes more apparent in this paper that the existence of positive Lyapunov exponent is related, on a mathematical basis, to the theory of ergodicity of dynamical systems.

As stated in § 4, the method of the Lyapunov exponents employed in this paper is influential not only in characterizing irregular motions in a quantitative manner, but also in applying to problems of bifurcation of attractors.

In closing this paper, we would like to propose that chaotic motions in 4 and /or higher dimensional dissipative dynamical systems should be classified according to the spectral type of the complete set of the Lyapunov exponents $\lambda_i (i=1, 2, \cdots)$. In 4-dimensional case, it is easily considered that there should exist two kinds of chaotic motions clearly distinguishable from each other, because there are two types of the Lyapunov spectra such as $(+, +, 0, -)$ and $(+, 0, -, -)$.

Acknowledgements

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Appendix A

——*Method for Numerical Estimation of the k-Dimensional Lyapunov Exponent*——

There is a technical problem, when we try to evaluate Lyapunov exponents directly, on the basis of the definition (6), by integrating 1-st variational equations. Namely, there occurs a overflow trouble in computer calculations, because 1-st variational equations have, at least, an exponentially diversing solution.

In order to keep calculations from this trouble, we exchange the base, after each time integration, in the following manner:

$$\begin{aligned}
\boldsymbol{e}_1^{j+1} &= U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_1^{j}/\|U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_1^{j}\|,\\
\boldsymbol{e}_2^{j+1} &= U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_2^{j}-(\boldsymbol{e}_1^{j+1}\cdot U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_2^{j})\cdot\boldsymbol{e}_1^{j+1}/\|U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_2^{j}-(\boldsymbol{e}_1^{j+1}\cdot U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_1^{j+1})\cdot\boldsymbol{e}_1^{j+1}\|,\\
\boldsymbol{e}_3^{j+1} &= U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_3^{j}-(\boldsymbol{e}_1^{j+1}\cdot U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_3^{j})\cdot\boldsymbol{e}_1^{j+1}-(\boldsymbol{e}_2^{j+1}\cdot U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_3^{j})\cdot\boldsymbol{e}_2^{j+1}/\|U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_3^{j}\\
\vdots &\qquad\qquad -(\boldsymbol{e}_1^{j+1}\cdot U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_3^{j})\cdot\boldsymbol{e}_1^{j+1}-(\boldsymbol{e}_2^{j+1}\cdot U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_3^{j})\cdot\boldsymbol{e}_2^{j+1}\|,\\
\boldsymbol{e}_k^{j+1} &= U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_k^{j}-(\boldsymbol{e}_1^{j+1}\cdot U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_k^{j})\boldsymbol{e}_1^{j+1}\cdots-(\boldsymbol{e}_{k-1}^{j+1}\cdot U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_k^{j})\cdot\boldsymbol{e}_{k-1}^{j+1}/\|U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_k^{j}\\
&\qquad\qquad -(\boldsymbol{e}_1^{j+1}\cdot U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_k^{j})\cdot\boldsymbol{e}_1^{j+1}\cdots-(\boldsymbol{e}_{k-1}^{j+1}\cdot U^{\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_k^{j})\cdot\boldsymbol{e}_{k-1}^{j+1}\|.
\end{aligned}\tag{A·1}$$

Using the chain rule (5) and exchanging the base $\{U^{\tau}_{\boldsymbol{x}_j}\boldsymbol{e}_i^{j}\}_i$ into $\{\boldsymbol{e}_i^{j+1}\}_i$ $(j=0, 1, \cdots, (n-1)$ and $i=1, 2, \cdots, k)$, we obtain the following equation:

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n\tau}\log\frac{\|\bigwedge_i U^{n\tau}_{\boldsymbol{x}_0}\boldsymbol{e}_i^{0}\|}{\|\bigwedge_i \boldsymbol{e}_i^{0}\|}\\
&\qquad=\lim_{n\to\infty}\frac{1}{n\tau}\sum_{j=0}^{n-1}\log\frac{\|\bigwedge_i U^{\tau}_{\boldsymbol{x}_j}\boldsymbol{e}_i^{j}\|}{\|\bigwedge_i \boldsymbol{e}_i^{j}\|}\,.
\end{aligned}\tag{A·2}$$

The procedure of exchange of bases leading to Eq. (A·2) is justified by using the following property of exterior product; if $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{f}_i\}$ generate the same k-dimensional subspace, then the relation

$$\frac{\|\bigwedge_i U\boldsymbol{e}_i\|}{\|\bigwedge_i \boldsymbol{e}_i\|}=\frac{\|\bigwedge_i U\boldsymbol{f}_i\|}{\|\bigwedge_i \boldsymbol{f}_i\|}$$

holds.

Now, it may be useful to note the relation between our present method and the approximate method developed in Ref. 7) by Benettin et al. Benettin's procedure in Ref. 7) will converge to our method under the restrictions $\tau \ll 1$ and $\|e_i^j\| \ll 1$. Under these restrictions, an approximate relation

$$U_{x_j}^{\tau} e_i^j \cong T^{\tau}(x_j + e_i^j) - T_{x_j}^{\tau} \tag{A·3}$$

holds. We can, therefore, obtain approximately the Lyapunov exponents as follows:

$$\lambda(e^k, x_0) \cong \lim_{n\to\infty} \frac{1}{n\tau} \sum_{j=0}^{n-1} \log \frac{\| \bigwedge_i (T^{\tau}(x_j + e_i^j) - T^{\tau} x_j) \|}{\| \bigwedge_i e_i^j \|}, \tag{A·4}$$

where T is the map which has appeared in Eq. (2).

The special case ($k=1$) of the expression (A·4) has been utilized first by Benettin et al.,[7] who gave discussion for estimating the K-entropy of Hamiltonian systems. The authors applied their method, in the previous paper,[8] to a dissipative dynamical system, and pointed out the possibility that turbulent phenomena in dissipative dynamical systems would be discussed in a generalized framework of the ergodic theory of classical dynamical systems.

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