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A NUMERICAL EMPIRICAL BAYES PROCEDURE FOR FINDING AN INTERVAL ESTIMATE

Summary

A numerical procedure is outlined for obtaining an interval estimate of a parameter in an empirical Bayes estimation problem. The case where each observed value x has a binomial distribution, conditional on a parameter ζ, is the only case considered. For each x, the parameter estimated is the expected value of ζ given x. The main purpose is to throw some light on the effectiveness of empirical Bayes estimation in samples of various sizes. Illustrative numerical results are presented.

1. Introductory Numerical Example

The following example illustrates a type of situation in which the usual estimator for a binomial parameter can be very misleading. A sample of $n = 10$ independent observations was drawn from a Bernouilli distribution with probability of success π. This was repeated for $N = 10,000$ different values of π, these being obtained by random sampling from a prechosen distribution of π's. The sample proportion p of successes was found for each of the 10,000 samples. The resulting sample distribution $Nf(p)$ of the 10,000 values of p is shown below:

p	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
$Nf(p)$	7	102	477	1140	2053	2476	2027	1175	437	98	10
$(\underline{\pi}, \bar{\pi})$	.189-.505		.469-.505		.490-.505		.493-.509		.493-.527		.493-.840

The usual estimator of the value of π giving rise to any particular sample of 10 is the observed p for that sample, this being a sufficient statistic if the sample of 10 is considered by itself. For even values of $10p$, the table also shows interval estimates $(\underline{\pi}, \bar{\pi})$ of the regression of π on p, obtained by the empirical Bayes approach to be described in this paper.

The purpose of the above numerical example is to display the disagreement between the usual estimates p of π and the empirical Bayes interval estimates shown. The reason for such sharp disagreement is that the population distribution of π chosen to generate these artificial data, until now concealed from the reader, actually had a negligible standard deviation so that $Nf(p)$ is in effect a random sample of 10,000 from a binomial distribution with $\pi = 0.5$. Although this information was not available for estimation purposes, the empirical Bayes estimation procedure partially recovered equivalent information from the sample $Nf(p)$. A pleasing result is that each interval estimate here turns out to include the true value $\pi = 0.5$.

2. Mathematical Formulation

Suppose that for each randomly drawn observation, there is not only an observed value x of the discrete random variable X, but also a corresponding unobservable value ζ of the continuous random variable Z. For any given ζ, x is drawn at random from the binomial distribution

$$h(x|\zeta) \equiv \text{Prob}(X = x|\zeta) = \binom{n}{x}\zeta^x(1 - \zeta)^{n-x} \quad , \qquad x = 0,1,\ldots, n \; , \qquad (1)$$

where n is known. Given randomly drawn observations $x_1, x_2, \cdots, x_N$, the

ζ corresponding to any particular observation x_a can be estimated by empirical Bayes point-estimation methods (for example, Robbins, 1956; Maritz, 1966; Copas, 1969; Griffin & Krutchkoff, 1971). More needs to be known about the effectiveness of such methods where n and N are not both very large.

Denote by $G(\zeta)$ the unknown cumulative distribution function of Z for the population from which the unobservable values are drawn. In view of (1), it will be assumed that the range of ζ is $0 \leq \zeta \leq 1$. Ordinarily, $G(\zeta)$ is thought of as continuous, but we will not exclude the possibility that it is a step function.

The (unconditional) probability distribution of X for the population can be written

$$\phi_G(x) = \int_0^1 h(x|\zeta)\, dG(\zeta) \quad , \qquad x = 0,1,\ldots,n \quad . \tag{2}$$

The observed sample distribution of $x_1, x_2, \ldots, x_N$, to be denoted by $f(x)$, is the distribution of a random sample from $\phi_G(x)$.

If $G(\zeta)$ were known, the value of ζ corresponding to any observed x would in common practice be estimated by the regression function

$$\mu_{Z|x} \equiv \mathcal{E}_G(Z|x) = \frac{1}{\phi_G(x)} \int_0^1 \zeta\, h(x|\zeta)\, dG(\zeta) \quad , \quad x = 0,1,\ldots,n \quad , \tag{3}$$

usually called the Bayes estimator. When $G(\zeta)$ is unknown, as here, the problem of finding a point estimator $\hat{\zeta}$ for the ζ corresponding to an observed x is a standard empirical Bayes problem. Typically, a point estimate ($\hat{\mu}_{Z|x_a}$, say) of $\mathcal{E}(Z|x_a)$, the regression of Z on x evaluated at x_a , is used as the empirical Bayes estimator of ζ for observation a . The binomial case considered here is of particular interest (as compared, for example, to the Poisson) since in the binomial case $G(\zeta)$ is unidentifiable--complete knowledge of $\phi_G(x)$ is sufficient to determine only the first n moments of $G(\zeta)$ (Skellam, 1948).

The present note is explicitly concerned with an interval estimator $(\underline{\mu}_{\alpha x}, \bar{\mu}_{\alpha x})$ of $\mu_{Z|x}$, $x = 0,1,\ldots,n$, $G(\zeta)$ being unknown. Given the empirical Bayes model already specified and a sample with the observed distribution $f(x)$, what range of values for $\hat{\mu}_{Z|x}$, $x = 0,1,\ldots,n$, is reasonably consistent with the observed $f(x)$?

Consider the set Γ_α consisting of all distribution functions $G(\zeta)$ such that the chi square (χ^2_G) between $\phi_G(x)$, defined by (2), and the given $f(x)$ is less than $\chi^2_{1-\alpha}$, the $1 - \alpha$ percentile of the chi square distribution:

$$\chi^2_G \equiv \sum_{x=0}^{n} \frac{[f(x) - \phi_G(x)]^2}{\phi_G(x)} \leq \chi^2_{1-\alpha} \quad . \tag{4}$$

For each possible value of x , find $\bar{\mu}_{\alpha x} \equiv \underset{\Gamma_\alpha}{\mathrm{Max}}(\mu_{Z|x})$ and $\underline{\mu}_{\alpha x} \equiv \underset{\Gamma_\alpha}{\mathrm{Min}}(\mu_{Z|x})$, the maximum and minimum values of (3) in Γ_α . For any given x , any value of $\mu_{Z|x}$ in the interval $(\underline{\mu}_{\alpha x}, \bar{\mu}_{\alpha x})$ is consistent with the data; values outside this interval will be considered implausible.

The foregoing is not an ideal way to set up an interval estimate. Until better methods are implemented, however, it can throw some light on the accuracy of empirical Bayes point estimation. For the standard empirical Bayes problem of estimating ζ , any point estimate $\hat{\zeta}$ in the interval

$(\underline{\mu}_{\alpha x}, \bar{\mu}_{\alpha x})$ would minimize the squared errors of estimation for some $G(\zeta)$ in Γ_α. Any estimate outside this interval would not.

If desired, the interval described above can be interpreted as a confidence interval for $\mu_{Z|x}$ with confidence level $> 1 - \alpha$. This may be seen as follows. For any given $G(\zeta)$, Γ_α is a random variable so defined as to include $G(\zeta)$ $1 - \alpha$ of the time. Thus $(\underline{\mu}_{\alpha x}, \bar{\mu}_{\alpha x})$ must include the true $\mu_{Z|x}$ at least $1 - \alpha$ of the time.

3. Bounds for the Regression of Z on x

Substituting (1) in (2) and expanding, we have

$$\phi(x) = \binom{n}{x} \int_0^1 \zeta^x (1 - \zeta)^{n-x} \, dG(\zeta)$$

$$= \binom{n}{x} \sum_{r=0}^{n-x} \binom{n-x}{r} (-1)^r \mu'_{r+x} \quad , \qquad x = 0,1,\ldots,n \quad , \qquad (5)$$

where μ'_s is the moment about the origin of order s for the distribution $G(\zeta)$. Ruling out the degenerate case where Z takes no values other than 0 or 1 (and where consequently $\phi(x) = 0$ unless $x = 0$ or n), we have that $\mu'_s > 0$ for all s. Similarly, from (3),

$$\mu_{Z|x} = \frac{\sum_{r=0}^{n-x} \binom{n-x}{r} (-1)^r \mu'_{r+x+1}}{\sum_{r=0}^{n-x} \binom{n-x}{r} (-1)^r \mu'_{r+x}} = \frac{\sum_{R=x}^{n} \binom{n-x}{R-x} (-1)^{R-x} \mu'_{R+1}}{\sum_{R=x}^{n} \binom{n-x}{R-x} (-1)^{R-x} \mu'_R} \quad , \quad x = 0,1,\ldots,n \quad . \qquad (6)$$

Consider first a restricted case where $\mu_1', \mu_2', \cdots, \mu_n'$ are fixed. In this special case, (6) is seen to be maximized (minimized) by maximizing (minimizing) μ_{n+1}' if $n - x$ is even, by minimizing (maximizing) μ_{n+1}' if $n - x$ is odd. A theorem of Markov (see Possé, 1886, sections V8 and V9; or Karlin & Shapley, 1953) shows that if $\mu_1', \mu_2', \cdots, \mu_n'$ (considered fixed) are the moments of some frequency distribution, then the maximum (minimum) value of μ_{n+1}' is uniquely attained when $G(\zeta)$ is a certain specified step function. If n is even, all the frequency is concentrated at exactly $n/2 + 1$ different values of ζ, including $\zeta = 1$ if μ_{n+1}' is maximized, $\zeta = 0$ if μ_{n+1}' is minimized. The situation for odd n is similar, but need not be detailed here.

This leads to the key conclusion that in order to find $G(\zeta)$ maximizing or minimizing (6) for fixed x, when $\mu_1', \mu_2', \cdots, \mu_n'$ are given, it is only necessary to determine $M \equiv n/2$ unknown values $\zeta_0, \zeta_1, \cdots, \zeta_{M-1}$ or $\zeta_1, \zeta_2, \cdots, \zeta_M$, together with the corresponding M unknown frequencies $g_0, g_1, \cdots, g_{M-1}$ or $g_1, g_2, \cdots, g_M$. It is <u>not</u> necessary to admit to consideration any of the continuous frequency distributions on $(0,1)$ nor any discrete distribution with more than M unknown values of ζ.

In our actual problem, we wish to find the probability distribution $\bar{g}_{\alpha x}(\zeta)$, say, maximizing (6) or the $g_{\alpha x}(\zeta)$ minimizing (6) subject to the restriction that the corresponding cumulative distribution function is in Γ_α. This restriction does not change the key conclusion stated above, since holding $\mu_1', \mu_2', \cdots, \mu_n'$ fixed also holds $\phi_G(0), \phi_G(1), \cdots, \phi_G(n)$ fixed, because of (5), and thus fixes χ_G^2. Thus the extremizing $G(\zeta)$ will still be a step function with exactly $M + 1$ different values of ζ, as before. (For example, let $\underline{\mu}_1', \underline{\mu}_2', \cdots, \underline{\mu}_n'$ be the first n moments of

$g_{\alpha x}$. Markov's theorem states that the minimizing $G(\zeta)$ with these moments is of the special form described. Thus if $g_{\alpha x}$ exists, it is of this special form.)

It is, of course, always possible that Γ_α is empty. This situation has not yet arisen in practical application. The smaller the value of α chosen, the less likely it is that Γ_α will be empty.

For a discrete $G(\zeta)$ such as $g_{\alpha x}(\zeta)$ the first line of (5) can be written

$$\phi(x) = \binom{n}{x} \sum_{m=0}^{M} g_m \zeta_m^x (1 - \zeta_m)^{n-x} \quad , \tag{7}$$

where either $g_0 = 0$ or $g_M = 0$. Similarly, (3) becomes

$$\mu_{Z|x} = \frac{\binom{n}{x}}{\phi(x)} \sum_{m=0}^{M} g_m \zeta_m^{x+1} (1 - \zeta_m)^{n-x} \quad . \tag{8}$$

Formulas derived by Markov (not given here) provide the explicit solution (if any) to the extremization problem when $\mu_1', \mu_2', \ldots, \mu_n'$ are fixed. These formulas do not help with the more general problem to be solved here, which seems to require the numerical methods of mathematical programming.

4. Numerical Methods

Note that the use of a step function for $G(\zeta)$ here is required by the problem itself, not imposed for the convenience of the writer. For simplicity, the following discussion deals only with minimization. Maximization is essentially similar.

Thanks to Markov, our problem is now to find ζ_m and g_m', $m = 0,1,\ldots,M-1$ or $m = 1,2,\ldots,M$, so as to minimize (8), subject to the restrictions

$$0 \leq \zeta_m \leq 1 \ , \quad g_m \geq 0 \ , \text{ and } \ \Sigma g_m \leq 1 \quad , \tag{9}$$

and also subject to (4). The problem thus stated can be solved numerically for any given set of data by mathematical programming techniques.

Actually, the inequality ($\leq$) restriction in (4) can without loss of generality be replaced by equality. A proof is given in the appendix for situations where at least two or three of the g_m are nonzero.

The writer is indebted to Martha Hamilton who developed the computer program to carry out the required minimizations and maximizations numerically for various sets of data. The program implements a sequential unconstrained minimization technique (SUMT) of Fiacco and McCormick (1968, chapter 4). The constraint on χ^2 was handled by use of a penalty function; other constraints were dealt with by simpler means. The unconstrained minimizations required for SUMT were carried out by a program developed by Jöreskog (1967, section 8) and modified by Hamilton, implementing the Fletcher-Powell-Davidon (1963) method.

All computer runs were made in double precision on an IBM 360/65. As a check, each of the 44 extremization problems dealt with in Table 1 was run with two different starting points, one of which was completely random within the limitations $0 \leq g_m$ and $0 \leq \zeta_m \leq 1$ ($m = 0,1,\ldots,M$) and $\Sigma g_m = 1$. When $N = 12,990$, the agreement between the $\underline{\mu}_{\alpha x}$ or $\bar{\mu}_{\alpha x}$ reached from two different starting points was to at least four decimal places; when $N = 130$, the agreement was to at least three decimal places. This suggests that the intervals obtained represent global, not just local, maxima and minima, at least to a three-decimal-place approximation. Hundreds of other checks were made to be sure that small changes in $\underline{g}_{\alpha x}(\zeta)$ or in $\bar{g}_{\alpha x}(\zeta)$ would not give more extreme $\underline{\mu}_{\alpha x}$ or $\bar{\mu}_{\alpha x}$, respectively. All such checks were satisfied.

5. Numerical Results

The procedure described was applied to the real data shown in the second column of Table 1. This column shows the frequency distribution of $N = 12,990$ independent observations (actually, test scores of 12,990 students--their number of correct answers on a psychological test composed of $n = 20$ questions). A separate study (see below) shows that this distribution is compatible with the mathematical model given by (1) and (2).

The fifth column shows interval estimates of $\mu_{\zeta|x}$, obtained for these data by the method outlined, with $\alpha = .05$. These empirical Bayes intervals, unlike ordinary interval estimates, are wider at extreme values of x than at middle values. In the middle, the intervals shown for $N = 12,990$ are happily short.

Although it is not obvious from a look at the table, no straight-line regression of ζ on x can be fitted inside the intervals shown for $N = 12,990$. The indicated nonlinearity is not rigorously demonstrated by the methods described here. However, linearity of regression implies

Table 1. Observed frequency distributions f(x) and corresponding interval estimates (α = .05) for the regression of Z on x .

x	Nf(x) N = 12,990	Nf(x) N = 130	$\hat{\mu}_{Z\|x}$	Interval Estimate of $\mu_{Z\|x}$ N = 12,990	Interval Estimate of $\mu_{Z\|x}$ N = 130
20	63	2	.898	.852-.952	.690-1.000
19	141	2	.863		
18	220	2	.823	.780-.846	.615-.943
17	319	2	.773		
16	424	6	.716	.690-.743	.573-.855
15	622	4	.663		
14	776	8	.619	.605-.646	.510-.749
13	1001	4	.583		
12	1203	9	.553	.534-.564	.440-.636
11	1443	19	.526		
10	1550	13	.500	.485-.511	.404-.556
9	1409	12	.475	.461-.487	
8	1235	13	.451	.436-.463	.363-.491
7	1052	18	.426		
6	696	8	.402	.387-.420	.305-.491
5	471	4	.379		
4	226	3	.356	.340-.399	.199-.491
3	98	1	.334		
2	27	0	.314	.233-.383	.049-.491
1	12	0	.296	.107-.380	
0	2	0	.280	.010-.374	.000-.491
	12,990	130			

that $\phi_G(x)$ is a negative hypergeometric distribution (Lord & Novick, 1968, section 23.6); thus linearity can be tested by determining whether $f(x)$ can be considered a random sample from such a distribution.

In order to investigate the effects of sample size, a random sample of 130 observations was drawn from the 12,990. The resulting $f(x)$ is shown in the table along with the corresponding interval estimates of $\mu_{Z|x}$. The intervals are of course much wider than for $N = 12,990$, but fortunately not 10 times as wide. (Grouping of frequencies was used in the calculations where necessary so that the denominator of (4) should never be less than 1.)

The fourth column of the table is included as a partial check on the validity of the intervals obtained. This column shows the regression of ζ on x corresponding to a certain $G(\zeta)$ which was found (in a separate study) to provide a good fit to the $f(x)$ in column two, the obtained chi square between $\phi_G(x)$ and $f(x)$ being near the 50th percentile of the tabled distribution of chi square for 20 degrees of freedom. It is pleasant to find that these regression values all lie well within the interval estimates shown in column five.

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Appendix

It will be shown here that $g_{\alpha x}(\zeta)$, the $G(\zeta)$ that minimizes (6) or (8) subject to (4) and (9) must lie on the boundary where $\chi^2_G = \chi^2_{1-\alpha}$. To avoid tediousness, the proofs are limited to situations where at least two or three (as convenient) of the ζ_m are distinct and have nonzero frequency.

It will be shown first that $\mu_{Z|x}$, considered as a function of the g_m and the ζ_m, has no unrestricted minimum satisfying (9). This is proved by showing that if we are given any set with $K \geq 2$ of g_m and ζ_m satisfying (9), we can reduce the $\mu_{Z|x}$ computed from (8) by a change in the g_m and ζ_m that does not violate (9). Thus if a restricted minimum with $K \geq 2$ is found by the methods of section 4, it must lie on the only remaining boundary ($\chi^2_G = \chi^2_{1-\alpha}$), imposed by restriction (4), since otherwise $\mu_{Z|x}$ could still be reduced by the methods outlined. We will treat explicitly only the case where x is even. Thus $\zeta_0 = 0$ by Markov's theorem. (Similar proofs will apply when x is odd.) Let $\zeta_1 < \ldots < \zeta_K$ be the $K \geq 2$ distinct nonzero values at which $g_{\alpha x}(\zeta)$ has nonzero frequency.

All possibilities will now be covered by the following four cases.

Case 1: $g_0 > 0$, $0 < x < n$.

Consider the derivative of (8) with respect to ζ_i:

$$\frac{\partial \mu_{Z|x}}{\partial \zeta_i} \propto g_i \zeta_i^x (1 - \zeta_i)^{n-x-1} [x + 1 - (n + 1)\zeta_i] \sum_{m=0}^{K} g_m \zeta_m^x (1 - \zeta_m)^{n-x}$$

$$- g_i \zeta_i^{x-1} (1 - \zeta_i)^{n-x-1} (x - n\zeta_i) \sum_{m=0}^{K} g_m \zeta_m^{x+1} (1 - \zeta_m)^{n-x}$$

$$\propto g_i \zeta_i^{x-1}(1 - \zeta_i)^{n-x-1}$$

$$\cdot \sum_{m=0}^{K} g_m \zeta_m^x (1 - \zeta_m)^{n-x}[\zeta_i\{x + 1 - (n + 1)\zeta_i\} - \zeta_m(x - n\zeta_i)]$$

$$\propto g_i \zeta_i^{x-1}(1 - \zeta_i)^{n-x-1}$$

$$\cdot \sum_{m=0}^{K} g_m \zeta_m^x (1 - \zeta_m)^{n-x}[\zeta_i(1 - \zeta_i) + (\zeta_i - \zeta_m)(x - n\zeta_i)] \quad . \qquad (A1)$$

Take $i = 0$ and suppose for the moment that $K \geq 2$ and that ζ_0 $(= 0)$ is replaced by a very small positive quantity. The first term in the brackets can be neglected. The second term in the brackets is zero when $m = 0$ but is negative otherwise. Since all quantities outside the brackets are nonnegative and some, at least, are positive, the derivative (A1) will be negative. Thus $\mu_{Z|x}$ can be reduced by replacing ζ_0 by a small positive quantity (this can also be seen intuitively).

<u>Case 2</u>: $g_0 = 0\,, \quad 0 < x < n$.

Consider the effect on $\mu_{Z|x}$ of a small change in g_i and a small compensating change in g_K , holding all other g_m fixed. Treating g_K as a function of g_i defined by the equation

$$g_K = 1 - \sum_{m=0}^{K-1} g_m \qquad (A2)$$

and using the formula

$$\frac{d\mu_{Z|x}}{dg_i} = \frac{\partial\mu_{Z|x}}{\partial g_i} + \frac{\partial\mu_{Z|x}}{\partial g_K}\frac{dg_K}{dg_i} \quad , \qquad (A3)$$

we find from (8) that when the g_m $(m \neq i,K)$ are fixed,

$$\frac{d\mu_{Z|x}}{dg_i} \propto \sum_{m=0}^{K} g_m\zeta_m^x(1 - \zeta_m)^{n-x}[\zeta_i^{x+1}(1 - \zeta_i)^{n-x} - \zeta_K^{x+1}(1 - \zeta_K)^{n-x}]$$

$$- \sum_{m=0}^{K} g_m\zeta_m^{x+1}(1 - \zeta_m)^{n-x}[\zeta_i^x(1 - \zeta_i)^{n-x} - \zeta_K^x(1 - \zeta_K)^{n-x}]$$

$$\propto \sum_{m=0}^{K} g_m\zeta_m^x(1 - \zeta_m)^{n-x}[\zeta_i^{x+1}(1 - \zeta_i)^{n-x}$$

$$- \zeta_K^{x+1}(1 - \zeta_K)^{n-x} - \zeta_m\zeta_i^x(1 - \zeta_i)^{n-x} + \zeta_m\zeta_K^x(1 - \zeta_K)^{n-x}]$$

$$\propto \sum_{m=1}^{K} g_m\zeta_m^x(1 - \zeta_m)^{n-x}[\zeta_i^x(1 - \zeta_i)^{n-x}(\zeta_i - \zeta_m)$$

$$+ \zeta_K^x(1 - \zeta_K)^{n-x}(\zeta_m - \zeta_K)] \quad . \tag{A4}$$

The last expression in (A4) makes use of the fact that since $g_0 = 0$ for Case 2, any summation in (A4) can be written either including or excluding $m = 0$.

The second term in the brackets is never positive. Now take $i = 1$. The first term in brackets is now zero when $m = 1$ and negative otherwise. All quantities outside the brackets are nonnegative; if either $\zeta_K < 1$, or if $K \geq 3$, then some of these quantities are positive. If so, the derivative of (A4) is necessarily negative for $i = 1$. Thus $\mu_{Z|x}$ can be reduced by shifting some frequency from ζ_1 to ζ_K .

(If $\zeta_K = 1$ and $K = 2$, all terms under the summation in (A4) are zero. This special case can be dealt with by using (A1) again with $i = 1$. Since $g_0 = 0$, $1 - \zeta_K = 0$, and $\zeta_i - \zeta_1 = 0$, the last expression in

(A1) simplifies to $d\mu_{Z|x}/d\zeta_1 \propto g_1^2\zeta_1^{2x}(1 - \zeta_1)^{2n-2x}$. This derivative of $\mu_{Z|x}$ is necessarily positive for $x < n$. Thus $\mu_{Z|x}$ can be reduced by decreasing ζ_1 .)

Case 3: $x = 0$

In this case (8) becomes

$$\mu_{Z|x} = \frac{\sum_{m=0}^{M} g_m\zeta_m(1 - \zeta_m)^n}{\sum_{m=0}^{M} g_m(1 - \zeta_m)^n} \quad .$$

Using (A2) and (A3) as before, but with $i = 0$, we find, since $\zeta_0 = 0$, that

$$\frac{d\mu_{Z|x}}{dg_0} \propto \sum_{m=0}^{M} g_m(1 - \zeta_m)^n[- \zeta_K(1 - \zeta_K)^n]$$

$$- \sum_{m=0}^{M} g_m\zeta_m(1 - \zeta_m)^n[(1 - \zeta_0)^n - (1 - \zeta_K)^n]$$

$$\propto \sum_{m=0}^{M} g_m(1 - \zeta_m)^n[- \zeta_K(1 - \zeta_K)^n - \zeta_m + \zeta_m(1 - \zeta_K)^n]$$

$$\propto \sum_{m=0}^{M} g_m(1 - \zeta_m)^n[- \zeta_m - (\zeta_K - \zeta_m)(1 - \zeta_K)^n] \quad . \qquad \text{(A5)}$$

Since (A5) is always negative, an increase in g_0 together with a corresponding decrease in g_K will reduce $\mu_{Z|x}$ in Case 3.

Case 4: $x = n$.

In this case

$$\mu_{Z|x} = \frac{\sum_{m=1}^{M} g_m \zeta_m^{n+1}}{\sum_{m=1}^{M} g_m \zeta_m^{n}} \quad .$$

Using (A2) and (A3) as before with $i = 0$, we have

$$\frac{d\mu_{Z|x}}{dg_0} \propto \sum_{m=1}^{M} g_m \zeta_m^{n}(-\zeta_K^{n+1}) + \sum_{m=1}^{M} g_m \zeta_m^{n+1}(\zeta_K^{n})$$

$$\propto \zeta_K^{n} \sum_{m=1}^{M} g_m \zeta_m^{n}(\zeta_m - \zeta_K) \quad . \tag{A6}$$

Since (A6) is always negative, an increase in g_0 together with a compensating decrease in g_K will reduce $\mu_{Z|x}$ in Case 4.

The four cases listed are exhaustive, provided $K \geq 2$. Similar proofs could be written to eliminate this proviso. Consequently, a minimum for $\mu_{Z|x}$ cannot occur except on the boundary established by (4).

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