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A numerical method for solving Linear Non-homogenous Fractional Ordinary Differential Equation

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Abstract: In this paper, a numerical method for solving LNFODE (Linear Non-homogenous Fractional Ordinary Differential Equation) is presented. The method presented is based on Bernstein polynomials approximation. The operational matrices of integration, differentiation and products are introduced and utilized to reduce the LNFODE problem in order to solve algebraic equations. The method is general, easy to implement, and yields very accurate results. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Keywords: Bernstein polynomials, Operational matrices, fractional Ordinary Differential Equation, numerical method.

1. Introduction

One of the major advantages of fractional calculus is that it can be considered as a super set of integer-order calculus. Thus, fractional calculus has the potential to accomplish what integer-order calculus cannot. We believe that many of the great future developments will come from the applications of fractional calculus to different fields.
In the last two decades, the fractional differentiation has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, control theory and signal and image processing.[1-5] One of the most important problems we face is how to solve fractional differential equations; therefore for this purpose, different techniques have been proposed. The most commonly used ones are Adomian decomposition method (ADM) [6], Variational Iteration Method (VIM) [7], Fractional Differential Transform Method (FDTM) [8], Operational Matrix Method [9], Homotopy Analysis Method [10,11], Fractional Difference Method (FDM) [12] and Power Series Method [13]. Also there are some classical solution techniques, e.g. Laplace Transform Method [14]. The operational Matrix Method has been one of the techniques that researchers have focused on recently. The operational Matrix Method is based on the application of orthogonal functions. Typical orthogonal functions that have been applied so far are: The Walsh functions [15, 16], block pulse functions [17–20], Generalized block pulse functions [21], Legendre polynomials [22–24], Chebyshev polynomials [25], Laguerre polynomials [26,27], and Fourier series [28,29]. In this article, the method of deriving the Bernstein operational matrices and the method of solving fractional differential equation by the Bernstein operational matrices has been tried.

2. Fractional operators

Equations in which an unknown function $y(x)$ is contained under the sign of a derivative of fractional order, i.e. equation of the form

$$F(x, y(x), D_{a_1}^{\alpha_1}\omega_1(x)y(x), D_{a_2}^{\alpha_2}$$
$$\omega_2(x)y(x), \cdots, D_{a_n}^{\alpha_n}\omega_n(x)y(x)) = g(x) \quad (1)$$

where $D_{a_j}^{\alpha_j} = D_{a_j+}^{\alpha_j} = (\frac{d}{dx})^{\alpha}$
or $D_{a_j}^{\alpha_j} = D_{a_j-}^{\alpha_j} = (-\frac{d}{dx})^{\alpha}, j = 1, 2, ..., n$ are called ordinary differential equation of fractional order. By analogy with the classical theory of differential equations, differential equations of fractional order are divided into linear, homogeneous and inhomogeneous equations with constant

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and variable coefficients. Differential equation of fractional order are studied both in the space of regular functions, i.e. functions summable to a certain power and continuous and differentiable up to certain order in a classical sense, and in various spaces of generalized functions.[30]

Definition 1: Let $f(x) \in L_1(a,b)$.The integrals

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt,\ \ x>a \tag{2}$$

$$I^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}\,dt,\ \ x<b \tag{3}$$

where $\alpha > 0$, are called Riemann-Liouville fractional integral of order α.

Definition 2:

For functions $f(x)$ given in interval $[a,b]$, each of the expressions

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n\int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\,dt, \tag{4}$$

$$D^{\alpha}_{b-}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}(\frac{d}{dx})^n\int_x^b \frac{f(t)}{(t-x)^{\alpha-n+1}}\,dt, \tag{5}$$

where $n = [\alpha]+1$, are called Riemann-Liouville fractional derivative of order α.

Definition 3:

Let $Re\alpha > 0$. A function $f(x) \in L_1(a,b)$ is said to have a summable fractional derivative $D^{\alpha}_{a+}f$, if $I^{\alpha}_{a+}f \in AC^n([a,b])$, $n = [Re\alpha]+1$.

Theorem 1:

Let $Re\alpha > 0$, then the equality

$$D^{\alpha}_{a+}I^{\alpha}_{a+}\varphi = \varphi(x) \tag{6}$$

is valid for any summable function $\varphi(x)$ while

$$I^{\alpha}_{a+}D^{\alpha}_{a+}f = f(x) \tag{7}$$

is satisfied for

$$f(x) \in I^{\alpha}_{a+}(L_1) \tag{8}$$

If we assume that instead of (8) a function $f(x) \in L_1(a,b)$ has a summable derivative $D^{\alpha}_{a+}f$(in the sense of Definition1), then (7) is not true in general and is to be replaced by the result

$$I^{\alpha}_{a+}D^{\alpha}_{a+}f = f(x) - \sum_{k=0}^{n-1}\frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)}f^{(n-k-1)}_{n-\alpha}(a)\,, \tag{9}$$

Where $n = [Re\alpha]+1$ and $f_{n-\alpha}(x) = I^{n-\alpha}_{a+}f$. In particular we have

$$I^{\alpha}_{a+}D^{\alpha}_{a+}f = f(x) - \frac{f_{1-\alpha}(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}\,, \tag{9}$$

for $0 < Re\alpha < 1$.[30]

3. The Properties of Bernstein polynomials

The Bernstein polynomials of mth-degree are defined on the interval $[a,b]$ as follows

$$B_{i,m}(x) = mi\frac{(x-a)^i(b-x)^{m-1}}{(b-a)^m};0\le i\le m \tag{10}$$

in which

$$mi = \frac{m!}{i!(m-i)!}.$$

These Bernstein polynomials form a basis over the interval $[a,b]$. [31, 32]

There are $m+1$, mth-degree polynomials. For convenience, we assume $B_{i,m}(x) = 0$, if $i<0$ or $i>m$. Moreover, the recursive definition for the Bernstein polynomials over the interval $[a,b]$ is as follows:

$$B_{i,m}(x) = \frac{b-x}{b-a}B_{i,m-1}(x) + \frac{x-a}{b-a}B_{i-1,m-1}(x)\,. \tag{11}$$

It can easily be shown that each of the Bernstein polynomials is positive and for all real x over the interval $[a,b]$, the sum of all the Bernstein polynomials is equal to unity, i.e.

$$\forall x\in[a,b];\qquad \sum_{i=0}^{m}B_{i,m}(x) = 1.$$

4. The orthonormalization of the Bernstein polynomials

Using gram-Schmidt orthonormalization process on the Bernstein polynomials and normalizing them on the interval $[a,b]$, we obtain a class of orthonormal polynomials naming them as

$$b_{0,m}, b_{1,m}, \ldots, b_{m,m}$$

in which m is the order of Bernstein polynomials in the base.

For example, if we let $m=4$,$b_{0,4}, b_{1,4}, \ldots, b_{4,4}$ orthogonal polynomials over interval $[a,b]$ are given by

$$b_{0,4} = 3(b-x)^4(b-a)^{-\frac{9}{2}}$$

$$b_{1,4} = -\sqrt{7}(b-x)^3(b-a)^{-\frac{9}{2}}$$

$$b_{2,4} = \sqrt{5}(b-x)^2(b-a)^{-\frac{9}{2}}(b^2-16bx+14ab-56ax$$
$$+36x^2+21a^2)$$

$$b_{3,4} = -\sqrt{3}(b-x)(b-a)^{-\frac{9}{2}}(b^3+18ab^2$$
$$-21b^2x+45a^2b+84bx^2-126abx$$
$$-105a^2x-84x^3+20a^3+168ax^2)$$

$$b_{4,4} = (b-a)^{-\frac{9}{2}}(b^4+20ab^3-24b^3x+60a^2b^2$$
$$-180ab^2x+126b^2x^2-240ba^2x-224bx^3+40a^3b$$
$$+420abx^2+210a^2x^2+5a^4-280ax^3-60a^3x+126x^4.$$

A function $y(x)$ which is integrable in the interval $a \leq x \leq b$, can be expanded into Bernstein series by

$$y(x) = \underset{m\to\infty}{Lim} \sum_{i=0}^{m-1} c_{i\,m-1} b_{i\,m-1}(x) \tag{12}$$

where

$$c_{i\,m-1} = \int_a^b y(x) b_{i\,m-1}(x)\,dx \tag{13}$$

Eq. (13) can be written into the discrete form by

$$y(x) \approx Y_m(x) = C^T B_m(x) \tag{14}$$

where

$$B_m(x) = [b_{0m}(x), b_{1m}(x), \cdots, b_{mm}(x)]^T$$

is the Bernstein matrix and

$$C = [c_{0m}, c_{1m}, \cdots, c_{mm}]^T$$

is the coefficient matrix of Y.

5. The Bernstein operational matrices for fractional calculus

Fractional calculus is a generalization of integration and differentiation to non-integer order. For finding the operational matrices of fractional integration we use

$$y(x) \approx Y_m(x) = C^T B(x) \tag{15}$$

and

$$D^\alpha B(x) = P_B^{-\alpha} B(x) \tag{16}$$

where P_B is the operational matrix for integration of the Bernstein polynomials.[31] By substituting (16) into (17) yields: $D^\alpha y(x) = D^\alpha \{C^T B(x)\}$

$$= C^T D^\alpha B(x) = C^T P_B^{-\alpha} B(x). \tag{17}$$

According to the property of fractional calculus, $P_B^\alpha P_B^{-\alpha} = I$, where P_B^α is the fractional operational matrix of the Bernstein polynomials for integration with the order α,we can get matrix P_B^α by inverting the $P_B^{-\alpha}$matrix.

For example, let $m = 10$, over interval $[0, 2]$

$$P_B = \begin{pmatrix} 0.0950 & 0.184 & 0.168 & \cdots & 0.0436 \\ -0.00473 & 0.0850 & 0.169 & \cdots & 0.0412 \\ 0.000494 & -0.00887 & 0.0750 & \cdots & 0.0388 \\ -0.0000811 & 0.00146 & -0.0123 & \cdots & 0.0359 \\ 0.0000186 & -0.000335 & 0.00283 & \cdots & 0.0339 \\ -0.00000562 & 0.000101 & -0.000854 & \cdots & 0.0281 \\ 0.00000213 & -0.0000382 & 0.000323 & \cdots & 0.0303 \\ -9.67\times10^{-7} & 0.0000174 & -0.000147 & \cdots & 0.0163 \\ 4.99\times10^{-7} & -0.00000898 & 0.0000759 & \cdots & 0.0244 \\ -2.36\times10^{-7} & 0.00000424 & -0.0000358 & \cdots & 0.00500 \end{pmatrix}_{10\times10}$$

The operational matrices P_B^α and $P_B^{-\alpha}$ are easily computable through using mathematical softwares.

6. The solution of the fractional differential equation by the Bernstein operational matrix

In this section, we are concerned with providing a numerical solution to Cauchy-type problem inhomogeneous differential equation of fractional order of the form

$$\frac{d^\alpha}{dx^\alpha} y(x) - \lambda y(x) = h(x), \quad n-1 < \alpha \leq n, \tag{18}$$

with initial conditions

$$\left.\frac{d^{\alpha-k}}{d\,x^{\alpha-k}} y(x)\right|_{x=0} = b_k, \quad k = 1, 2, ...n$$

We apply the fractional integral $I_{a+}^\alpha$with respect x to the system (19) to obtain

$$I_{a+}^\alpha D_{a+}^\alpha y(x) - \lambda I_{a+}^\alpha y(x) = I_{a+}^\alpha h(x), \; n-1 < \alpha \leq n,$$

thus, by using (9), this system gives

$$y(x) - \lambda I_{a+}^\alpha y(x) = I_{a+}^\alpha h(x)$$

$$+ \sum_{k=0}^{n-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} y_{n-\alpha}^{(n-k-1)}(a)$$

for $n - 1 < \alpha \leq n$,
here, we define a new function as follows:

$$g(x) = I_{a+}^\alpha h(x) + \sum_{k=0}^{n-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} y_{n-\alpha}^{(n-k-1)}(a), \tag{19}$$

by rewriting the equation (19), we obtain

$$y(x) - \lambda I_{a+}^\alpha y(x) = g(x) \tag{20}$$

the $g(x)$ and $I_{a+}^\alpha y(x)$ must be expanded by the Bernstein polynomials as

$$g(x) \approx G_m(x) = C_g^T B_m(x) \tag{21}$$

$$I_{a+}^\alpha y(x) \approx Y_m(x) = C^T P_B^\alpha B_m(x) \tag{22}$$

where C_g is a known $m \times 1$ column vector but $C = [c_0, c_1, \cdots, c_{m-1}]$ is an unknown $m \times 1$ column vector.
With substituting Eq.(22)-(23) into (21),we have

$$C^T B_m(x) - \lambda C^T P_B^\alpha B_m(x) = C_g B_m(x)$$

according to the properties of orthogonal Bernstein polynomials, we have

$$C^T - \lambda C^T P_B^\alpha = C_g$$

or

$$C^T (I - \lambda P_B^\alpha) = C_g$$

Solving the system of algebraic equation, we can obtain the coefficients C^T.

$$C^T = C_g (I - \lambda P_B^\alpha)^{-1}$$

Then, we can get

$$y(x) \approx Y_m(x)$$

$$= C^T B_m(x) = C_g (I - \lambda P_B^\alpha)^{-1} B_m(x) \;,$$

7. Illustrative examples

Example 1: consider the fractional equation

$$\frac{d^\alpha}{dx^\alpha}y(x)+y(x)=h(x) \qquad (23)$$

With initial condition

$y(0)=0\,.$

These equations have relevance to, e.g., mechanical system with fractional order damping and under slow loading (where inertia plays a negligible role), such as in creep test [33]. The solution of (23) can be obtained by using Laplace transforms. In particular, if $h(x)=1$ then for $\alpha=0.5$ the exact solution is

$y(x)=1-e^{x}erfc(\sqrt{x}).$

By using (20) and initial condition $y(0)=0$, we have

$$g(x)=I^{\alpha}_{0+}h(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{h(t)}{(x-t)^{1-\alpha}}\,dt$$

Therefore, for $h(x)=1$, we obtain

$$g(x)=\frac{x^\alpha}{\Gamma(\alpha+1)}$$

Then, for $\lambda=-1$ and by replacing in (21), we get

$$y(x)\ +I^{\alpha}_{0+}y(x)=\frac{x^\alpha}{\Gamma(\alpha+1)}$$

if $\alpha=0.5$,our approximate solutions for $y(x)$ for $m=10$ and $m=43$ and exact solution for $y(x)$ over interval $[0,25]$ is shown in fig.1.

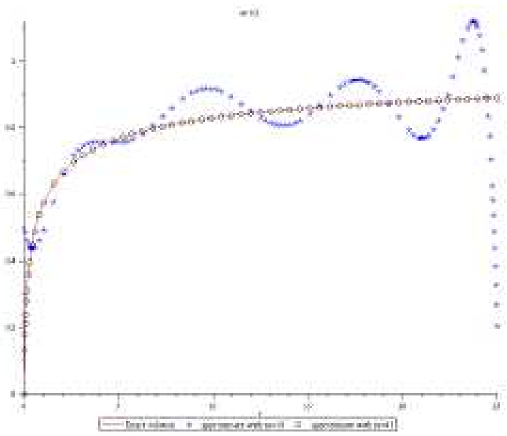

Fig.1. *Approximate solutions of $y(x)$ for $m=10, m=43$ and $\alpha=0.5$ and exact solution*

Example 2: Finally, we indicate that the simplest Cauchy problems for differential equation of fractional order

$$D^{\alpha}_{0+}y(x)-\lambda y(x)=0,\quad x>0,\ \ 0<\alpha<1 \qquad (24)$$

With initial condition,

$$I^{1-\alpha}_{0+}y(x)\big|_{x=0}=1$$

Solved by a similar method, our approximate solutions of $y(x)$ for variety α and $\lambda=1$ is shown by fig.2.

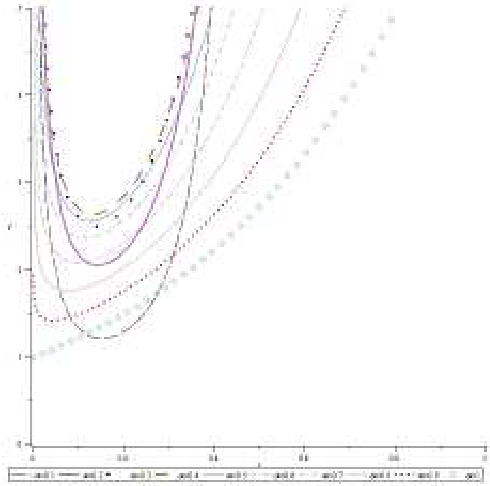

Fig.2. *Approximate solutions of $y(x)$ for $m=43$ and $\alpha=0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9$ and $\alpha=1$*

8. Conclusion

A lot of physical phenomena in different sciences have been presented by differential equations with fractional derivatives which are of more importance compared to integer derivatives. This is so because the solutions of these equations interpret the phenomena better. Recently, a lot of methods have been proposed to solve these equations. In this paper, a new method to solve fractional differential equation based on Bernstein polynomials has been presented. This method is a very simple one with high accuracy and programming capability with computer. Another advantage of this method is that with an increase in the number of base polynomials involving Bernstein polynomials, not only there is no deficiency in convergence but also the accuracy of calculations increases.

References

[1] E. Laroche, D. Knittel, An improved linear fractional model for robustness analysis of a winding system, Control Eng. Pract. 13 (2005) 659–666.

[2] A. Calderon, B. Vinagre, V. Feliu, Fractional order control strategies for power electronic buck converters, Signal Process. 86 (2006) 2803–2819.

[3] J. Sabatier, M. Aoun, A. Oustaloup, G. Grgoire, F. Ragot, P. Roy, Fractional system identification for lead acid battery state of charge estimation, Signal Process. 86 (2006) 2645–2657.

[4] V. Feliu-Batlle, R. Perez, L. Rodriguez, Fractional robust control of main irrigation canals with variable dynamic parameters, Control Eng. Pract. 15(2007) 673–686.

[5] B. Vinagre, C. Monje, A. Calderon, J. Suarez, Fractional PID controllers for industry application. A brief introduction, J. Vib. Control 13 (2007) 1419–1430.

[6] N.T. Shawagfeh, Analytical approximate solutions for nonlinear fractional differential equations, Appl. Math. Comput. 131 (2002) 517–529.

[7] S. Das, Analytical solution of a fractional diffusion equation by variational iteration method, Comput. Math. Appl. 57 (2009) 483–487.
[8] A. Arikoglu, I. Ozkol, Solution of fractional differential equations by using differential transform method, Chaos Solitons Fractals 34 (2007) 1473–1481.
[9] A. Saadatmandi, M. Dehghan, A new operational matrix for solving fractional order differential equations, Comput. Math. Appl. 59 (2010) 1326–1336.
[10] M. Dehghan, J. Manafian, A. Saadatmandi, Solving nonlinear fractional partial differential equations using the homotopy analysis method, Numer. Methods Partial Differential Equations 26 (2010) 448–479.
[11] M. Dehghan, J. Manafian Herris, A. Saadatmandi, The Solution of the linear fractional partial differential equations using the homotopy analysis method, Z. Naturforsch. 65a (2010) 935–949.
[12] S. Momani, Z. Odibat, Numerical comparison of methods for solving linear differential equations of fractional order, Chaos Solitons Fractals 31 (2007) 1248–1255.
[13] Z. Odibat, N. Shawagfeh, Generalized Taylor's formula, Appl. Math. Comput. 186 (2007) 286–293.
[14] I. Podlubny, The Laplace transform method for linear differential equations of the fractional order, 1997, eprint arXiv:funct-an/9710005.
[15] C. Chen, C. Hsiao, A state-space approach to Walsh series solution of linear systems, Int. J. Syst. Sci. 6 (1975) 833–858.
[16] W. Glabisz, Direct Walsh-wavelet packet method for variational problems, Appl. Math. Comput. 159 (2004) 769–781.
[17] H. Chyi, S. Yen-Ping, Solution of population balance equations via block pulse functions, Chem. Eng. J. 25 (1982) 39–45.
[18] A. Deb, G. Sarkar, M. Bhattacharjee, S.K. Sen, All-integrator approach to linear SISO control system analysis using block pulse functions (BPF), J. Franklin Inst. 334 (1997) 319–335.
[19] K. Maleknejad, M. Shahrezaee, H. Khatami, Numerical solution of integral equations system of the second kind by block-pulse functions, Appl. Math. Comput. 166 (2005) 15–24.
[20] E. Babolian, Z. Masouri, Direct method to solve Volterra integral equation of the first kind using operational matrix with block-pulse functions, Comput. Math. Appl. 220 (2008) 51–57.
[21] Yuanlu Li, Ning Sun, Numerical solution of fractional differential equations using the generalized block pulse operational matrix, Computers and Mathematics with Applications, in press.
[22] C. Hwang, M. Chen, Analysis and optimal control of time-varying linear systems via shifted Legendre polynomials, Internat. J. Control 41 (1985) 1317–1330.
[23] P. Paraskevopoulos, Legendre series approach to identification and analysis of linear systems, IEEE Trans. Automat. Control 30 (1985) 585–589.
[24] X.T. Wang, Numerical solutions of optimal control for time delay systems by hybrid of block-pulse functions and Legendre polynomials, Appl. Math. Comput. 184 (2007) 849–856.
[25] K. Maleknejad, S. Sohrabi, Y. Rostami, Numerical solution of nonlinear Volterra integral equations of the second kind by using Chebyshev polynomials, Appl. Math. Comput. 188 (2007) 123–128.
[26] C. Hwang, Y. Shih, Parameter identification via Laguerre polynomials, Int. J. Syst. Sci. 13 (1982) 209–217.
[27] S. Zaman, A. Jha, Parameter identification of non-linear systems using Laguerre operational matrices, Int. J. Syst. Sci. 16 (1985) 625–631.
[28] M. Razzaghi, S.D. Lin, Identification of time-varying linear and bilinear systems via Fourier series, Comput. Electron. Eng. 17 (1991) 237–244.
[29] M. Razzaghi, A. Arabshahi, S.D. Lin, Identification of nonlinear differential equations via Fourier series operational matrix for repeated integration, Appl. Math. Comput. 68 (1995) 189–198.
[30] samko G.s, Kilbas AA, Marichev OI, Fractional Integral and Derivatives,: Theory and Applications. Gordon and Breach, Amsterdam. (1993)
[31] Yousefi, S. A. and Behroozifar, M., 'Operational matrices of Bernstein polynomials and their applications', International Journal of Systems Science, 41:6, 709 — 716,(2010)
[32] B.N. Mandal, Subhra Bhattacharya, Numerical solution of some classes of integral equations using Bernstein polynomials, Applied Mathematics and Computation 190 (2007) 1707–1716
[33] J.sabatier, O.P.Agrawal, J.A.Tenreiro Machado, Advances in Fractional Calculus, Theoretical Developments and Applications in Physics Engineering, Springer (2007)

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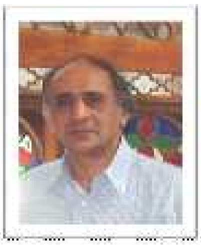

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