

CHAPTER SEVENTY TWO

A NUMERICAL SOLUTION OF BOUSSINESQ TYPE WAVE EQUATIONS

H. SCHAPER

W. ZIELKE

Institut of Fluid Mechanics
University of Hannover
Fed. Republic of Germany

1. INTRODUCTION

Numerical models of short waves in shallow water, which are of particular interest for the calculation of the wave climate in harbours and coastal areas, have been presented by Abbott et al. (1978) and by Hauguel (1980). These models are based on the solution of the Boussinesq or Serre type equations. A recent discussion of the range of application for the equations has been presented by McCowan (1982).

Nevertheless, there is some uncertainty as to which terms in the differential equations are of importance, and how they are to be approximated. Therefore, no final judgement can presently be made on the accuracy and credibility of the solutions. Research on such models is still in progress and is of high theoretical and practical interest.

Some of the aspects of current research relate to the handling of nonlinear terms, the non-reflecting boundary conditions and the transfer capability of the models for spectral input. This paper will reflect on these points.

2. BASIC EQUATIONS

For simplicity, we will restrict ourselves to the one-dimensional case, for which the Boussinesq type equations read

$$(1) \quad \frac{\partial p}{\partial t} + \frac{\partial}{\partial x} \left(\frac{p^2}{h} \right) + g h \frac{\partial \zeta}{\partial x} = \frac{D}{2} \frac{h}{\partial x^2} \frac{\partial^3}{\partial t} \left(\frac{dp}{h} \right) - \frac{D^2 h}{6} \frac{\partial^3}{\partial x^2 \partial t} \left(\frac{p}{h} \right)$$

$$(2) \quad \frac{\partial \zeta}{\partial t} + \frac{\partial p}{\partial x} = 0$$

with D = mean water depth, ζ = surface elevation, $h = D + \zeta$
 p = flux per unit width, g = acceleration due to gravity.

For the actual calculation with a finite difference method, the third order terms on the right hand side of (1) have to be rewritten such that each term contains only derivatives of either p or ζ . Two third order terms remain, as well as some 15 other terms with products of first and second order derivatives:

$$(1') \quad \dots = \frac{D^2}{3} \frac{\partial^3 p}{\partial x^2 \partial t} - \frac{D^2}{3} \frac{p}{h} \frac{\partial^3 \zeta}{\partial x^2 \partial t} + \text{product terms}$$

We assume that the product terms are small in comparison to the third order terms (Abbott et al., 1978). Equations (1') and (2) can be collected in matrix form

$$(3) \quad \frac{\partial w}{\partial t} + A \frac{\partial w}{\partial x} + B \frac{\partial^3 w}{\partial x^2 \partial t} = R$$

with

$$w = \begin{bmatrix} p \\ \zeta \end{bmatrix} \quad A = \begin{bmatrix} \frac{2p}{h} & gh - \frac{p^2}{h^2} \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -\frac{D^2}{3} & \frac{D^2 p}{3h} \\ 0 & 0 \end{bmatrix} \quad R = \begin{bmatrix} \frac{D^2}{h^2} D, x \\ 0 \end{bmatrix}$$

3. NUMERICAL APPROXIMATION

In order to implement the third order terms of (3), a two level scheme with central differences is used, the difference star being given in Fig. 1. The finite difference approximations of the partial derivatives are given by equations (4.a) - (4.c).

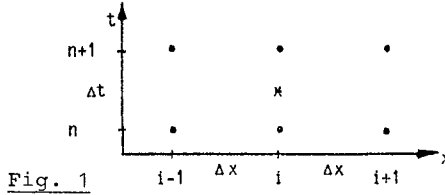


Fig. 1

$$(4.a) \quad \left. \frac{\partial w}{\partial t} \right|_i^{n+1/2} = \frac{w_i^{n+1} - w_i^n}{\Delta t} - \frac{\Delta t^2}{24} \frac{\partial^3 w}{\partial t^3} + O(\Delta^3)$$

$$(4.b) \quad \left. \frac{\partial w}{\partial x} \right|_i^{n+1/2} = \frac{w_{i+1}^{n+1} - w_{i-1}^{n+1} + w_{i+1}^n - w_{i-1}^n}{4\Delta x} - \frac{\Delta t^2}{8} \frac{\partial^3 w}{\partial x \partial t^2} - \frac{\Delta x^2}{6} \frac{\partial^3 w}{\partial x^3} + O(\Delta^3)$$

$$(4.c) \quad \left. \frac{\partial^3 w}{\partial x^2 \partial t} \right|_i^{n+1/2} = \frac{(w_{i+1}^{n+1} - 2w_i^{n+1} + w_{i-1}^{n+1}) - (w_{i+1}^n - 2w_i^n + w_{i-1}^n)}{\Delta x^2 \Delta t} + O(\Delta^3)$$

The third order correction terms are detected by Taylor series expansion of the used grid points.

At the boundary nodes 0 and N one sided difference stars have to be used and one coefficient of the correction terms is changed

$$(4.d) \quad \left. \frac{\partial w}{\partial x} \right|_0^{n+1/2} = \frac{(-3w_0 + 4w_1 + w_2)^{n+1} + (-3w_0 + 4w_1 - w_2)^n}{4\Delta x} - \frac{\Delta t^2}{8} \frac{\partial^3 w}{\partial x \partial t^2} + \frac{\Delta x^2}{3} \frac{\partial^3 w}{\partial x^3} + O(\Delta^3)$$

$$(4.e) \quad \left. \frac{\partial w}{\partial x} \right|_N^{n+1/2} = \frac{(w_{N-2} - 4w_{N-1} + 3w_N)^{n+1} + (w_{N-2} - 4w_{N-1} + 3w_N)^n}{4\Delta x} - \frac{\Delta t^2}{8} \frac{\partial^3 w}{\partial x \partial t^2} + \frac{\Delta x^2}{3} \frac{\partial^3 w}{\partial x^3} + O(\Delta^3)$$

Clearly all the correction terms cannot be approximated with the used 6-point star, because only a derivative of the kind $\partial^3/\partial x^2 \partial t$ is possible. So the correction terms needed to be transformed into this form. As pointed out in literature (Peregrine, 1967, Abbott et al., 1978) the linearised equations (5) can be used and the transformations are given in (5.a) - (5.c).

$$(5) \quad \frac{\partial w}{\partial t} + L \frac{\partial w}{\partial x} = 0 \quad \text{with} \quad L = \begin{bmatrix} 0 & gD \\ 1 & 0 \end{bmatrix}$$

$$(5.a) \quad \frac{\partial^3 w}{\partial t^3} = L^2 \frac{\partial^3 w}{\partial x^2 \partial t}$$

$$(5.b) \quad \frac{\partial^3 w}{\partial t^2 \partial x} = -L \frac{\partial^3 w}{\partial x^2 \partial t}$$

$$(5.c) \quad \frac{\partial^3 w}{\partial x^3} = -L^{-1} \frac{\partial^3 w}{\partial x^2 \partial t}$$

Substitution of the finite difference approximations into equation (3) leads to

$$(6) \quad w_{,t} - \frac{\Delta t^2}{24} L^2 w_{,xxt} + A w_{,x} + \frac{\Delta t^2}{8} \underbrace{A L}_{L^2} w_{,xxt} + \frac{\Delta t^2}{6} \underbrace{A L^{-1}}_I w_{,xxt} + B w_{,xxt} = R$$

The product AL^{-1} can be replaced by the unit matrix, because L is the linearised form of A . The product AL is linearised to L^2 , and all third order terms can be collected in matrix C .

$$(7) \quad w_{,t} + A w_{,x} + \underbrace{\left(B + \frac{\Delta X^2}{12} L^2 + \frac{\Delta X^2}{6} I \right)}_C w_{,xxt} = R$$

Now equation (7) can be rearranged with respect to the used grid points

$$\begin{aligned}
(8) \quad & \left[-\frac{\Delta t}{4\Delta x}A + \frac{1}{\Delta x^2}C \right] w_{i-1}^{n+1} + \left[I - \frac{2}{\Delta x^2}C \right] w_i^{n+1} + \left[\frac{\Delta t}{4\Delta x}A + \frac{1}{\Delta x^2}C \right] w_{i+1}^{n+1} \\
& = \left[\frac{\Delta t}{4\Delta x}A + \frac{1}{\Delta x^2}C \right] w_{i-1}^n + \left[I - \frac{2}{\Delta x^2}C \right] w_i^n + \left[-\frac{\Delta t}{4\Delta x}A + \frac{1}{\Delta x^2}C \right] w_{i+1}^n + \Delta t R
\end{aligned}$$

and the coefficients of the unknowns can be collected in the matrices D , E , F , while the right hand side of (8) is collected in the vector G .

$$(9) \quad D_i w_{i-1}^{n+1} + E_i w_i^{n+1} + F_i w_{i+1}^{n+1} = G_i$$

Equation (9) is applied to all internal grid points $1 \dots N-1$. Similar equations can be performed for the boundary nodes 0 and N .

Neglecting the nonlinearity and the implementation of boundary conditions, which are described in detail later, a linear equation system of the following form arises.

$$\begin{bmatrix}
D_0 & E_0 & F_0 & & & & & & & & \\
D_1 & E_1 & F_1 & & & & & & & & \\
& D_2 & E_2 & F_2 & & & & & & & \\
& & D_3 & E_3 & F_3 & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & & & & \\
& & & & & & & D_{N-1} & E_{N-1} & F_{N-1} & \\
& & & & & & & D_N & E_N & F_N &
\end{bmatrix}
*
\begin{bmatrix}
w_0 \\
w_1 \\
w_2 \\
w_3 \\
\vdots \\
\vdots \\
w_{N-1} \\
w_N
\end{bmatrix}
=
\begin{bmatrix}
G_0 \\
G_1 \\
G_2 \\
G_3 \\
\vdots \\
\vdots \\
G_{N-1} \\
G_N
\end{bmatrix}$$

This system is solved very efficiently by a double sweep algorithm, taking into account the matrices F_0 and D_N .

4. NONLINEAR TERMS

The differential equations are solved in the points $*(i, n+1/2)$. The matrices A and B, therefore, have to be evaluated at the time level $n+1/2$. A linearization by using the time level n can only be accepted, when calculating sufficiently long and shallow waves. The unknown values of $W^{n+1/2}$, arising in A and B, can be calculated explicitly from a Taylor series expansion

$$(10) \quad w_i^{n+1/2} = w_i^n + \frac{\Delta t}{2} \left(A \frac{\partial w}{\partial x} + B \frac{\partial^3 w}{\partial x^2 \partial t} - R \right) + \frac{\Delta t^2}{8} \frac{\partial^2 w}{\partial t^2} + \dots$$

by substitution of the differential equations (3) for $\partial w / \partial t$.

Some remarks should be made to this point:

- the third order terms of (3) are negligible, because of less influence;
- the second order term in (10) can also be neglected
- the convective parts of the first order terms of (3) cannot be neglected, especially in the calculation of high waves.

5. BOUNDARY CONDITIONS

It is of great interest for the practical application of numerical wave models, to introduce the waves at the boundary of the model using a time serie of the water elevation.

Such a condition is introduced by substitution of

$$(11) \quad \zeta_0 = f(t)$$

for either the momentum or the continuity equation at the node 0.

In this case, for example, the coefficient matrices will take the following form, where the coefficients of the remaining original equation are denoted by x

$$\begin{matrix} \begin{bmatrix} x & x \\ 0 & 1 \end{bmatrix} w_0^{n+1} & + & \begin{bmatrix} x & x \\ 0 & 0 \end{bmatrix} w_1^{n+1} & + & \begin{bmatrix} x & x \\ 0 & 0 \end{bmatrix} w_2^{n+1} & = & \begin{bmatrix} x \\ f^{n+1} \end{bmatrix} \\ D_0 & & E_0 & & F_0 & & G_0 \end{matrix}$$

In the same way, a condition of total reflection may be introduced at the node N, which means

$$(12) \quad p_N = 0$$

$$\begin{matrix} \begin{bmatrix} 0 & 0 \\ x & x \end{bmatrix} w_{N-2}^{n+1} & + & \begin{bmatrix} 0 & 0 \\ x & x \end{bmatrix} w_{N-1}^{n+1} & + & \begin{bmatrix} 1 & 0 \\ x & x \end{bmatrix} w_N^{n+1} & = & \begin{bmatrix} 0 \\ x \end{bmatrix} \\ D_N & & E_N & & F_N & & G_N \end{matrix}$$

If waves are reflected from within the domain, non reflecting boundary conditions have to be used, which allow waves to leave the domain undisturbed. In this case, we use the equation given by Hanguel (1980), which he adopted from long wave theory.

$$(13) \quad p_n - c \zeta = -2c f(t)$$

with p_n flux normal to the boundary, ζ water elevation, c wave celerity, $f(t)$ water elevation of the incoming wave

Equation (13) may be slightly changed for simulation of partial reflection, occurring at breakwaters, for example. Naturally $f(t)$ is equal to zero in this case, and with a reflection coefficient γ we get:

$$(14) \quad p_n - (1-\gamma) c \zeta = 0$$

Both conditions (13) and (14) are as well easily introduced as (11) and (12).

6. NUMERICAL RESULTS

Regular as well as irregular waves may be input, using the time function $f(t)$. For test calculations waves of permanent form, cnoidal waves, as well as their limiting case, solitary waves, are of particular interest.

Figures 2 and 3 show calculations of solitary waves where the coefficients are based only on the old time level n . As shown in Fig.2 for waves with an amplitude of 1 m, the errors due to the nonlinearity may be suppressed by using a very small time step (Fig. 2b). This was not possible for 2 m waves (Fig. 3).

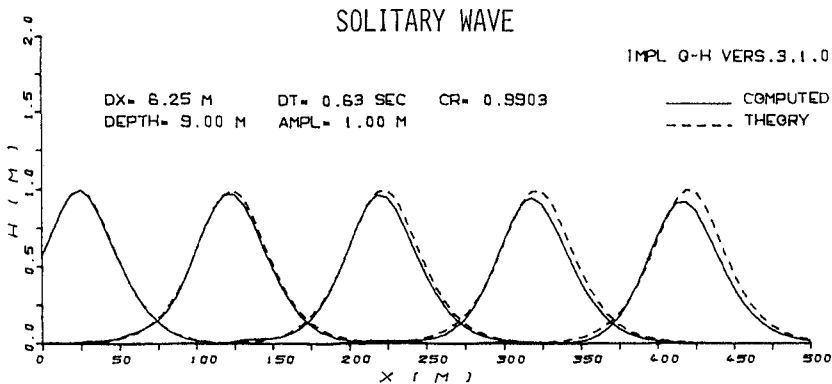


Fig. 2 a

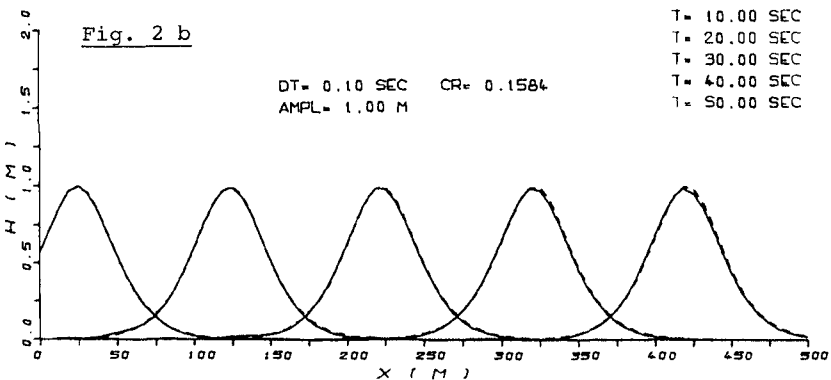


Fig. 2 b

SOLITARY WAVE

IMPL Q-H VERS.3.1.1.0

DX= 6.25 M DT= 0.63 SEC CR= 0.9903
 DEPTH= 8.00 M AMPL= 2.00 M

———— COMPUTED
 - - - - - THEORY

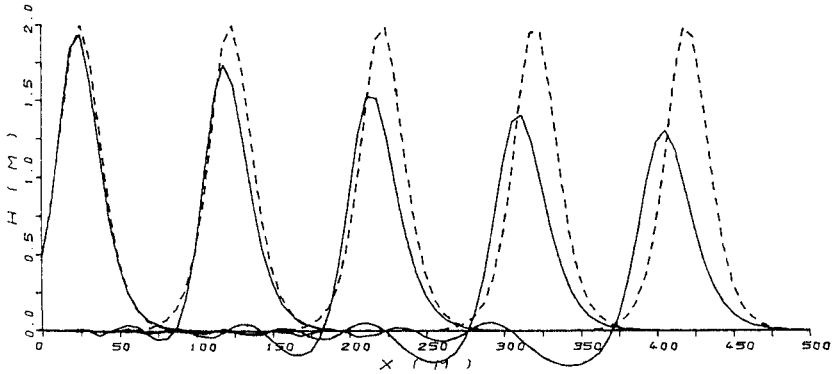


Fig. 3 a

DX= 6.25 M DT= 0.10 SEC CR= 0.1584
 DEPTH= 8.00 M AMPL= 2.00 M

T= 10.00 SEC
 T= 20.00 SEC
 T= 30.00 SEC
 T= 40.00 SEC
 T= 50.00 SEC

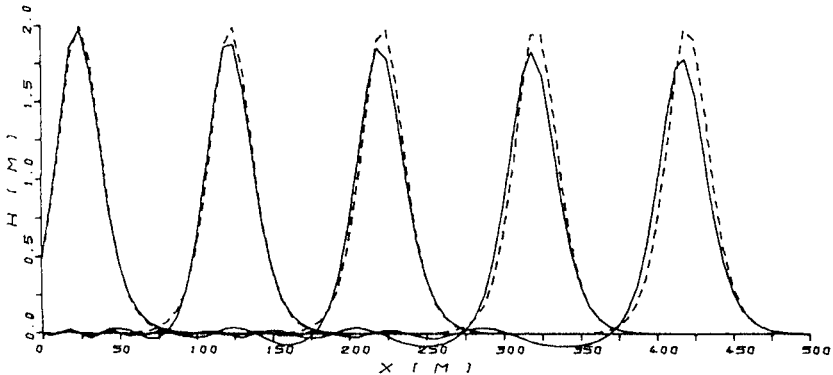


Fig. 3.b

The advantage of determining the coefficients from the time level $n + 1/2$ (equation (10)) is shown in Fig. 4, for the case of 2 m waves and a time step of 0,625 sec.

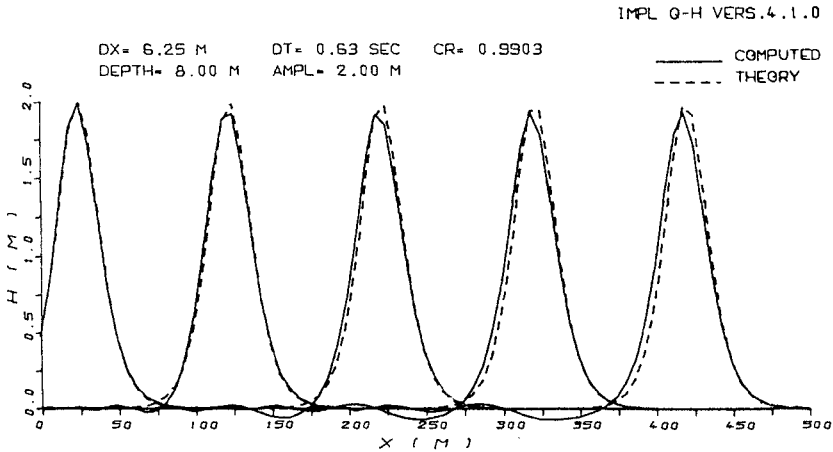


Fig. 4

Fig. 5 shows calculations for a solitary wave of limited height. Again it should be noted, that the wave is fed into the system using a time function for the water elevation only.

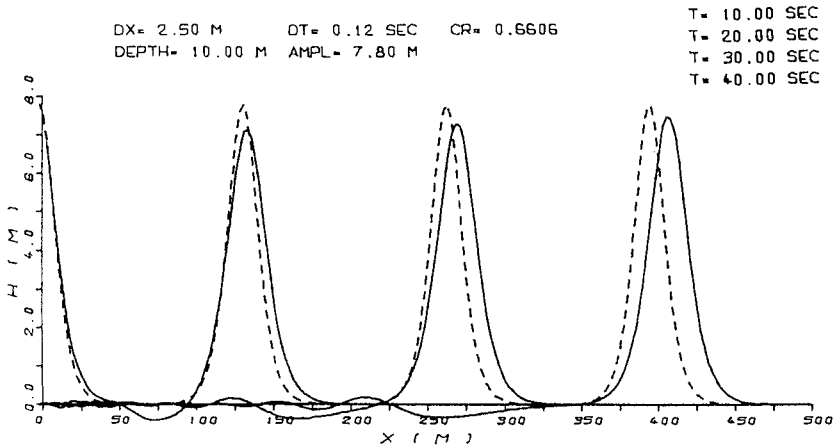
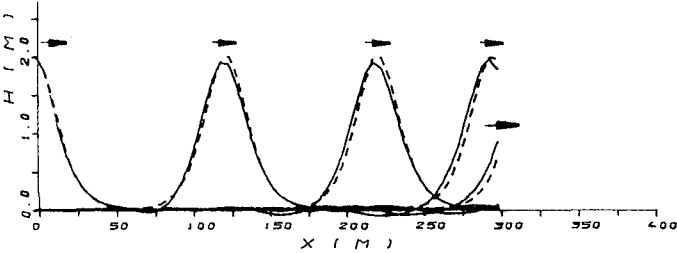


Fig. 5

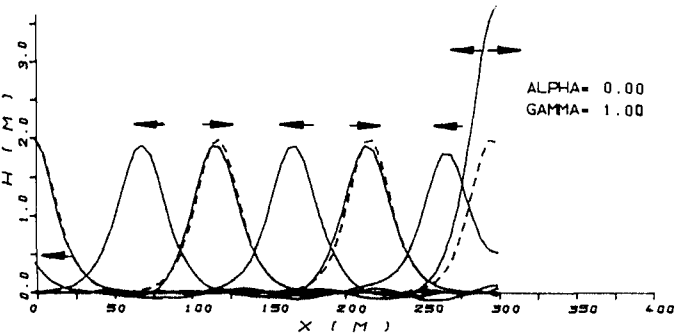
The behavior of the non-reflecting boundary conditions is shown in Fig. 6. In case (a) the reflection coefficient is zero and the wave leaves through the right boundary, located at $x = 300$ m. In case (b) where γ is equal to one, the wave is totally reflected at the boundary and leaves the solution domain through the left boundary. In case (c) the reflection coefficient is 0,7 and the wave is partially reflected.

DX= 5.00 M DT= 0.50 SEC CR= 0.9903 ALPHA= 0.00 _____ COMPUTED
 DEPTH= 8.00 M AMPL= 2.00 M GAMMA= 0.00 - - - - - THEORY



T= 7.50 SEC
 T= 20.00 SEC
 T= 30.00 SEC
 T= 37.50 SEC
 T= 40.00 SEC
 T= 42.50 SEC
 T= 50.00 SEC

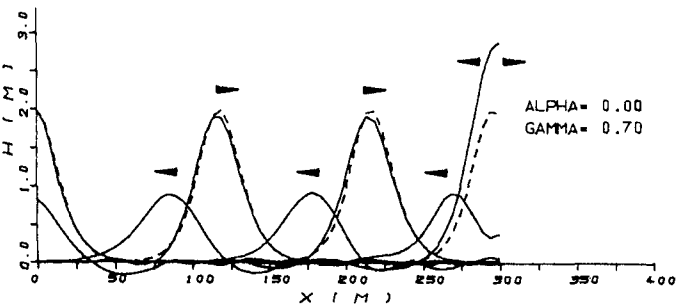
Fig. 6 a



ALPHA= 0.00
 GAMMA= 1.00

T= 8.00 SEC
 T= 20.00 SEC
 T= 30.00 SEC
 T= 38.00 SEC
 T= 42.00 SEC
 T= 52.00 SEC
 T= 62.00 SEC
 T= 72.00 SEC

Fig. 6 b



ALPHA= 0.00
 GAMMA= 0.70

T= 8.00 SEC
 T= 20.00 SEC
 T= 30.00 SEC
 T= 38.00 SEC
 T= 42.00 SEC
 T= 52.00 SEC
 T= 62.00 SEC
 T= 72.00 SEC

Fig. 6 c

Figure 7 shows the transformation of a solitary wave into several waves, due to its passing over a sloped bottom onto a shelf. These results are in good agreement with those of Hauguel (1980).

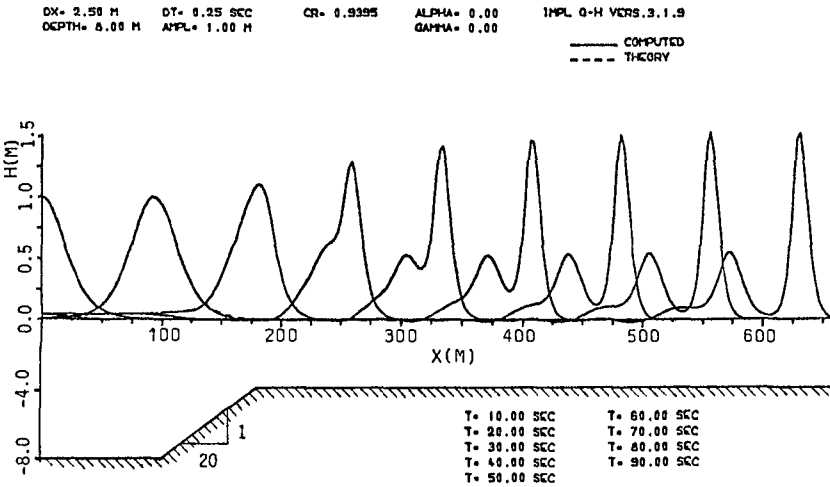


Fig. 7

Figures 8 a, b show the calculation of a cnoidal wave with a wave height of 2 m and a period of 12 sec. The water depth was held constant at 10 m.

Case (a), with a zero reflection coefficient, is again a test for the radiation condition at the right boundary.

In case (b), with total reflection, a standing wave results with a wave height of 4.30 m. The period of 12 sec and the wave length of 115 m remain unchanged. This example is a severe test of the left boundary condition.

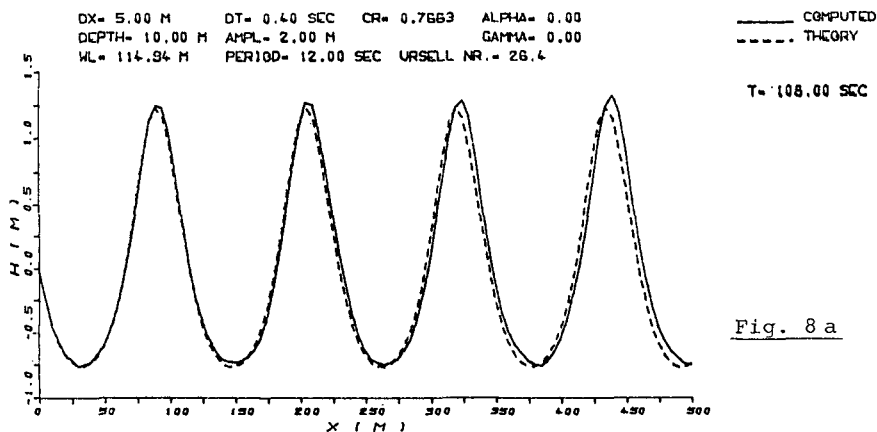


Fig. 8 a

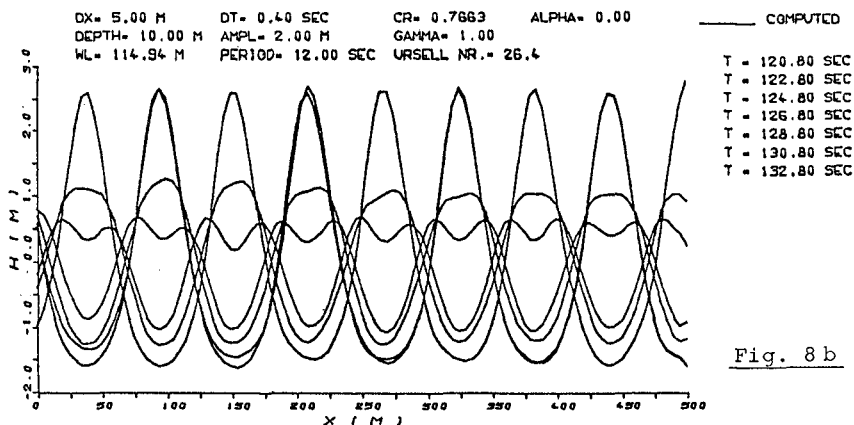
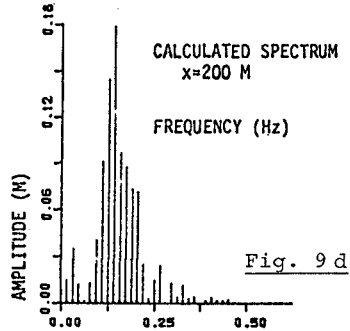
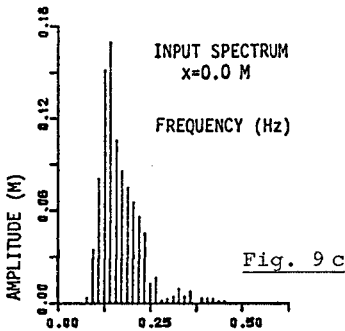
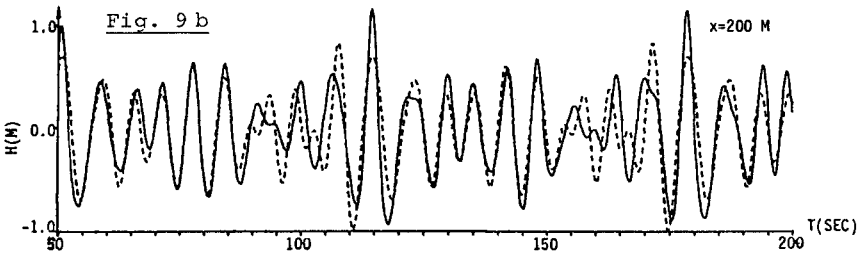
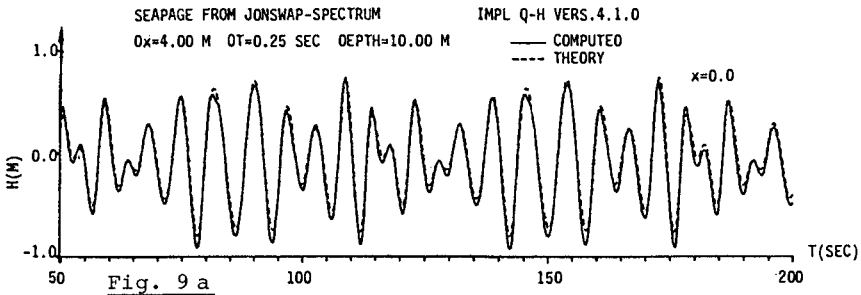


Fig. 8 b

One of the attractive features of a Boussinesq type wave model is the ability to calculate nonlinear irregular waves.

The transformation of a Jonswap spectrum into a time series is used as input to the numerical model (Fig. 9 a, c). The wave length at the peak frequency was approximately 50 m and the shortest components less than 30 m. The water depth was held constant at 10 m. Figure 9 b shows the water elevations at $x = 200$ m and Figure 9 d the spectra, calculated with Fast Fourier Transformation.

The dotted lines in Figs. 9 a, b are obtained analytically from a superposition of the single frequency components. One can observe that some interaction between frequencies occurs in the numerical model. The results can be better understood, however, if one considers the frequency domain, which includes lower frequencies (Fig. 9 c). Only a comparison with experimental data can verify, whether these frequencies are physically meaningful. Experiments to obtain such data are planned for the near future.



The last test illustrates the essential differences between the results of a Boussinesq type wave equation and a shallow water wave equation. Fig.10 shows, that a hydraulic surge, under certain conditions, is transformed into a moving undular hydraulic jump.

These undulations are a result of the vertical accelerations included in Boussinesq type wave equations. Shallow water wave equations, which neglect vertical accelerations, would produce only a step in the water elevation, as indicated by the dotted line in Fig. 10.

The frequencies, as well as the amplitudes of the undulations, are in good agreement with experimental results given by Favre (1935).

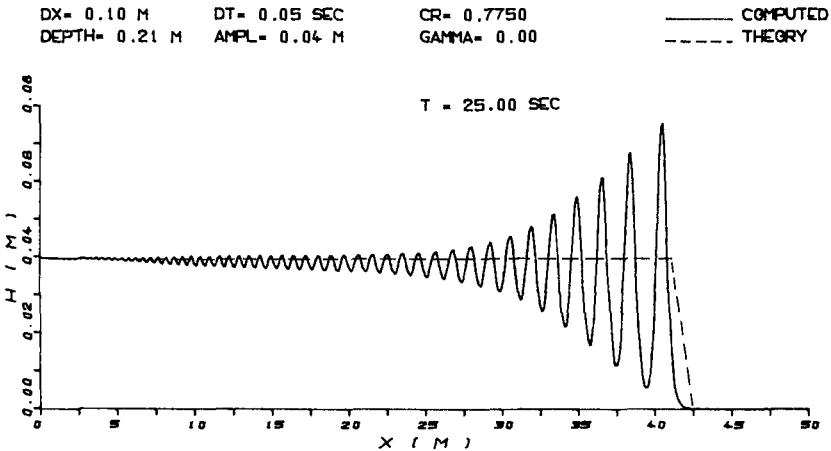


Fig. 10

7. CONCLUSIONS

Numerical solutions of Boussinesq type wave equations provide a powerful tool for the prediction of short waves in shallow water. They cover a wide range, including intermediate depth conditions and wave heights just before breaking.

Solitary and cnoidal waves, as well as irregular waves have been calculated. The necessary boundary conditions, such as total reflection, partial reflection and non-reflection have been successfully incorporated in the model.

The results presented in this paper are limited to the one dimensional case. Fig. 11 shows a result of the two dimensional version, currently under investigation.

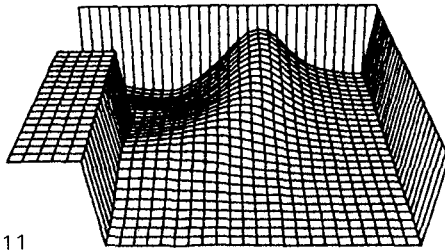


Fig. 11

8. REFERENCES

- Abbot, M.B., Petersen, H.M. and Skovgaard, O. (1978): On the numerical modelling of short waves in shallow water. J.Hyd.Res., Vol. 16, No. 3.
- Favre, H. (1935): Etude théorique et expérimentale des ondes de translation. Dunod, Paris.
- Hauguel, A. (1980): Adaption of tidal numerical models to shallow water wave problems. Proc. 17th Conf. on Coastal Eng., Sydney.
- McCowan, A.D. (1982): Numerical short wave modelling - an evaluation. Abstracts 18th Conf. on Coastal Eng., Cape Town.
- Peregrine, D.H. (1967): Long Waves on a Beach. J. Fluid Mech., Vol. 27, No. 4.

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